

Asset Pricing With Linear Collateral Constraints *

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Abstract

We study the pricing of financial securities in the presence of linear collateral constraint in a finite dimensional multiperiod partial equilibrium setup. We take the collateral constraint as given exogenously and focus on its impact on asset prices. Securities differ in their cashflows and usefulness as collateral and agents can not borrow more than the collateral value of the securities they purchase. We show that, absent arbitrage, the value of any security is the sum of its discounted future cash flow and its future collateral service value. We also show that the expected excess return on any security is the sum of risk premium and solvency premium. The risk premium is proportional to covariance of the return with that of a “pricing portfolio” whereas the solvency premium is proportional to the potential arbitrage profit in the absence of collateral constraints. We express some of these relationships in the form of a collateral-adjusted CAPM. In addition, we establish existence of optimal intertemporal consumption for all investors and show that an investor’s marginal utility at his optimal consumption and his shadow price of the collateral constraint can be used to price, respectively, the future cash flow and the future collateral service of any security. We use this result to study the equilibrium riskfree interest rate and extend Breeden’s (1979) consumption CAPM to a collateral adjusted consumption CAPM model. We also discuss the empirical implication of such an extension.

1 Introduction and Summary

In this paper we study the pricing of financial securities in the presence of trading constraints. We study markets with two characteristics. First, securities can be purchased on margin and the margin requirements vary across securities. A position in a high quality security can be financed by borrowing a high fraction of its value whereas only a smaller fraction can be borrowed to purchase a lower quality security. Second, traders are required to maintain financial solvency at all times. Solvency is maintained when the net value of the trader's position usable as collateral is nonnegative. When all securities have the same collateral quality, this constraint is simply that wealth is always nonnegative.

Our study contributes to a growing literature that addresses the effects of market frictions on asset prices. Asset pricing theories that assume a frictionless complete market failed to explain the relationship between consumption and market return data; see Hansen and Singleton (1982), Mehra and Prescott (1985), and Hansen and Jagannathan (1991).

Heaton and Lucas (1992) and Aiyagari and Gertler (1990) showed that the combined effects of uninsurable income shocks, transactions costs, and borrowing and short sale constraints can help explain part of the equity premium puzzle of Mehra and Prescott (1985). Lucas (1991), however, showed that, when transactions costs are zero, the combined effect of transitory income shock and short sale constraint does not help explain the equity premium puzzle.¹ Luttmer (1991), He and Modest (1992), Jouini and Kallal (1993), and Chen (1994) studied the implication of the absence of arbitrage in securities markets with short sale constraints. The conclusion of these studies is that arbitrage-free prices are greater than or equal to the discounted future payoffs of the securities. This inequality relationship between asset prices and payoffs is insufficient for studying the systematic effect of market frictions on cross sectional asset returns.

In another strand of the literature, He and Pearson (1990) and Xu and Shreve (1992) used duality approach to study the optimal consumption and portfolio policy problem in

¹As Constantinides and Duffie (1992) pointed out, part of the reason is that transitory income shocks do not prevent consumers from effectively smoothing consumption by saving.

markets with short sale constraint. Cuoco (1994) studies the same problem when there is uncertain labor income and the holdings of risky securities are constrained to be in a general convex set while Cvitanić and Karatzas (1992) study the portfolio optimization problem for the case in which the holdings of all securities are constrained to be in a general convex set. Brown (1988) argued that consumption-based CAPM does not hold in the presence of borrowing constraint against non-marketable future income. He and Pagès (1993), however, showed that the nonnegative market wealth constraint in the presence of uninsurable labor income shocks does not invalidate the consumption-based CAPM.

In this paper, we contribute to this literature on two dimensions. First, we study more realistic trading constraints. Second, we analyze the systematic effect of the constraint on the cross-sectional variation in asset returns. The extant studies that analyze the impact of constraints on asset prices have assumed either short sale constraint on the risky assets or nonnegative market wealth constraint. Short sales constraints on risky assets do not formally restrict the investor from borrowing to purchase securities. Nonnegative wealth constraints do not discriminate between risky assets of varying liquidity and collateral value. We study a linear class of collateral constraints that captures the market practice that investors can take short positions in financial market by posting collaterals and that different securities can have different collateral qualities. We note that such a constraint is a reasonable description of the kind of constraint that exist in, for instance, the Repo market. We will show that the linear collateral constraint can have systematic effects on asset returns that can help explain part of the inconsistency between market data and frictionless asset pricing theories.

We emphasize that the approach we take in this paper is of a partial equilibrium nature — we take the existence of collateral constraint and the collateral qualities of different securities as exogenously given and focus on their impact on asset prices. Collateral constraints exist in the market because a collateral allows the lender to extend a loan without expending any resources to establish the creditworthiness of the borrower. This increases the depth of the borrowing market. Securities differ in the ease by which their market

value can be determined and the daily volatility of their prices. As a result, lenders might demand different levels of protection for money lent against different securities. Ideally, we would like to endogenize the collateral constraint and collateral qualities of securities in a general equilibrium model with default and understand how the collateral quality of a security depends on its characteristics such as riskiness and liquidity. General equilibrium models with default, however, are in general difficult to characterize; see, for example, Dubey, Geanakoplos, and Shubik (1989). In this paper, we sidestep the issue of endogeneity of collateral constraints and focus on analyzing the effects of collateral constraint on asset prices. This approach, in addition to simplifying our analysis, allows our conclusions to remain valid in any analysis of the source of the collateral constraint assuming that no individual investor's action can affect the collateral qualities or market prices of securities.

In Section 2, we introduce the market, describe the securities, and discuss the trading constraints. For simplicity of exposition, we work with a finite number of trading dates and a finite number of unknown states. Hindy (1995) studies the implication of viability on security prices in the context of a continuous time infinite-dimensional state space.

We study the market from a number of view points. We analyze in Section 3 the implications of absence of arbitrage. We define a collateral constrained arbitrage opportunity, CC-arbitrage henceforth, as a trading strategy that requires no investment at any time, satisfies the solvency requirements, and produces strictly positive cash flow with strictly positive probability. We show that, in the absence of CC-arbitrage, there must exist two stochastic discount factors. The first factor discounts the future uncertain cash flows and the second discounts the future collateral value. The price of any security can be written as the sum of its discounted future cash flows and its discounted future collateral value. This shows that, in the presence of solvency constraints, the price of any security is not due solely to its future cash flows. Rather, a part of the value of the security is attributable to its future solvency services. As a result, two securities with the same cash flows but with different collateral value will trade at different prices without creating unlimited arbitrage

opportunities.²

In Section 4 we study the cross sectional variation in expected excess returns. The traditional single factor Sharpe CAPM (Sharpe (1964)) states that the excess expected return on any security is proportional to the covariance of its return with that of a “pricing portfolio”. The pricing portfolio is that combination of assets with the maximum possible correlation with the stochastic discount factor. In many versions of the CAPM, the pricing portfolio is the market portfolio. We show that the traditional single factor CAPM does not generally hold in the presence of solvency restrictions. The expected excess return on any security has two components. The first component is proportional to the covariance of the return with that of a pricing portfolio. This is the risk premium. The second component is inversely proportional to the collateral value of the security. This is the solvency premium of the security. Securities with relatively higher collateral value command relatively higher prices, and, hence, relatively lower expected excess return. We also show that a collateral-adjusted CAPM obtains if one defines the total return on a security to include its solvency services in addition to its dividends and capital gains.

In order to quantify the size of the solvency premium, we quantify the magnitude of the collateral discount factor. In Section 4.2, we introduce the notion of “potential arbitrage” which is the riskless profit with no investment that would obtain if the solvency constraint were not imposed. The magnitude of potential arbitrage for every dollar of the potential arbitrage strategy is a measure of the deviation from the law of one price. we show that the magnitude of the collateral discount factor is proportional to the size of the potential arbitrage profit. With relatively small potential arbitrage profits and typical margin requirements on securities we show that the solvency premium on securities can be as high as 2–3%.

In Section 5, we turn our attention to the relationship between aggregate consumption

²In making this statement, we assume that the collateral quality of a security can depend on factors other than its future cash flows, such as its liquidity. We do not study the issue of what determines a security’s collateral quality. The answer to this question requires a model with endogenous collateral constraint.

and the excess expected returns on securities. We show that, in the finite-dimensional model we consider, every agent will find an optimal consumption and portfolio policy. We also show that a pair of stochastic discount factors for cash flows and collateral can be taken to be, respectively, proportional to the marginal utility of consumption and the shadow price of the collateral constraints for every agent. With these results in hand, we first show that the equivalence between the riskfree return and the inverse of consumers' expected intertemporal marginal rate of substitution in a frictionless market no longer holds in a market with collateral constraints. The difference is equal to the solvency premium. We then proceed to construct a collateral adjusted consumption-based CAPM. Similar to Breeden's (1979) model, we show that the risk premium on any security is related to the correlation of its returns with the aggregate consumption levels. In contrast to Breeden (1979), however, we show that the risk premium is but one component of the total expected excess return. The balance is accounted for by the solvency premium. The empirical implication of this extension is also discussed.

Section 6 concludes the paper. The appendix contains some technical lemmas and all proofs.

2 Formulation

We consider economies with uncertainty about a finite number of possible future events. Wealth is transferred and traded in a multiperiod market for financial assets. There is a finite number of possible trading dates that also serve as points at which uncertainty is partially resolved. Agents exchange the financial assets in unlimited quantities and with no transaction costs. Agents, however, are required to remain financially solvent as long as they participate in the market. Details of the resolution of uncertainty, financial securities, and trading limitations are given next.

2.1 Uncertainty

Consider a securities market that is open for trade on $T + 1$ dates $\{0, 1, \dots, T\}$. We model uncertainty and its resolution by an event tree. Any branch ω in the event tree describes one complete history of the exogenous uncertain environment. The collection Ω of all branches is all possible states of nature from time 0 to time T . Information is resolved over time through a sequence of successively finer partitions³ of the set Ω . We will use \mathbf{F} to denote the sequence of partitions, or *filtration*, $\{\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_T\}$. The filtration \mathbf{F} is the common information structure with which agents are endowed. Individuals agree on the flow of information in the economy but may have heterogeneous beliefs about the probabilities of future events. We require, however, that all agents assign strictly positive probability to each state of nature. In that case, agents are said to have *equivalent* probability beliefs.

For simplicity, and with no loss of generality, we can take $\mathcal{F}_0 = \{\Omega\}$ and \mathcal{F}_T to be the partition generated by all individual states. In other words, individuals only know, at time zero, that the state of nature is one of the elements of Ω and learn the true state by time T . To describe the events at each time t , we use $E_{i,t} \in \mathcal{F}_t, i = 1, 2, \dots, l_t$ to denote the l_t distinct events (or tree branches) at t .

An adapted process is a sequence $X = \{X_0, \dots, X_T\}$ such that, for each t , X_t is a random variable whose value depends only on the information available at time t . All processes to appear in the paper are adapted and we henceforth drop this qualifier. We use \mathcal{L} to denote the space of all processes. It is useful to note that \mathcal{L} is an Euclidean space whose dimension equals the sum of the number of events in the partitions $\mathcal{F}_t; t = 0, 1, \dots, T$.

2.2 Securities

There are N long-lived securities in the market. Security n , where $n = 1, \dots, N$, is characterized by its dividend process D^n , with D_t^n denoting the dividend paid by the security at time t , and by its “collateral indicator” process α^n , which we will soon define.

³A partition is a collection of pairwise disjoint subsets, or events, of Ω whose union is equal to Ω . A partition \mathcal{F}_i is finer than partition \mathcal{F}_j if any event in \mathcal{F}_j is the union of events in \mathcal{F}_i .

The price process of security n is S^n , with S_t^n denoting the *ex-dividend* price of one share of the security. Since prices are quoted ex-dividends, it is natural to assume that, for all n , $S^n(T) = 0$. At each t , security n pays the dividend D_t^n and is then available for trade at price S_t^n . We use $D \equiv (D^1, \dots, D^N)$ and $S \equiv (S^1, \dots, S^N)$ to denote, respectively, the \mathbf{R}^N -valued dividend process and price process for the N securities.

A trading strategy θ is \mathbf{R}^N -valued process with $\theta_t = (\theta_t^1, \dots, \theta_t^N)$ representing the portfolio held after trading at time t . From the dynamic budget constraint, we define the dividend process D^θ generated by a trading strategy θ as

$$D_t^\theta = \theta_{t-1}^\top \cdot (S_t + D_t) - \theta_t^\top \cdot S_t, \quad \text{for } t = 0, \dots, T, \quad (1)$$

with $\theta_{-1} \equiv 0$ by convention and $^\top$ denoting the transpose of a vector. A negative dividend is, of course, addition of outside funds.

2.3 Collateral Constraints

We assume that there are trading constraints similar to those imposed on repurchase agreements or “repo” transactions. A repo transaction is a spot market sale of a security together with a forward contract to repurchase the security at some future date at a specified forward price. The trader who buys the security in the spot market and sells it forward is said to have engaged in a “reverse repo” transaction. A repo transaction, which is collateralized borrowing, is frequently used by securities dealers to finance their inventory of securities. A reverse repo, which is collateralized lending, is frequently used in connection with short selling strategies. A trader can short a security by engaging in a reverse repo transaction to obtain the security and then sell it in the spot market.

For our purposes, the important feature of the repo transactions is that the lender extends a loan on only a fraction of the market value of the security. The remainder, which is colorfully known as the “haircut”, represents some form of protection against default by the borrower. Haircuts, or margin requirements, vary across securities. For example, margin requirements in dealer-to-dealer transactions could be 0.25% for Treasury bills,

notes and bonds, 2% on commercial paper and about 3% on mortgage-backed securities. In nondealer-to-dealer transactions, margins on U.S. bonds can be as high as 2%. Such margin requirements also change from time to time. For more details, see Stigum (1989, page 217). Similar, but more stringent, restrictions are imposed on individuals who may finance their purchases of securities via a margin account with a broker. A typical margin account requires that the investor maintain a balance of at least 50% of the value of purchased securities.

We capture the collateral value of security n by a process α^n , with $0 \leq \alpha^n \leq 1$ at all times and in all states.⁴ We interpret α^n as follows: when an agent holds one share of security n at time t , he can borrow up to $H_t^n \equiv \alpha_t^n S_t^n$ using that share as a collateral. For example, using the above margin requirements, $\alpha = 0.97$ for a mortgage-backed security. We call the process H^n the “collateral” value of security n . Note that by taking α^n to be a process, we allow the collateral value of \$1 worth of every security to be state and time dependent. We assume that there exists one general borrowing or repo rate in the market. The same rate applies to all securities. We do not address the case of special repo rates and refer the reader to Duffie (1993) for an analysis of this phenomenon.

In addition to the dynamic budget constraint, traders are required to be solvent in all events and at all times. Consider, for example, a trader with a long position in mortgage-backed securities whose market value is one hundred million dollars. The trader can finance this position by a repo transaction. With a margin requirement of 3%, however, the trader has to invest three million dollars of equity to initiate the position. We assume that the equity of the trader is required to be at least 3% of the value of the total position at each trading date. If the market price of the securities declines, the trader would either liquidate the position or add more equity to maintain solvency.

The margin requirements also affect traders with short positions. Consider again a trader with a short position in mortgage-backed securities with market value of one hundred

⁴We can assume that all processes α^n have 1 as an upper bound without losing generality because we will impose a linear collateral constraint.

million dollars. One common way of initiating such a short position is through a reverse repo transaction. The trader can obtain the securities to be sold short by lending ninety seven million dollars and receiving securities worth one hundred million dollars. The net result, after selling the securities in the spot market, is that the trader has at his discretion three million dollars of excess cash that can be utilized to purchase other assets.

From the previous examples, it is easy to see that the higher the margin requirement on a security, the *more* the equity required to initiate a long position in that security. Furthermore, the higher the margin requirement, the more the “free cash” available from a short position constructed through a repo transaction. In order to capture both these effects, we define the collateral process R^θ associated with the trading strategy θ as $R_t^\theta = \theta_t^\top \cdot H_t$, where $H_t = (\alpha_t^1 S_t^1, \dots, \alpha_t^N S_t^N)$. Finally, in order to capture the solvency requirements we impose the constraint that agents can only use trading strategies θ that satisfy the condition:

$$R_t^\theta \geq 0 \quad \text{for } t = 0, 1, \dots, T - 1. \quad (2)$$

Note that, when all securities have the same collateral qualities $\alpha^n = 1$, the linear collateral constraint collapses to the nonnegative market wealth constraint studied by Brown (1988) and He and Pagès (1993).

A numerical example here should make the collateral constraint more concrete. We consider again the trader who wishes to use the repo market to establish a long position in mortgage-backed securities whose market value is 100 million dollars. The trader invests 3 million dollars of equity, borrows 97 million dollars from repo market, and uses the equity and proceeds from borrowing to purchase 100 million dollars of mortgage-backed securities to post as collaterals for the borrowing. The net portfolio of the trader is a long position of 100 million dollars of mortgage-backed securities with $\alpha = 0.97$ and a short position of 97 million dollars of riskless security with $\alpha = 1$.⁵ The collateral value of this portfolio is

$$R^\theta = 0.97 \times 100 \text{ million} - 1.0 \times 97 \text{ million} = 0,$$

⁵We assume that the repo market lending is a riskless security with $\alpha = 1$. This security will later be introduced as the first security.

which means that the collateral constraint is indeed binding for the portfolio that requires the minimum amount of equity when the trader wants to borrow to have 100 million dollars of mortgage-backed securities. It is easy to show that the collateral constraint will be satisfied if the trader puts in more than the minimum 3 million dollars of equity.

3 Implications of No Arbitrage

In this section we study the implications of absence of arbitrage in the securities market described in Section 2.

3.1 Collateral Constrained Arbitrage

We would like to study the structure of prices of financial securities in markets where solvency constraints are imposed. Given a dividend-collateral-price triplet (D, α, S) for N securities, a trading strategy θ is a *collateral-constrained arbitrage*, CC-arbitrage henceforth, if $D^\theta > 0$ and $R^\theta \geq 0$.⁶ In other words, a CC-arbitrage portfolio requires no net investment at any date, provides strictly positive dividend at some state and date, and satisfies the collateral constraint at all dates. A trading strategy θ is henceforth called unconstrained arbitrage if $D^\theta > 0$. Unconstrained arbitrage is an arbitrage opportunity that can be exploited in the absence of the collateral constraint but is not necessarily realizable when it is imposed.

Let Θ denote the space of trading strategies. It is easy to show that the marketed subspace $M = \{(D^\theta, R^\theta): \theta \in \Theta\}$ of dividend-collateral processes generated by trading strategies is a linear subspace of $\mathcal{L} \times \mathcal{L}$, where \mathcal{L} is the space of processes.

The following result is the first step towards the decomposition of security prices. It shows that, absent arbitrage in the financial market, the price of any financial security, or a portfolio thereof, can be decomposed into two sources of value. The first source is the

⁶We use the following convention of inequalities for vectors (or processes) $x, y \in \mathbf{R}^n$. $x \geq y$ means that $x - y$ is non-negative in every coordinate (or at every time and state). $x > y$ means that $x \geq y$ and $x \neq y$. $x \gg y$ means that $x - y$ is strictly positive in every coordinate (or at every time and state).

cash flows provided by the portfolio over its life time and the second source is the life-time collateral service value of the portfolio. This generalizes the result in frictionless markets in which security prices are attributed solely to cash flows. The proposition also shows that failure to rationalize the price of financial securities on the grounds of cash flow and collateral values is a clear sign that there are CC–arbitrage opportunities in the market.

Proposition 1 *There is no CC-arbitrage if and only if there exists a nontrivial linear functional $F: \mathcal{L} \times \mathcal{L} \rightarrow \mathbf{R}$, such that $F(D^\theta, R^\theta) = 0$ for all $\theta \in \Theta$. We can write $F(D, R) = \psi(D) + \phi(R)$, where ψ is a strictly positive linear functional on \mathcal{L} and ϕ is a non-negative linear functional on \mathcal{L} .*

It is important to know the conditions under which the collateral service value of securities, represented by ϕ at this stage, is different from zero. Intuition suggests that the collateral service of securities is guaranteed to have positive value if two portfolios with identical cash flows and different collateral values command different prices. On the other hand, if the solvency constraint does not hinder creating an arbitrage strategy, then the collateral service of securities may not have value.

We say that the collateral constraint is *arbitrage binding* if there would be arbitrage opportunity for the same set of prices in the absence of the constraint. In other words, the collateral constraint is arbitrage binding if there exists unconstrained arbitrage but no CC-arbitrage. The following proposition shows the intuitive result that ϕ is nontrivial for every pair (ψ, ϕ) if and only if the collateral constraint is arbitrage binding. Again, we assume that there is no CC-arbitrage opportunity in the market.

Proposition 2 *ϕ is nontrivial for every pair of nontrivial linear functionals (ψ, ϕ) that satisfy $\psi(D^\theta) + \phi(R^\theta) = 0$ for all θ in Θ if and only if the collateral constraint is arbitrage binding.*

3.2 Stochastic Discount Factors

We proceed to the next level of decomposition of the price of a financial security. At that level, we rationalize the price of the security by studying its cash flows and its collateral service in each time and event. The next proposition shows that in a market without CC–arbitrage, there must exist strictly positive cash flow discount factors and nonnegative collateral discount factors. The price of any security is the sum of its future cash flows, adjusted by the cash flow discount factors, and its future collateral value, adjusted by the collateral discount factors.

Proposition 3 *The dividend–collateral–price triplet (D, α, S) admits no CC–arbitrage if and only if there exist a strictly positive process $g \in \mathcal{L}$ and a non-negative process $u \in \mathcal{L}$ such that, for any $\theta \in \Theta$, either of the following equivalent conditions holds*

$$\theta_t^\top \cdot S_t = \frac{1}{g_t} \mathbf{E}_t \left[\sum_{s=t+1}^T g_s D_s^\theta + \sum_{s=t}^{T-1} u_s R_s^\theta \right], \quad (3)$$

in particular,

$$S_t = \frac{1}{g_t} \mathbf{E}_t \left[\sum_{s=t+1}^T g_s D_s + \sum_{s=t}^{T-1} u_s R_s \right]. \quad (4)$$

Furthermore, for any such pair (g, u) , the process u is non-zero if the collateral constraint is arbitrage-binding.

We call g and u , the process of *dividend stochastic discount factors* and *collateral stochastic discount factors*, respectively. The interpretation of proposition 3 is that the price of any cash flow plan can be thought of as the sum of two sources of value: the cash flow value and the collateral service value. For each “dollar” that the plan distributes at time t in event $E_{i,t} \in \mathcal{F}_t$, the market imputes a discount factor of $g_t(E_{i,t})$ per unit of probability of occurrence of $E_{i,t}$. In addition, the market imputes a discount factor of $u_t(E_{i,t})$ for each dollar of collateral value that the assets of the plan provide at t in event $E_{i,t}$.

Intuitively, readers can think of the nonnegative collateral stochastic discount factor u as multipliers for the collateral constraint. Since the arbitrage argument does not depend on any investor's optimal consumption and portfolio policy, this intuitive analogy can be made formal only after we study the optimal consumption and portfolio policy problem later in the paper.

Our result in Proposition 1 and Proposition 3 is analogous to the result in Jouini and Kallal (1993) for short sale constraint and is consistent with the result in Chen (1994) for general market frictions and convex portfolio constraints. They conclude that the absence of arbitrage in the presence of market portfolio constraints implies the existence of a strictly linear functional lying below the price functional on the feasible payoff set.⁷ Analogously, in our notation, equation 3 implies that there exists a strictly positive linear functional described by discount factor g that lies below the price functional for any feasible portfolio θ with $R^\theta \geq 0$. In addition to this general inequality result, however, by focusing on the simple linear collateral constraint, we are able to identify the difference between the price functional and the cash flow linear functional as the nonnegative collateral service value component which is simply related to the collateral value of securities.⁸ This simple identification and the resulting equality relationship enable us to explore the effects of linear collateral constraint on the cross sectional risk premia of securities.

Note that, in this market, it is possible for two plans with identical cash flows to have different prices with the difference due to the discrepancy between their "collateral" characteristics as reflected by the parameter α for the securities that comprise the plan. This is different from the case of markets with no collateral restrictions. In such markets, assets have no value beyond cash distributions and identical cash flows have the same price.

⁷The feasible payoff set is the set of payoffs generated by portfolios that satisfy the trading constraints.

⁸Similar identification can also be made for a general convex portfolio constraints. For instance, a n -face polyhedral convex cone portfolio constraint can be viewed as a combination of n linear portfolio constraints. Under such a portfolio constraint in the absence of arbitrage, the difference between the price functional and the low-lying strictly positive linear functional can be written as n terms representing the shadow prices of n linear constraints. But the economic intuition behind these n shadow price terms is in general not as simple as in the case of linear collateral constraint.

In our set-up, a security may have value, beyond cash distribution, because it contributes to the solvency of its holder.

Of course, the above discussion about securities with identical cash flows but different collateral qualities is relevant only if two securities with identical cash flows can have different collateral qualities. Such a scenario can exist in the case of, for example, an on-the-run and an off-the-run treasury securities with identical remaining cash flows. It is commonly observed that such two securities can have different collateral qualities. Cornell and Shapiro (1989) observed that two such securities can also have significantly different prices for an extended period of time. We acknowledge here that a big part of the price difference of such two securities is possibly due to market microstructure factors such as squeezes and special repo rates; see, for example, Duffie (1993). We note, however, that our study indicates that different collateral qualities of two securities caused by factors such as different liquidities can also have an effect on the prices of the two securities.

4 Cross Sectional Risk Premia

In this section we explore the implications of absence of CC-arbitrage on cross sectional risk premia of securities. We will assume the existence of an asset that serves as a store of value.

Assumption 1 *At each date t , there is short-term riskless borrowing and lending: \$1 cash deposited at time t pays back $\$(1 + r_t)$ cash at time $t + 1$, where r_t is the short-term riskless interest rate and is known at time t . One unit of security 1 is \$1 deposited at time 0 and rolled over repeatedly in riskless lending each period until time T , with the price process $S_t^1 = \prod_{s=0}^{t-1} (1 + r_s)$. The collateral indicator of this security is $\alpha_1 = 1$.*

Applying the result of proposition 3 to the first security, we have

$$S_t^1 = \frac{1}{g_t} \mathbf{E}_t \left[g_{t+1} S_{t+1}^1 + u_t S_t^1 \right] = \frac{1}{g_t} \left[\mathbf{E}_t [g_{t+1}] S_t^1 (1 + r_t) + u_t S_t^1 \right],$$

from which we conclude that

$$1 + r_t + \frac{u_t}{\mathbf{E}_t[g_{t+1}]} = \frac{g_t}{\mathbf{E}_t[g_{t+1}]}.$$
 (5)

This equation suggests the intuition that the inverse of an investor's expected intertemporal marginal rate of substitution, $\frac{g_t}{\mathbf{E}_t[g_{t+1}]}$, is usually larger than the riskless return in a market with collateral constraints, with the difference due to the collateral service value. This intuition will be made precise later in an equilibrium analysis.

We now proceed to calculate the expected excess return on other securities relative to the locally riskless security. we apply Proposition 3 to security n and obtain

$$S_t^n = \frac{1}{g_t} \mathbf{E}_t \left[g_{t+1} (S_{t+1}^n + D_{t+1}^n) + u_t \alpha_t^n S_t^n \right] \quad n = 2, \dots, N,$$
 (6)

from which we express the expected return on security n , denoted $\mathbf{E}_t[\mu_{t+1}^n]$, as

$$\mathbf{E}_t \left[\mu_{t+1}^n \right] = -\frac{1}{\mathbf{E}_t(g_{t+1})} \left[\mathbf{E}_t(g_{t+1}) - g_t + \text{Cov}_t \left(g_{t+1}, \frac{S_{t+1}^n + D_{t+1}^n}{S_t^n} \right) + \alpha_t^n u_t \right].$$

The expected excess return for security n is thus

$$\mathbf{E}_t[\mu_{t+1}^n] - r_t = -\frac{\text{Cov}_t \left(\mu_{t+1}^n, g_{t+1} \right)}{\mathbf{E}_t(g_{t+1})} + \frac{(1 - \alpha_t^n) u_t}{\mathbf{E}_t(g_{t+1})} \quad n = 2, \dots, N.$$
 (7)

The expected excess return on security n is due to two sources: a risk premium and a solvency premium. The risk premium is proportional to the covariance of the return with the stochastic pricing factor g . This is similar to the case of market with no constraints and the intuition is the same. Securities that payoff relatively more in states when the pricing factor g is higher than average are relatively more valuable and hence relatively lower expected excess return is required for holding them. The solvency premium is inversely proportional to the collateral quality of the security. The higher quality securities are valued relatively higher, *ceteris paribus*, and hence lower expected excess return is demanded for holding them.

Note that the solvency return of any security is computed relative to that of the riskless asset. Securities with the same quality as the riskless asset, that is securities with $\alpha = 1$,

do not command any solvency return. This is a consequence of the fact that the solvency services that the riskless security provides can also be obtained from such assets.

4.1 Collateral Adjusted CAPM

It is useful to state the determinants of excess return on securities relative to the excess return on some benchmark portfolio in a CAPM-like relationship. We next develop a *collateral-adjusted* CAPM formula.

Simple algebra shows that (7) also applies to any portfolio of marketed securities and hence

$$\mathbf{E}_t[\mu_{t+1}^\theta] - r_t = -\frac{\text{Cov}_t(\mu_{t+1}^\theta, g_{t+1})}{\mathbf{E}_t(g_{t+1})} + \frac{(1 - \alpha_t^\theta)u_t}{\mathbf{E}_t(g_{t+1})}, \quad (8)$$

where the collateral indicator for portfolio θ is value-weighted average of that of the individual securities and defined as $\alpha_t^\theta = \frac{\sum_n \alpha_t^n \theta_t^n S_t^n}{\sum_n \theta_t^n S_t^n}$ for $\sum_n \theta_t^n S_t^n \neq 0$. Define portfolio θ^* as a portfolio whose return μ_{t+1}^* has the highest correlation with g_{t+1} . Then we can show, see Lemma 2 in the appendix for detail, that, for an arbitrary portfolio θ , the covariance between its return μ_{t+1}^θ and g_{t+1} is proportional to the covariance between μ_{t+1}^θ and μ_{t+1}^* . Applying (8) to the portfolio θ^* , we arrive at the following *collateral-adjusted* CAPM relationship:

$$\mathbf{E}_t[\mu_{t+1}^\theta] - r_t - (1 - \alpha_t^\theta) \frac{u_t}{\mathbf{E}_t(g_{t+1})} = \beta_t^\theta \left(\mathbf{E}_t(\mu_{t+1}^*) - r_t - (1 - \alpha_t^{\theta^*}) \frac{u_t}{\mathbf{E}_t(g_{t+1})} \right), \quad (9)$$

where $\beta_t^\theta \equiv \text{Cov}(\mu_{t+1}^\theta, \mu_{t+1}^*) / \text{Var}(\mu_{t+1}^*)$.

The collateral-adjusted CAPM formula (9) is very similar to the traditional CAPM formula in frictionless markets if one defines the total return on a security to include its solvency services in addition to its dividends and capital gains. Define the *gross return* of any portfolio θ , denoted $\tilde{\mu}^\theta$, as the sum of the cash return plus the collateral service return as $\tilde{\mu}_{t+1}^\theta \equiv \mu_{t+1}^\theta + \alpha_t^\theta u_t / \mathbf{E}_t(g_{t+1})$, and, in particular, $\tilde{r}_t \equiv r_t + u_t / \mathbf{E}_t(g_{t+1})$. Then, these gross

returns satisfy the usual CAPM formula:

$$\begin{aligned} \mathbf{E}_t[\tilde{\mu}_{t+1}^\theta] - \tilde{r}_t &= \beta_t^\theta \left(\mathbf{E}_t(\tilde{\mu}_{t+1}^*) - \tilde{r}_t \right) \\ \text{with } \beta_t^\theta &= \frac{\text{Cov}_t(\mu_{t+1}^\theta, \mu_{t+1}^*)}{\text{Var}_t(\mu_{t+1}^*)} = \frac{\text{Cov}_t(\tilde{\mu}_{t+1}^\theta, \tilde{\mu}_{t+1}^*)}{\text{Var}_t(\tilde{\mu}_{t+1}^*)}. \end{aligned} \quad (10)$$

The above result again shows that, in a security market with collateral constraints, the gross return, which is the sum of a security's cash return plus its collateral service return, captures more precisely the value of a security and satisfies the regular CAPM formula. Our model then shows a zero intercept from a regression of the gross return of any portfolio versus the gross return of the portfolio θ^* which has the highest correlation with g_{t+1} .⁹ The regression of the *cash* return of a portfolio versus the *cash* return of portfolio θ^* , however, should produce a nonzero intercept as indicated by rewriting (9) as follows,

$$\mathbf{E}_t[\mu_{t+1}^\theta] - r_t = \left[(1 - \alpha_t^\theta) - \beta_t^\theta (1 - \alpha_t^{\theta^*}) \right] \frac{u_t}{\mathbf{E}_t(g_{t+1})} + \beta_t^\theta \left[\mathbf{E}_t[\mu_{t+1}^*] - r_t \right]. \quad (11)$$

4.2 Collateral Service Value and Potential Arbitrage

In this section, we turn our attention to the determinants of the collateral service return of securities. We start from Proposition 2, which states that the collateral service of a security is guaranteed to have value only when the collateral constraint is arbitrage-binding. This suggests that, if the collateral constraint is arbitrage-binding, the value of collateral services should be closely related to the degree to which the collateral constraint precludes otherwise feasible arbitrage. Put differently, the value of collateral services should be related to the potential arbitrage profit should the collateral constraint be removed. We will show next how to quantify this link between the potential for unconstrained arbitrage profits and the collateral service value of securities. The analysis depends on the assumption that the collateral constraint is arbitrage-binding.

The value of the collateral service can be represented by the collateral service return, which is determined by $\frac{u_t}{\mathbf{E}_t(g_{t+1})}$. This same variable also appears in the intercept in the

⁹Note, however, that this maximum correlation portfolio θ^* is usually not unique because the dividend shadow process g_t is usually not unique.

collateral-adjusted CAPM formula for cash returns in (11). Suppose that the collateral constraint is arbitrage-binding so that there exists an unconstrained arbitrage portfolio θ at some event in \mathcal{F}_t between time t and $t+1$.¹⁰ More specifically, suppose that the arbitrage portfolio at time t has market price of zero¹¹ and generates a constant liquidating payoff δ at time $t+1$.¹² We write the unconstrained arbitrage portfolio as $\theta_t = \theta_t^+ - \theta_t^-$, where θ_t^+ represents all positive holdings while θ_t^- represents the portfolio of short positions. By assumption, $V_t^\theta = V_t^{\theta^+} - V_t^{\theta^-} = 0$. The fact that there is no CC-arbitrage implies that $R_t^\theta \equiv (\alpha_t^{\theta^+} - \alpha_t^{\theta^-})V_t^{\theta^+} < 0$, where $\alpha_t^{\theta^+}$ and $\alpha_t^{\theta^-}$ are, respectively, the value weighted average collateral indicator of portfolios θ^+ and θ^- . Observe that when there is no CC-arbitrage, the unconstrained arbitrage portfolio involves shorting high collateral quality securities and investing the proceeds in low collateral quality securities.

Applying (24) to the portfolio θ , we have

$$0 = V_t^\theta = \frac{1}{g_t} \mathbf{E}_t [g_{t+1} V_{t+1}^\theta + u_t R_t^\theta] = \frac{1}{g_t} [\mathbf{E}_t(g_{t+1} \delta) + u_t (\alpha_t^{\theta^+} - \alpha_t^{\theta^-}) V_t^{\theta^+}],$$

from which we conclude that

$$\frac{u_t}{\mathbf{E}_t(g_{t+1})} = \frac{\delta}{V_t^{\theta^+} (\alpha_t^{\theta^-} - \alpha_t^{\theta^+})}. \quad (12)$$

Note that $\delta/V_t^{\theta^+}$ is the potential unconstrained arbitrage profit for every dollar of the “long” securities. This quantity, expressed in percentage or basis points, is a measure of the arbitrage profit, relative to the portfolio size, that could be realized by an unconstrained investor. Equation (12) shows that the collateral service return is equal to the ratio of the

¹⁰In our finite dimensional event tree set-up, one can show that the existence of unconstrained arbitrage portfolio across any number of time periods implies the existence of unconstrained arbitrage opportunity at one event at a time t between time t and $t+1$ (we call it a mini-tree). This can be proved by noting that if no mini-tree alone has unconstrained arbitrage opportunity then there exists a (strictly positive) dividend stochastic state price process for that mini-tree because each mini-tree is a one-period problem. A multiperiod dividend state price process can then be constructed by connecting these mini-tree dividend state prices mini-tree by mini-tree, which in turn implies that there exists no unconstrained arbitrage in the multiperiod security market.

¹¹This can be assumed without loss of generality.

¹²This assumption can be relaxed and δ can be understood as a weighted average payoff.

arbitrage profit to the spread in the collateral indicator of the short and long portfolios that define the potential arbitrage strategy.

We use (12) to get some idea of the size of the solvency returns. Assume, for illustration, that a market allows for a potential arbitrage profit of 0.2% using portfolios where the short and long positions require margins of 2% and 10%, respectively. In that market, $\frac{u_t}{\mathbb{E}_t(g_{t+1})} = 2.5\%$. As discussed following (5) and shown in the next section, one of the immediate implications of this result is that the collateral service return drives a wedge in the amount of 2.5% between $1 + r_t$ and the inverse of the expected intertemporal marginal rate of substitution while there would be no difference between the two in a frictionless market. The fact that such relatively large returns are generated in a market with rather small (20 basis points) potential arbitrage suggests that the solvency component could represent a significant portion of the total return on financial securities.

Note that, if the collateral constraint is not arbitrage-binding, the above analysis does not imply that the solvency premium is zero. In fact, collateral services can still have value if investors are bound by the collateral constraint in making optimal portfolio and consumption decisions. In general, securities with higher collateral services help investors achieve better efficiency in risk management when making consumption and portfolio choice in the presence of solvency constraints. One could estimate the solvency return on securities by studying the added value of their collateral services in reducing the impact of the solvency constraints on the life-time utility of constrained consumers. Understanding this general effect of collateral constraints on asset prices, however, requires an equilibrium analysis, which we study in the next section.

5 Consumption and Equilibrium Asset Pricing

In this section we turn our attention to the consumption policies of individuals who transfer their wealth over time using the financial securities. We also study the equilibrium effect of collateral constraints on asset prices. We have three objectives in mind. First, we would

like to understand the relationship between the dividend state price and the collateral state prices of section 3.2 on one hand, and the marginal utility of consumption and the shadow price of the collateral constraint on the other. This connection can be helpful in determining the optimal consumption policies *given* security prices in a market without CC-constrained arbitrage. Second, we would like to relate the interest rate to the marginal utilities of consumers and show the difference between this relationship in a frictionless market and that in a market with collateral constraints. Third, we would like to relate the excess expected return on securities to the level of aggregate consumption and hence extend Breeden's (1979) consumption-based CAPM results.

5.1 Existence of Optimal Solution

In this section, we establish the existence of an optimal intertemporal consumption plan for any investor with a continuous utility function when there is no CC-arbitrage.

Consider an investor in the securities market who faces the collateral constraints. The problem of optimal intertemporal consumption is

$$\begin{aligned} \max_{c \in \mathcal{C}(e)} U(c) \quad \text{where} \\ \mathcal{C}(e) = \{c \in \mathcal{L} \mid c \geq 0 \ \& \ \exists \theta \in \Theta : c = D^\theta + e, R^\theta \geq 0\}, \end{aligned} \tag{13}$$

where U is the investor's utility function over consumption streams $c \in \mathcal{L}$.

Proposition 4 *If there is no CC-arbitrage, then there exists a solution to the intertemporal consumption problem (13) when the utility function $U(c)$ is continuous.*¹³

5.2 Marginal Utility and State Prices

It is well known that in a frictionless market, the marginal utility of an investor, evaluated at the optimal consumption choice, is proportional to the state price vector. This section

¹³Technically, we need to assume that the endowment stream is such that the consumption feasible set $\mathcal{C}(e)$ is not empty.

studies the relationship between marginal utility of consumption and state prices in the presence of solvency constraints.

We consider an investor with a utility function U that is strictly increasing, strictly concave, and differentiable. We assume that there exists an optimal strictly positive consumption c^* for the investor. Then the investor's optimal consumption and portfolio policy is the solution to the following problem:

$$\max_{\theta} U(D^{\theta} + e) \quad \text{s.t.} \quad R^{\theta} \geq 0. \quad (14)$$

It is then easy to show that the marginal utility at c^* is a dividend state price process \bar{g}_t and the shadow price of the collateral constraint is the corresponding collateral state price process \bar{u}_t .¹⁴ We remind the reader that the shadow price of the collateral constraint is a process that describes, for every time and state, the added life-time utility that the agent achieves if the collateral constraint were slightly relaxed at that time and state.

Proposition 5 *Suppose the optimization problem (13) has a strictly positive solution c^* and that $\nabla U(c^*)$ exists and is strictly positive. Then there is no CC-arbitrage and a dividend state price \bar{g}_t is given by $\nabla U(c^*)$ and the corresponding collateral state price \bar{u}_t is given by the investor's shadow price of the collateral constraint.*

Proposition 5 identifies a pair of dividend and collateral state prices of section 3.2 with the pair of marginal utilities of consumption and shadow prices of solvency for every agent that achieves an optimal solution in the market. This result is the key to studying the equilibrium interest rate and to linking the returns on securities to the aggregate level of consumption in the economy as we will show next.

5.3 Equilibrium Interest Rate and Solvency Premium

Let $m_{i,t}$ denote the intertemporal marginal rate of substitution (IMRS) for consumer i between time $t - 1$ and t , that is, the time t to time $t - 1$ ratio of the marginal utility for

¹⁴We have so far been working with the stochastic discount factors, which are the state price per unit of probability.

consumer i of one unit of consumption. In a frictionless market, the equilibrium riskfree interest rate at time t is directly related to the each consumer's IMRS by

$$1 + r_t = \frac{1}{\mathbf{E}_t[m_{i,t+1}]} \tag{15}$$

One immediate implication of Proposition 5 is that, in a market with collateral constraints, this relationship no longer holds and that the market riskfree interest rate can be lower than that implied by (15), with the difference given by the solvency premium. This can be shown by applying Proposition 5 to (5),

$$1 + r_t + \frac{u_t}{\mathbf{E}_t[g_{t+1}]} = \frac{1}{\mathbf{E}_t[m_{i,t+1}]} \tag{16}$$

This result is reminiscent of the observation made by Weil (1989), also called the *riskfree rate puzzle*, that the riskfree rate is significantly less than that implied by aggregate consumption data as suggested by (15) when the parameters of consumers' preference were adjusted so as to fit the equity premium or to satisfy the Hansen-Jagannathan IMRS volatility bound test (see Cochrane and Hansen (1992)). Our numerical example in Section 4.2, following (12), shows that this wedge between the riskfree rate and that suggested by consumption data through (15) can be as high as 2–3% for relatively small potential arbitrage profits and typical margin requirements on securities. Of course, our number here is relevant only if we can extend (16) to a relationship between the interest rate and the *aggregate* consumption. For the above numerical example, equation (12) shows that $\frac{u_t}{\mathbf{E}_t[g_{t+1}]}$ is the same for every consumer in the economy. In this case, for time-additive utility functions with linear risk tolerance, as shown by Luttmer (1991), the individual IMRS, $m_{i,t+1}$, in equation (16), can be further replaced by the the IMRS at the per-capita consumption, of a utility function that has risk aversion that is equal to some weighted average of the risk aversions of all investors.

5.4 Collateral Adjusted Consumption CAPM

With the characterization of optimal solutions for every agent with strictly increasing differentiable utility function, we can use aggregation to relate the excess expected return

on risky securities to aggregate consumption. Assuming that agents have time additive quadratic utility functions, we show that the “risk premium” on any security is related to the covariance of its return with aggregate consumption. This generalizes Breeden’s consumption CAPM to markets with solvency constraints. The difference in such markets is that the “risk premium” is only one component of expected excess return. The balance is provided by the *aggregate solvency premium*. Note that the term *aggregate solvency premium* in this section differs from the solvency premium in last section in two ways. First, it is measured against a different bench mark, aggregate consumption in this case. Secondly, it represents the aggregate effect of collateral service value of a security (or portfolio) in reducing the impact of the solvency constraints on the life-time utility of each consumer.

To proceed, we assume that all investors have homogeneous beliefs about the probabilities of future events and each investor i in the economy has the following von Neumann–Morgenstern time additive quadratic utility function:¹⁵

$$U_i(c) = \mathbf{E} \left[\sum_{t=0}^T u_{i,t}(c_t) \right] = \mathbf{E} \left[\sum_{t=0}^T (a_{i,t-1}c_t - b_{i,t-1}c_t^2) \right]. \quad (17)$$

Let $c_{i,t}^*$ denote the strictly positive optimal consumption for individual i at time t , and let $\lambda_{i,t}$ denote collateral constraint shadow price per unit of probability for individual i at time t : $\lambda_{i,t}(E_{j,t}) = \bar{\lambda}_{i,t}(E_{j,t})/p(E_{j,t})$. Proposition 5 then implies that $(u'_{i,t}(c_{i,t}^*), \lambda_{i,t}) = (a_{i,t-1} - 2b_{i,t-1}c_{i,t}^*, \lambda_{i,t})$ is a pair of dividend–collateral stochastic discount factors, and (8) takes the form of

$$\mathbf{E}_t(a_{i,t} - 2b_{i,t}c_{i,t+1}^*) \left[\mathbf{E}_t(\mu_{t+1}^\theta) - r_t \right] = -\text{Cov}_t(\mu_{t+1}^\theta, a_{i,t} - 2b_{i,t}c_{i,t+1}^*) + (1 - \alpha_t^\theta)\lambda_{i,t}, \quad \forall i, t.$$

Divide both sides by $2b_{i,t}$ and then sum the above equation over all individuals in the economy to get

$$\mathbf{E}_t(\mu_{t+1}^\theta) - r_t = \frac{\text{Cov}_t(\mu_{t+1}^\theta, C_{t+1}^*)}{\sum_i \tau_{i,t}} + (1 - \alpha_t^\theta) \frac{\sum_i \lambda_{i,t} a_{i,t} / 2b_{i,t}}{\sum_i \tau_{i,t}}, \quad \forall t \quad (18)$$

¹⁵We assume that $c_{i,t} < \frac{a_{i,t-1}}{2b_{i,t-1}}$ for all i, t and all states so that all investors are not satiated.

where C_{t+1}^* is the time $t+1$ aggregate consumption and $\tau_{i,t} = -\mathbf{E}_t[u'_{i,t+1}(c_{i,t+1})/u''_{i,t+1}(c_{i,t+1})]$ is individual i 's expected time $t+1$ risk tolerance.

We can also relate the risk premium of every portfolio to its beta with the portfolio that has the most correlation with aggregate consumption. Define portfolio θ^c as a portfolio whose return μ_{t+1}^c has the highest correlation with C_{t+1}^* , then the same algebra leading from (8) through (10) leads from (18) to the following:

$$\begin{aligned} \mathbf{E}_t[\hat{\mu}_{t+1}^\theta] - \hat{r}_t &= \hat{\beta}_t^\theta \left(\mathbf{E}_t(\hat{\mu}_{t+1}^c) - \hat{r}_t \right) \quad \forall t \\ \text{with} \quad \hat{\mu}_{t+1}^\theta &\equiv \mu_{t+1}^\theta + \alpha_t^\theta \frac{\sum_i \lambda_{i,t} a_{i,t} / 2b_{i,t}}{\sum_i \tau_{i,t}}, \quad \hat{r}_t \equiv r_t + \frac{\sum_i \lambda_{i,t} a_{i,t} / 2b_{i,t}}{\sum_i \tau_{i,t}}, \\ \text{and} \quad \hat{\beta}_t^\theta &= \frac{\text{Cov}_t(\mu_{t+1}^\theta, \mu_{t+1}^c)}{\text{Var}_t(\mu_{t+1}^c)} = \frac{\text{Cov}_t(\hat{\mu}_{t+1}^\theta, \hat{\mu}_{t+1}^c)}{\text{Var}_t(\hat{\mu}_{t+1}^c)}. \end{aligned} \quad (19)$$

Note that our result is an extension of Breeden's consumption-based CAPM in the presence of the linear collateral constraint. Equation (18) shows that the expected excess return of any portfolio has two components. The first is the risk premium of the portfolio in a traditional consumption CAPM model. The second is the aggregate solvency premium of the portfolio in the presence of collateral constraint.

The magnitude of the aggregate solvency premium of a security increases as the number of investors in the economy who are bound by the solvency constraint increases. It is also monotonically related to how severely the collateral constraint binds for investors in the economy. These two factors, however, uniformly affect the solvency premia of all securities. Different securities have different aggregate solvency premia because they have different collateral qualities. The magnitude of the aggregate solvency premium of a security is proportional to its collateral quality difference with that of the riskless security. When all securities have the same collateral quality with $\alpha_t^n = 1$, i.e., when the linear collateral constraint collapses to the nonnegative market wealth constraint, the aggregate solvency premia for all securities are zero and the consumption-based CAPM is again valid. This special case of our result is consistent with He and Pagès (1993) who showed that consumption-based CAPM remains valid in the presence of uninsurable labor income shocks and borrowing constraint against future income.

Qualitatively, our result is consistent with a possible partial explanation of the equity premium puzzle. It is well documented; see Mehra and Prescott (1985), that the large difference between stocks' expected cash return and the riskless interest rate is too high to be reconciled with the relatively smooth aggregate consumption data and a reasonable range of investor risk aversion in equilibrium models. Our analysis indicates that riskless securities provide higher collateral service due to their relatively higher collateral values. It is then reasonable that investors would tolerate lower expected cash return from the riskless securities in return for their relatively higher collateral services. Such services are "priced" in the market because they help investors satisfy the solvency constraints.

The question of whether our analysis bears any relevance to a potential resolution of the equity premium puzzle, however, has to be answered by an empirical estimation. On the surface, the difficulty to observe investors' shadow prices for the collateral constraints seems to prevent us from drawing any empirically testable implication from (18). In fact, however, as discussed above, the term containing the weighted sum of these shadow prices applies to all securities uniformly, and thus we can cancel it from (18) to obtain the following relationship for any two portfolios θ and θ' ,

$$\mathbf{E}_t \left[\frac{\mu_{t+1}^\theta - r_t}{1 - \alpha_t^\theta} - \frac{\mu_{t+1}^{\theta'} - r_t}{1 - \alpha_t^{\theta'}} \right] = \frac{1}{\sum_i \tau_{i,t}} \text{Cov}_t \left(\frac{\mu_{t+1}^\theta - r_t}{1 - \alpha_t^\theta} - \frac{\mu_{t+1}^{\theta'} - r_t}{1 - \alpha_t^{\theta'}}, C_{t+1}^* \right). \quad (20)$$

This relationship can be tested together with Breeden's consumption CAPM,

$$\mathbf{E}_t(\mu_{t+1}^\theta - r_t) = \frac{\text{Cov}_t(\mu_{t+1}^\theta, C_{t+1}^*)}{\sum_i \tau_{i,t}}. \quad (21)$$

It is important to note that, although (21) implies (20), the converse needs not be true. So a rejection of (21) does not imply a rejection of (20). This observation is useful, especially given numerous past studies (see, for example, Hansen and Singleton (1982), Grossman, Melino, and Shiller (1987), and Breeden, Gibbons, and Litzenberger (1989)) that have rejected (21). In fact, a rejection of (21) coupled with failure to reject (20) would be consistent with the prediction of our result. Since some models with credit constraints that apply uniformly to all securities, such as the market wealth constraints, also yield (21) (He

and Pagès (1993)), it is then interesting to know whether our study, by considering the effect on asset prices of different collateral qualities of different securities, better describes the effect of trading constraints on equilibrium asset prices.

6 Concluding Remarks

In this paper we studied how solvency constraints in a multiperiod model affect the way asset prices are determined. In particular, we showed how the law of one price could be violated without creating riskless arbitrage profits. We also demonstrated how to construct a dynamic general equilibrium model in the presence of solvency constraints. In our analysis, we assumed that the extent to which a particular security can be used as collateral is exogenously given. We also assumed, without further justification, that agents are required to maintain solvency at all times. Understanding the determinants of the collateral usefulness of securities and the solvency constraints on agents are important issues that deserve further inquiry.

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Appendix

The first lemma in the appendix is an extension in a special case of the lemma “Linear Separation of Cones” in Appendix B in Duffie (1992), which studied the separation of two cones that intersect precisely at zero. Here we obtain partial separation for two special cones whose intersection lies on a subspace of \mathbf{R}^n . This result is used in establishing Proposition 1.

Lemma 1 *Let $L_1 = \mathbf{R}^{n_1}$, $L_2 = \mathbf{R}^{n_2}$, $L = L_1 \times L_2$, and n_1, n_2 are positive integers. Let $K = L_+$ and let M be a linear subspace of L . Define $\Pi_1: L \rightarrow L_1$ as the projection operator onto the L_1 subspace. If $\Pi_1(M \cap K) = \{0\}$, then there exists a linear functional $F: L \rightarrow \mathbf{R}$ such that $F(x) \leq F(y), \forall x \in M, y \in K$, and $F(x) < F(y), \forall x \in M, y \in K$ with $\Pi_1(y) \neq 0$.*

Proof. We begin by describing the approach of the proof. First, we need to show strict separation of M with K' where K' is the set of all elements in K with nonzero L_1 space components: $K' \equiv \{(u, v): u \in L_1, v \in L_2, u > 0, v \geq 0\} \subset K$. Note that K' is neither open nor closed. To prove separation between M and K' , we find an open convex set J that contains K' such that $J \cap M = \emptyset$. The separating linear functional for J and M is then the required linear functional F . The bulk of the proof consists of searching for such an open set J . As it turns out, J can be constructed as an open convex cone generated by a closed convex set D plus a small ball.

We first construct D . Note that $K' \subset K$. Let $C \equiv \{z \in L: \|\Pi_1(z)\| = 1\}$. Then $K \cap C \subseteq K'$. Define D as the convex hull of $K \cap C$, i.e., the smallest convex set that contains $K \cap C$. Since K' is convex and contains $K \cap C$, we have $D \subseteq K'$. From the assumption that $\Pi_1(K \cap M) = \{0\}$, $K' \cap M = \emptyset$. So we have $D \cap M = \emptyset$. Next we show that D is closed by writing it out explicitly. Note that $K \cap C = (K \cap C \cap L_1) \times L_{2+}$. Using the relationship $\text{Conv}(X \times Y) = \text{Conv}(X) \times \text{Conv}(Y)$, where $\text{Conv}(X)$ denotes the convex hull of the set X , we have $D = \text{Conv}(K \cap C \cap L_1) \times L_{2+}$. Since $K \cap C \cap L_1$ is closed and bounded, $\text{Conv}(K \cap C \cap L_1)$ is closed. (The convex hull of a bounded closed set is also closed. See Rockafellar (1970, pp. 158)). Also note that L_{2+} is closed. So D is closed. (The direct product of two closed sets is also closed. See Rockafellar (1970, pp. 49)). In summary, D as constructed is convex, closed, and $D \cap M = \emptyset$.

Next, we construct J . To proceed, we first make a claim and finish the rest of the proof. We then come back to prove the claim.

Claim: $\epsilon \equiv \inf\{\|d-m\|: d \in D, m \in M\} > 0$. (Note that this is stronger than $D \cap M = \emptyset$.) If this is true, then we can define $J \equiv \text{Cone}(D + B_\epsilon) = \{\lambda(d + b): \lambda > 0, d \in D, b \in B_\epsilon\}$, where B_ϵ is the open ball centered at zero with radius ϵ . Then it is straight forward to verify that $K' \subseteq J$, J is open, convex, and $J \cap M = \emptyset$. So there exists a linear functional $F: L \rightarrow \mathbf{R}$ such that $F(x) < F(y), \forall x \in M, y \in J$. Since $K' \subseteq J$, we have $F(x) < F(y), \forall x \in M, y \in K'$. And since K is the closure of K' and that F is continuous, we have $F(x) \leq F(y), \forall x \in M, y \in K$. So F is the desired linear functional.

Proof of the claim: We know that D and M are closed and separated. But neither D nor M is bounded. So it is not immediately clear that the minimum distance between them should be positive. To understand and make use of the nature of their unboundedness, we need to use the concept of *recession cone* of a set. The definition of the recession cone of set D , denoted as 0^+D , is $0^+D \equiv \{y : x + \lambda y \in D \text{ for every } \lambda \geq 0 \text{ and } x \in D\}$. Intuitively, 0^+D is the set of all vectors in whose direction the set D goes to infinity. (See Rockafellar (1970) for detail.)

Note that $0^+D = L_{2+}$, $0^+(-M) = M$. We discuss two cases. In case one, we have $0^+D \cap 0^+(-M) = L_{2+} \cap M = \{0\}$. This is the case when the unbounded D and M are going to infinity at different directions. In this case, one expects that their unboundedness shouldn't change the intuitive result that their minimum distance is positive. Indeed, according to Corollary 9.1.1. in Rockafellar (1970), when D and M are both closed and have no common direction of recession other than 0, $D - M$ is also closed. And since $D - M \cap \{0\} = \emptyset$, we have $\epsilon > 0$. In case two, we have $0^+D \cap 0^+(-M) = L_{2+} \cap M \neq \{0\}$. This is the case when there are some common directions in which D and M go off to infinity. This case is possible because M can have intersection with L_2 . The fact that D and M go to infinity in some common directions, however, does not cause their minimum distance to go to zero. The reason is that D and M are both cones and go to infinity in "straight" lines. So the distance between them doesn't approach zero. This can be shown as follows. Define $S \equiv \{x \in L : \|\Pi_{L_2 \cap M}(x)\| \leq 1\}$, where $\Pi_{L_2 \cap M}$ denotes the projection onto the linear subspace $L_2 \cap M$. For any $m \in M \cap S^c$ and $d \in D$, we can find $m^* = m - \Pi_{L_2 \cap M}(m) \in M \cap S$ and $d^* = d - \Pi_{L_2 \cap M}(d) \in D$, such that $\|d^* - m^*\| \leq \|d - m\|$. So we have $\epsilon \equiv \inf\{\|d - m\| : d \in D, m \in M\} = \inf\{\|d^* - m^*\| : d^* \in D, m^* \in M \cap S\}$. By construction, we have $0^+D = L_{2+}$, $0^+(-M \cap S) = M \cap L_1$, so $0^+D \cap 0^+(-M \cap S) = \{0\}$. Applying Corollary 9.1.1. in Rockafellar (1970) as done above in case one again, we show that $D - (M \cap S)$ is closed. So we have $\epsilon > 0$. ■

Lemma 2 *Let g_{t+1} be an \mathcal{F}_{t+1} -measurable random variable and let μ_{t+1}^θ denote the return of a portfolio $\theta_t \in \mathbf{R}^N$: $\mu_{t+1}^\theta = (V_{t+1}^{\theta_t} - V_t^{\theta_t})/V_t^{\theta_t}$. Suppose that the return μ_{t+1}^* of portfolio θ_t^* has the highest correlation with g_{t+1} :*

$$\theta^* \equiv \arg \sup_{\theta_t \in \mathbf{R}^N} \text{corr}(\mu_{t+1}^\theta, g_{t+1}). \quad (22)$$

Then for any portfolio $\theta_t \in \mathbf{R}^N$, we have

$$\text{Cov}_t(\mu_{t+1}^\theta, g_{t+1}) = \frac{\text{Cov}_t(\mu_{t+1}^*, g_{t+1})}{\text{Var}_t(\mu_{t+1}^*)} \text{Cov}_t(\mu_{t+1}^\theta, \mu_{t+1}^*). \quad (23)$$

Proof. Since θ_t and S_t are constants conditioning on information available at time t , we have

$$\theta^* = \arg \sup_{\theta_t \in \mathbf{R}^N} \text{corr}(\mu_{t+1}^\theta, g_{t+1})$$

$$\begin{aligned}
&= \arg \sup_{\theta_t \in \mathbf{R}^N} \operatorname{corr} \left(\frac{V_{t+1}^{\theta_t} - V_t^{\theta_t}}{V_t^{\theta_t}}, g_{t+1} \right) \\
&= \arg \sup_{\theta_t \in \mathbf{R}^N} \operatorname{corr} (V_{t+1}^{\theta_t}, g_{t+1}).
\end{aligned}$$

Then for any portfolio $\theta \in \mathbf{R}^N$, we have

$$\begin{aligned}
0 &= \arg \sup_{\alpha \in \mathbf{R}} \operatorname{corr} (V_{t+1}^{\theta_t^* + \alpha \theta_t}, g_{t+1}) \\
&= \arg \sup_{\alpha \in \mathbf{R}} \operatorname{corr} (V_{t+1}^{\theta_t^*} + \alpha V_{t+1}^{\theta_t}, g_{t+1}) \\
&= \arg \sup_{\alpha \in \mathbf{R}} \frac{\operatorname{Cov}_t (V_{t+1}^{\theta_t^*} + \alpha V_{t+1}^{\theta_t}, g_{t+1})}{\sqrt{\operatorname{Var}_t (g_{t+1})} \sqrt{\operatorname{Var}_t (V_{t+1}^{\theta_t^*} + \alpha V_{t+1}^{\theta_t})}}.
\end{aligned}$$

The first order condition of the above equation yields

$$\operatorname{Cov}_t (V_{t+1}^{\theta}, g_{t+1}) = \frac{\operatorname{Cov}_t (V_{t+1}^{\theta_t^*}, g_{t+1})}{\operatorname{Var}_t (V_{t+1}^{\theta_t^*})} \operatorname{Cov}_t (V_{t+1}^{\theta_t}, V_{t+1}^{\theta}).$$

Rewriting the above equation in terms of returns then results in (23). \blacksquare

Proof for Proposition 1

Proof. Note that absence of CC–arbitrage implies that there exists no marketed dividend-collateral process that has a positive (meaning nonzero and non-negative) dividend process and a non-negative collateral process. Define the projection operator onto the dividend process linear subspace as $\Pi_D(D, R) = (D, 0)$. There is no CC–arbitrage if and only if $\Pi_D(M \cap (\mathcal{L}_+ \times \mathcal{L}_+)) = \{0\}$. Suppose there is no CC–arbitrage. Lemma 1 in the appendix implies that there exists a linear functional F on $\mathcal{L} \times \mathcal{L}$ such that $F(x) \leq F(y)$ for each x in M and each y in $\mathcal{L}_+ \times \mathcal{L}_+$ and the inequality holds strictly for any y with nonzero dividend process. Since M is a linear subspace, this implies that $F(x) = 0$ for all x in M , and thus that $F(y) \geq 0$ for each y in $\mathcal{L}_+ \times \mathcal{L}_+$ with strict inequality for y with nonzero dividend process.

Since \mathcal{L} is a Euclidean space, we can write $F(D, R)$ as $\psi(D) + \phi(R)$ where $\psi(D) \equiv F(D, 0)$ is defined on the dividend process linear subspace and is a strictly positive linear functional on \mathcal{L} , and $\phi(R) \equiv F(0, R)$ is defined on the collateral value linear subspace and is a non-negative linear functional on \mathcal{L} . For the converse, assume that there exists such a linear functional F . If there exists a CC–arbitrage trading strategy $\theta \in \Theta$ such that $D^\theta > 0$

and $R^\theta \geq 0$, then $F(D^\theta, R^\theta) = \psi(D^\theta) + \phi(R^\theta) > 0$, contradicting the assumption. So there is no CC-arbitrage. ■

Proof for Proposition 2

Proof. Absence of arbitrage in a frictionless market implies that the collateral constraint is not arbitrage binding if and only if there exists at least *one* pair of linear functionals (ψ, ϕ) with a strictly positive ψ and a trivial ϕ such that $\psi(D^\theta) + \phi(R^\theta) = 0$ for all θ in Θ . The proposition then follows. ■

Proof for Proposition 3

Proof. The proof here is similar to the proof given by Duffie (1992, §2C) for the case of a frictionless market. As mentioned before, we assume that $S(T) = 0$. We first show the equivalence of (3), (4), and the statement that the cumulative gain process $G_t = \sum_{s=1}^t g_s D_s + \sum_{s=0}^{t-1} u_s R_s$ is a martingale. The equivalence of the latter two can be proved by noting that $G_T = \sum_{s=1}^T g_s D_s + \sum_{s=0}^{T-1} u_s R_s$, and that $G_t - \mathbf{E}_t(G_T) = g_t S_t - \mathbf{E}_t \left[\sum_{s=t+1}^T g_s D_s + \sum_{s=t}^{T-1} u_s R_s \right]$. The equivalence of (3) and (4) can be shown as follows. Assume that (3) holds, then (4) follows by choosing $\theta^k = 0$ for $k \neq n$ and $\theta^k = 1$ for $k = n; n = 1, \dots, N$. Assume that (4) holds, then we can rewrite it as $g_s S_s = \mathbf{E}_s [g_{s+1}(S_{s+1} + D_{s+1}) + u_s R_s]$. For any $\theta \in \Theta$, with D^θ defined as in (1), we have

$$\begin{aligned} g_s \theta_s^\top \cdot S_s &= \mathbf{E}_s \left[g_{s+1} \theta_s^\top \cdot (S_{s+1} + D_{s+1}) + u_s R_s^\theta \right] \\ &= \mathbf{E}_s \left[g_{s+1} \theta_{s+1}^\top \cdot S_{s+1} + (g_{s+1} D_{s+1}^\theta + u_s R_s^\theta) \right]. \end{aligned}$$

So, for all $s \geq t$, we have $\mathbf{E}_t [g_s \theta_s^\top \cdot S_s] = \mathbf{E}_t [g_{s+1} \theta_{s+1}^\top \cdot S_{s+1} + (g_{s+1} D_{s+1}^\theta + u_s R_s^\theta)]$. Summing the above equation for $s = t, t+1, \dots, T-1$, we obtain (3).

With the above equivalence established, we only need to show that there is no arbitrage if and only if G_t is a martingale. Suppose there is no CC-arbitrage. Proposition 1 implies that there exist a strictly positive linear functional ψ on \mathcal{L} and a non-decreasing linear functional ϕ on \mathcal{L} such that $\psi(D^\theta) + \phi(R^\theta) = 0$ for all θ in Θ . By Riesz Representation Theorem; see Conway (1985, §1.3), there exist a strictly positive $g \in \mathcal{L}$ and a non-negative $u \in \mathcal{L}$ such that $\psi(D^\theta) + \phi(R^\theta) = \mathbf{E} \left[\sum_{t=0}^T (g_t D_t + u_t R_t) \right]$ for all $(D, R) \in \mathcal{L} \times \mathcal{L}$. This implies that

$$\mathbf{E} \left[\sum_{t=0}^T (g_t D_t^\theta + u_t R_t^\theta) \right] = 0 \quad \text{for all } \theta \in \Theta. \quad (24)$$

A process X is a martingale if and only if $\mathbf{E}(X_\tau) = X_0$ for any stopping time $\tau \leq T$. (See, for example, Duffie (1992), Appendix A) Consider, for an arbitrary security n and an arbitrary stopping time $\tau \leq T$, the trading strategy θ defined by $\theta_t^k = 0$, for $k \neq n$, for all t , and $\theta_t^n = 1, t < \tau$, with $\theta_t^n = 0, t \geq \tau$. Since $\mathbf{E} \left[\sum_{t=0}^T (g_t D_t^\theta + u_t R_t^\theta) \right] = 0$, we have

$$\mathbf{E} \left[-g_0 S_0^n + \sum_{t=1}^{\tau} g_t D_t^n + g_\tau S_\tau^n + \sum_{t=0}^{\tau-1} u_t R_t^n \right] = 0,$$

implying $\mathbf{E}(G_\tau^n) = G_0^n$. Since τ is arbitrary, G^n is a martingale. And since n is arbitrary, G is a martingale.

We have thus shown that absence of CC-arbitrage implies that G is a martingale and that (3) and (4) hold. The converse can be shown as follows. Suppose that G_t is a martingale for a strictly positive $g \in \mathcal{L}$ and a non-negative $u \in \mathcal{L}$. Then (3) and (4) hold for this pair of (g, u) . Suppose $\tilde{\theta}$ is a CC-arbitrage strategy. Applying (3) to $\tilde{\theta}$ at $t = 0$, we have $\tilde{\theta}_0^\top \cdot S_0 = \frac{1}{g_0} \mathbf{E}_0 \left[\sum_{s=1}^T g_s D_s^{\tilde{\theta}} + \sum_{s=0}^{T-1} u_s R_s^{\tilde{\theta}} \right] > 0$ because $D_s^{\tilde{\theta}}$ is positive and $R_s^{\tilde{\theta}} \geq 0$. This implies that strategy requires initial outside fund— a contradiction. The last assertion follows from Proposition 2. ■

Proof for Proposition 4

Proof. In order to make full use of the finite dimensionality and properties of polyhedral convex cones, we will rewrite the primitive of our model in a more convenient form used originally by Breeden (1987). Let l_t denote the number of events in \mathcal{F}_t . Note that $l_0 = 1$ because $\mathcal{F}_0 = \{\Omega\}$. Then $\mathcal{L} = \mathbf{R}^L$ with $L = l_0 + l_1 + \dots + l_T$. A process X can now be represented by a column vector $X \in \mathbf{R}^L$. Naturally, we allocate the first coordinate to X_0 , the next l_1 coordinates to time 1's X values, etc. Now we can use $c, e \in \mathbf{R}^L$ to denote consumption bundles and endowment income streams, respectively. We also use a column vector of dimension $W \equiv N(l_0 + l_1 + \dots + l_{T-1})$ to denote the trading strategy θ . Each coordinate of such a vector θ represents a long or a short position in one of the securities at some date $t \leq T-1$ and event $E_{i,t} \in \mathcal{F}_t$. By convention, we allocate the first N coordinates to the holdings at time 0, the next N coordinates to the holdings at the first event $E_{1,t}$ of \mathcal{F}_1 , and the next N to the holdings at the second event $E_{2,t}$ of \mathcal{F}_1 , etc.

The linear relation between a trading strategy and its payoff can be represented by a payoff matrix $Y \in \mathbf{R}^{L \times W}$. Thus we write $D^\theta = Y\theta$, where D^θ denotes the column vector $D^\theta \in \mathbf{R}^L$ of the dividend process from the trading strategy θ . Similarly, the collateral value matrix $Z \in \mathbf{R}^{L \times W}$, with the m th column $z_m \in \mathbf{R}^L$ of matrix Z represents the collateral value process of the m th investment opportunity. Note that since there is no trading at time T , the collateral value process of the last l_T investment opportunities is zero. Hence, both matrices Y and Z have the same dimension. Finally, using $R^\theta \in \mathbf{R}^L$ to denote the collateral value process of trading strategy θ , we have $R^\theta = Z\theta$.

With this new formulation of our finite dimensional problem, the proof of existence of optimal consumption can be easily made.

Since U is continuous, it is enough to show that $\mathcal{C}(e) = \{c \in \mathbf{R}^L : c = D^\theta + e \geq 0, R^\theta \geq 0 \text{ for some } \theta \in \mathbf{R}^W\}$ is closed and bounded. We first prove closure. Define $\Theta_R = \{\theta \in \mathbf{R}^W : R^\theta = Z\theta \geq 0\}$ and $D(\Theta_R) = \{D^\theta = Y\theta : \theta \in \Theta_R\}$. Then $\mathcal{C}(e) = (D(\Theta_R) + e) \cap \{c : c \geq 0\}$ and we only need to show that $D(\Theta_R)$ is closed. Since Θ_R is the intersection of a finite collection of closed half-spaces defined by $z_i \cdot \theta \geq 0$, where z_i is the i th row vector of Z , Θ_R is a polyhedral convex cone and is closed (see Rockafellar (1970) pp. 170-171). $D(\Theta_R)$ is the linear transformation of Θ_R under matrix Y and is therefore also a polyhedral convex cone and is closed (see Rockafellar (1970) pp. 174). So $\mathcal{C}(e)$ is closed.

We next prove boundedness. Suppose that $\mathcal{C}(e)$ is not bounded, then there is a sequence $d_n \in D(\Theta_R)$ such that $d_n + e \geq 0$ and $\|d_n\| \rightarrow \infty$. Let d^* be a cluster point of $d_n/\|d_n\| \in D(\Theta_R) \cap \bar{B}_1$, where $\bar{B}_1 = \{d \in \mathbf{R}^L : \|d\| = 1\}$. Since $D(\Theta_R) \cap \bar{B}_1$ is compact, $d^* \in D(\Theta_R) \cap \bar{B}_1$, which means that there exists a portfolio θ^* that generates cash flows d^* with $R^{\theta^*} \geq 0$. Note that $\|d^*\| = 1$, so $d^* \neq 0$. But from $d_n/\|d_n\| \geq -e/\|d_n\|$ and $\|d_n\| \rightarrow \infty$, we have $d^* \geq 0$. So $d^* > 0$, a contradiction of no arbitrage. ■

Proof for Proposition 5

Proof. We again adopt the new formulation that we used in proving Proposition 4. Simply put, there exists matrix Y and Z such that $D^\theta = Y\theta$ and $R^\theta = Z\theta$.

Since $\nabla U(c^*)$ is strictly positive, the investor is locally insatiable at c^* . No CC-arbitrage then follows. Note that constraint $c \geq 0$ is not binding since $c^* \gg 0$. So c^* is the solution of the following optimization problem:

$$\max_{\theta} U(Y\theta + e) \quad \text{s.t.} \quad Z\theta \geq 0. \quad (25)$$

Kuhn–Tucker Theorem then implies that there exists a non-negative row vector $\bar{\lambda} \in \mathbf{R}^L$ of Lagrange multipliers such that

$$\frac{\partial}{\partial \theta} [U(Y\theta + e) + \bar{\lambda} \cdot Z\theta] |_{c^*=Y\theta+e} = 0, \quad \text{which yields}$$

$$\nabla U(c^*) \cdot Y + \bar{\lambda} \cdot Z = 0.$$

So for any trading strategy θ ,

$$\nabla U(c^*) \cdot D^\theta + \bar{\lambda} \cdot R^\theta = (\nabla U(c^*) \cdot Y + \bar{\lambda} \cdot Z) \theta = 0.$$

Comparing this result with (24), and noting that $\nabla U(c^*)$ is strictly positive and $\bar{\lambda}$ is non-negative, we conclude that $(\nabla U(c^*), \bar{\lambda})$ is a pair of dividend–collateral state price processes. ■