Toeholds and Takeovers

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Toeholds have an enormous impact in "common-value" takeover battles, such as those between two financial bidders. This contrasts with the small impact of a toehold in a "private-value" auction. Our results are consistent with empirical findings that a toehold helps a buyer win an auction, sometimes very cheaply. A controlling minority shareholder may therefore be effectively immune to outside offers. A target may benefit by requiring "best and final" sealed-bid offers or by selling a cheap toehold or options to a "white knight." Our analysis extends to regulators selling "stranded assets," creditors bidding in bankruptcy auctions, and so forth.

I. Introduction

Buying a stake or "toehold" in a takeover target is a common and profitable strategy.\(^1\) The potential acquirer can gain either as a buyer

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that needs to pay a premium for fewer shares or as a losing bidder that sells out at a profit. A bidder that owns a toehold has an incentive to bid aggressively since every price it quotes represents not just a bid for the remaining shares but also an ask for its own holdings.

But this is the beginning of the story, not the end. There is a crucial difference between auctions among “strategic” private-value buyers, each of which has a different use in mind for the target assets, and auctions among “financial” common-value bidders, which primarily differ in their estimates of the returns from largely similar strategies. This paper focuses on the common-values case, where the implications of toeholds are dramatic.

Because a toehold makes a bidder more aggressive, it increases the winner’s curse for a nontoholder and makes it bid more conservatively in an ascending auction. This reduces the toeholder’s winner’s curse and allows it to be more aggressive still, creating a powerful feedback loop. So owning a toehold can help a bidder win an auction, and win very cheaply.

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2 Indicate that a large percentage of bidders own toeholds, often of 10–20 percent or more, at the time they make offers. (Betton and Eckbo’s highly comprehensive data set of 1,353 takeover attempts shows that about half of the initial bidders have toeholds.) We know of no data on options granted to friendly bidders such as Kohlberg, Kravis, and Roberts in its offer for Borden and U.S. Steel in its offer for Marathon Oil, or similar devices that can effectively serve as “toehold substitutes.” There is also little information on the differences between the types of bidders that acquire toeholds and those that do not.

For noncontrolling shareholders, stocks are almost entirely common-value assets. For competing leveraged buyout firms, which are likely to apply similar managerial and financing techniques to acquired companies, the common-value element probably dominates. When Wall Street analysts quote a company’s “breakup value,” they are essentially making common-value estimates of the value of a company’s business.

3 To focus clearly on the strategic effects, we assume pure common values. Of course, in reality takeover targets have both private-value and common-value components, so our model yields some results that are quantitatively implausible, even though we believe that they are qualitatively correct. (The equilibrium we find is continuous as small private-value components are added; Bulow, Huang, and Klemperer [1995] study the general partially common-value, partially private-value, case.)

4 In a related literature, Shleifer and Vishny (1986), Hirshleifer and Titman (1990), and Chowdhry and Jegadeesh (1994) focus on the use of toeholds by a single bidder to combat the free-rider problem described by Grossman and Hart (1980). Owning a toehold gives a bidder a profit from a successful takeover, even if it has to pay the expected full value for any shares bought in a tender offer. However, the free-rider problem is eliminated if a bidder that acquires a supermajority of the stock is able to force out nontendering shareholders. Also, if buyers do not have to buy out small, untendered minority stakes, the loss of liquidity in those shares may reduce their value, giving bidders an extra incentive to tender. We therefore ignore free-rider issues.

5 By contrast, with private values, a nontoholder will be unaffected by an opponent’s bidding; a toeholder will become more aggressive if it thinks that there is less chance of its opponent dropping out at any given price.

6 By contrast with our results, in the private-value models of Engelbrecht-Wiggans (1994), Burkart (1995), and Singh (1998), a small toehold has only a small effect,
But if two or more bidders have toeholds, a toeholder hoping to sell will be more aggressive if an opponent has a large toehold and can be expected to bid high. Our model predicts that if all bidders have toeholds of identical size, they will be more aggressive than if none had a toehold, and prices will be higher.

Our model can explain why bidders sometimes seem to overpay for the companies they take over, without appealing to stories of managerial hubris or of management’s pursuit of its own interests at the expense of shareholders. Here, bidding “too high” maximizes a bidder’s ex ante expected profits even though it sometimes loses money ex post.\(^7\)

Our results are consistent with empirical findings that toeholds increase bidders’ chances of winning takeover battles (Walkling 1985; Betton and Eckbo 1997), but it is unclear whether they decrease (Eckbo and Langohr 1989; Jarrell and Poulsen 1989), increase (Franks and Harris 1989), or have no effect on (Stulz, Walkling, and Song 1990) target returns.\(^8\)

and toeholds always raise prices. In the “free-rider” models of Shleifer and Vishny, Hirshleifer and Titman, and Chowdhry and Jegadeesh, small toeholds likewise imply small profits because, on average, all of a bidder’s profits are accounted for by gains on its toehold. A larger toehold reduces the price a bidder will have to pay in the Shleifer and Vishny and Hirshleifer and Titman models, but increases it in the Chowdhry and Jegadeesh model. None of these models can show how a toehold can make a competitor more conservative and so significantly raise a bidder’s expected profits while lowering prices. However, Hirshleifer (1995, sec. 4.5) shows that in the special case of full information, a small toehold can have a big effect on an ascending private-value auction. The firm with the lower value will drop out at a price just below the other bidder’s valuation if it has a small toehold (and if any bidding costs are small enough), but if it has no toehold it will bid no further than its own valuation (and will withdraw from the bidding if there are any bidding costs).

\(^7\) Burkart (1995) and Singh (1998) have made this point in the context of a private-value auction, but in their models a small toehold has only a small effect. Chowdhry and Nanda (1993) argue that an indebted firm may commit itself to aggressive bidding (and so sometimes deter competition) by committing to financing the acquisition through additional debt of equal or senior priority and that this might sometimes lead to overpayment.

The free-rider models provide a theoretical foundation for the conventional wisdom that acquirers do not make profits on average, judged by their subsequent stock market performance. However, Loughran and Vijh (1996) show that acquirers that pay cash do make profits whereas those that issue stock underperform the market, just as other nonacquiring equity issuers do. So market prices may overstate the consideration paid in stock takeovers, and market returns may underestimate the real profitability of these transactions. (Similarly, Rau and Vermaelen [1996] show that “value” companies appear to make profits on tender offers, whereas “glamour” companies, those whose shares sell at a high multiple of book value, decline in the extended period following the issuance of new equity in a takeover.) These papers are therefore consistent with the “bidding contest” models of toeholds, including ours, in which bidders make profits on average. Of course, there are many nonpublic investors, such as private entrepreneurs and leveraged buyout firms, that make a business of acquiring and reorganizing companies and appear to be very profitable on average.

\(^8\) The existing empirical literature does not distinguish clearly between the single- and multiple-toehold cases. The case of multiple bidders with toeholds seems rare,
The model also implies that an ownership stake of significantly less than 50 percent in a company may be sufficient to guarantee effective control; a toehold may make it much less likely that an outside bidder will enter a takeover battle. This result is consistent with the results of Walkling and Long (1984) and Jennings and Mazzeo (1993), who find that toeholds lower the probability of management resistance; of Stulz et al. (1990), who report much larger toeholds in uncontested than in contested takeovers; and of Betton and Eckbo (1997), who find that greater toeholds increase the probability of a successful single-bid contest by lowering both the chance of entry by a rival bidder and target management resistance.9

Our analysis also makes predictions that have not yet been tested because empirical work in the field has not distinguished between financial and strategic bidders or between the single- and multiple-toehold cases. Since a toehold has a much smaller effect on a private-value auction than on a common-value auction, the incentive for acquiring a toehold is much lower for a “strategic” bidder than for a “financial” bidder. A financial bidder should generally not compete with a strategic bidder unless it has a toehold or other financial inducement. And bidding should be particularly aggressive when multiple bidders have toeholds.

Since a basic message of the analysis is that if just one bidder has a substantial toehold then it may win an auction cheaply, we consider two ways for target management to counteract this effect. One way is to limit bidders to a single sealed “best and final offer,”10 so the toeholder cannot push up the competitor’s price by raising its own bid. Eliminating this incentive reduces the winner’s curse faced by a nontoeholder and creates a more competitive auction. A second approach is to “level the playing field” by giving a sec-

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9 Except that both Jennings and Mazzeo (1993) and Betton and Eckbo (1997) find that very small toeholds lead to more target management resistance than zero toeholds. This result would be explained if, as we argue next, financial bidders were more likely to acquire toeholds and, because they have no private-value advantage, were also more likely to be challenged.

10 While it may be legally difficult for a board to refuse to consider higher subsequent offers, it may be able to de facto create a “first-price” (i.e., “sealed-bid”) auction by awarding the highest sealed bidder a “breakup fee,” options to buy stock, or options to purchase some of the company’s divisions on favorable terms. (A breakup fee is a fee that would be payable to the highest sealed bidder in the event that it did not ultimately win the company.) Thus our analysis can justify the use of “lockup” provisions to support the credibility of a first-price auction. For previous analyses of the merits of allowing lockups, see Kahan and Klausner (1996) and the references cited therein.
ond bidder the opportunity to buy a toehold cheaply. The benefits
of a more competitive auction can easily swamp the "giveaway" aspect
of such a deal.

While we primarily focus on auctions of companies, our analysis
also applies to several related problems. Consider, for example, the
sale of "stranded assets" by state public utilities commissions, which
promise to reimburse the current owners for some percentage of the
difference between the assets' sale prices and their book values.
If the reimbursement is 80 percent, then the current owner effect-
ively has a toehold of 20 percent in the auctioned asset.11

As another example, our analysis applies to whether BSkyB,
Rupert Murdoch's satellite television company, should be allowed to
acquire Manchester United, England's most successful football
club. The U.K. government recently blocked the acquisition, in large
part because of concerns that by acquiring Manchester United,
which receives the biggest share of the Premier League's television
revenues (about 7 percent), BSkyB would be able to shut out other
television companies when the contract for the league's broadcast-
ing rights next comes up for auction (see Economist, March 20, 1999,

Other applications include the sharing of profits in bidding rings
(McAfee and McMillan 1992; Engelbrecht-Wiggans 1994), creditors' bidding in bankruptcy auctions (Burkart 1995), and the negotiation of a partnership's dissolution (Cramton, Gibbons, and Klemperer 1987). More generally, the theory lends insight into situations in which the loser cares how much the winner pays, as when a competitor in several auctions faces an aggregate budget constraint.12

Section II sets out our basic "common-values" model of two bid-
ders with toeholds that have independent private information about
the value of a target company. Were the bidders to completely share
information, they would have the same valuation for the target.

Section III solves for the unique equilibrium of an ascending auc-
tion between the bidders.13 Section IV derives its properties and
shows that asymmetric toeholds tend to lower sale prices.

11 That is, the current owner is 20 cents better off if the asset is sold to someone else for a dollar more and is only 80 cents worse off if it must bid an extra dollar to win the auction. So its position is strategically identical to that of a toeholder with a 20 percent stake.

12 The theory here is also very closely related to other examples in which one player has a small advantage (e.g., a small private-value advantage or a reputational advantage) in an otherwise pure common-value auction; see Bikhchandani (1988), Bulow and Klemperer (1999), and Klemperer (1998).

13 Note that with toeholds we obtain a unique equilibrium in the ascending English auction even with pure common values. It is well known that when bidders have no initial stakes in the object they are competing for, there is a multiplicity of
Sections V and VI show how a target can make bidding more competitive by using a "sealed-bid" auction (Sec. V) or by selling a cheap stake to the bidder with the smaller toehold (Sec. VI). Section VII extends to bidders that are differentially well informed and shows that most of our results are unaffected.

Section VIII presents conclusions.

II. The Model

Two risk-neutral bidders \( i \) and \( j \) compete to acquire a company. Bidder \( k \) (\( k = i, j \)) owns a share \( \theta_k \) of the company, \( 0 < \theta_k < \frac{1}{2} \), and observes a private signal \( t_k \). Bidders' shares are common knowledge\(^{14}\) and exogenous.\(^{15}\) Bidders' signals are independent, so without loss of generality we can normalize so that both the \( t_k \) are uniformly distributed on \([0, 1]\). That is, a signal of \( t_k = .23 \) is more optimistic than 23 percent of the signals \( k \) might receive and less optimistic than 77 percent. Conditional on both signals, the expected value of the company to either bidder is \( v(t_i, t_j) \). We assume that \( v(\cdot, \cdot) \) has strictly positive derivatives \( \partial v/\partial t_k \) everywhere.

The company is sold using a conventional ascending-bid (i.e., English) auction. That is, the price starts at zero and rises continuously.

\(^{14}\) This assumption is consistent with takeover regulations that require bidders to disclose their stakes.

\(^{15}\) Among the many factors that could affect the size of a bidder's toehold are the liquidity of the company's shares; institutional constraints such as the Williams Act and Securities and Exchange Commission rule 16(b), which may affect some bidders' ability to retain profits if a toehold of 10 percent or more is sold; the effect of accumulating shares on the likelihood of arranging a friendly deal (as in Freeman [1991]); the probability that management will find out that a toehold is being accumulated and the range of management response; the risk that information leakage about a potential offer will cause a prebid run-up in the stock price (Schwert [1996] shows that a prebid run-up forces a bidder to pay more to buy a company); and the amount of shares held by the bidder prior to any decision to make an offer for the company (many toeholders own large stakes accumulated years before a buyout offer).
When one bidder drops out, the other bidder buys the fraction of the company that it does not already own at the current price per unit.\textsuperscript{16} (If bidders quit simultaneously, we assume that the company is allocated randomly at the current price, though this assumption is unimportant.) Thus a (pure) strategy for bidder $k$ is a price $b_k(t_k)$ at which it will quit if the other bidder has not yet done so. We solve for the Nash equilibrium.\textsuperscript{17}

We assume that $v(\cdot, \cdot)$ is symmetric in $t_i$ and $t_j$. We define $i$’s “marginal revenue” as

$$
MR_i(t_i, t_j) = v(t_i, t_j) - (1 - t_i) \frac{\partial v}{\partial t_i}(t_i, t_j)
$$

and assume that the bidder with the higher signal has the higher marginal revenue, that is, $t_i > t_j \Rightarrow MR_i(t_i, t_j) > MR_j(t_i, t_j)$.\textsuperscript{18} This is a standard assumption in auction theory and monopoly theory; it corresponds to assuming that bidders’ marginal revenues are downward sloping in symmetric private-value auction problems and the corresponding monopoly problems. The assumption is a much stronger one for common-value auctions than for private-value auctions,\textsuperscript{19} but we note that the assumptions of this paragraph are required only for propositions 2 and 6.

We denote the price that the bidding has currently reached by $b$. We write bidder $k$’s equilibrium profits, conditional on its signal,

\textsuperscript{16} Thus all shareholders (including the two bidders) are assumed to be willing to sell out to the highest bidder, so we are ignoring any free-rider problems of the kind discussed by Grossman and Hart (1980). Also, all offers are assumed to be binding (which is supported by the legal environments of the European Community and the United States). Offers are made for all the outstanding shares. (Partial offers are legal under dominant U.S. law but only if they are nondiscriminating, and we would obtain similar results in this case.) See McAfee et al. (1993, p. 461) and Burkart (1995) for more legal details.

\textsuperscript{17} We shall see that the Nash equilibrium outcome is unique and is also the unique perfect Bayesian equilibrium.

\textsuperscript{18} In analyzing our auction using marginal revenues, we are following Bulow and Roberts (1989), who first showed how to interpret private-value auctions in terms of marginal revenues, and Bulow and Klemperer (1996), who extended their interpretation to common-values settings such as this one. Since the marginal revenue of a bidder is exactly the marginal revenue of the customer that is the same fraction of the way down the distribution of potential buyers in the monopoly model, this interpretation allows the direct translation of results from monopoly theory into auction theory and so facilitates the analysis of auctions and the development of intuition about them.

\textsuperscript{19} See Bulow and Klemperer (1999) for a discussion of when this assumption holds in the common-value case. See n. 30 for an example for which the assumption fails. See also Myerson (1981), who calls this the “regular” case in his largely private-value analysis; Bulow and Roberts (1989), who refer to this as downward-sloping marginal revenue in their private-value analysis; and Bulow and Klemperer (1996), who also (more loosely) refer to this as downward-sloping marginal revenue in the general case.
as \( \pi_k(t_k) \) and its unconditional profits (averaged across its possible signals) as \( \Pi_k \). We write the expected profits accruing to all the shareholders except the two bidders as \( \Pi_0 \).

III. Solving the Model

In this section, we first establish the necessary and sufficient conditions for the equilibrium strategies of our model (lemmas 1 and 2), next solve for the equilibrium (proposition 1), and then calculate the expected revenue of the bidders and the nonbidding shareholders.

By standard arguments, we obtain the following lemma. (All proofs are in the Appendix.)

**Lemma 1.** Bidders’ equilibrium strategies must be pure strategies \( b_i(t_i) \) and \( b_j(t_j) \) that are continuous and strictly increasing functions of their types, with \( b_i(0) = b_j(0) > v(0, 0) \) and \( b_i(1) = b_j(1) = v(1, 1) \).

We can therefore define “equilibrium correspondence” functions \( \phi_i(\cdot) \) and \( \phi_j(\cdot) \) by \( b_i(\phi_i(t_j)) = b_j(t_j) \) and \( b_j(\phi_j(t_i)) = b_i(t_i) \). That is, in equilibrium, type \( i \) of \( i \) and type \( \phi_j(t_j) \) of \( j \) drop out at the same price, and type \( i \) of \( j \) and type \( \phi_i(t_i) \) of \( i \) drop out at the same price. So bidder \( i \) will defeat an opponent of type \( t_j \) if and only if \( t_i \leq \phi_j(t_j) \), and \( \phi_j(t_i) \) is type \( t_i \)’s probability of winning the company.

Given \( i \)’s bidding function \( \hat{b}(\cdot) \), for any type \( t_j \) of \( j \), we can find \( t_j \)’s equilibrium choice of where to quit or, equivalently, \( t_j \)’s choice of which \( t_i \) to drop out at the same time as, by maximizing \( t_j \)’s expected revenues

\[
\max_{t_i} \left\{ \int_{t_j=0}^{t_i} [v(t, t_j) - (1 - \Theta_j) b_i(t)] dt + \Theta_j(1 - t_i) b_i(t_i) \right\}.
\]

(1)

The term in the integral is \( j \)’s revenue from buying, and the second term is \( j \)’s revenue from selling. Setting the derivative of (1) equal to zero\(^{20}\) and using the fact that \( t_j = \phi_j(t_i) \) in equilibrium yields

\(^{20}\) Making this argument assumes that \( b_i(t_i) \) is differentiable. Strictly, we should proceed by noting that type \( t_j = \phi_j(t_i) \) prefers quitting at \( b_i(t_i) \) to \( b_i(t_i + \Delta t_i) \). Therefore,

\[
\Theta_j [b_i(t_i + \Delta t_i) - b_i(t_i)] \left( 1 - \frac{\Delta t_i}{1 - t_i} \right) \leq \left( \frac{\Delta t_i}{1 - t_i} \right) [b_i(t_i) - v(t_i, \phi_j(t_i))] + o(\Delta b) + o(\Delta v),
\]

where \( o(\Delta b) \) and \( o(\Delta v) \) are terms of smaller orders than, respectively, \( \Delta b = b_i(t_i + \Delta t_i) - b_i(t_i) \) and \( \Delta v = \Delta t_i \cdot (\partial v/\partial t_i) \). So

\[
\lim_{\delta t_i \to 0} \sup b_i(t_i + \Delta t_i) - b_i(t_i) \leq \frac{1}{\Theta_j} \left( \frac{1}{1 - t_i} \right) [b_i(t_i) - v(t_i, \phi_j(t_i))].
\]

Using the fact that \( j \)’s type \( \phi_j(t_i + \Delta t_i) \) prefers quitting at \( b_i(t_i + \Delta t_i) \) to \( b_i(t_i) \) yields the same equation except with the inequality reversed and \( \lim \inf \) instead of \( \lim \sup \), so the right derivative of \( b(\cdot) \) exists and is given by (2). Examining the incentives
\[ b'_i(t_i) = \frac{1}{\theta_j} \frac{1}{1 - t_i} \left[ b_i(t_i) - v(t_i, \phi_j(t_i)) \right]. \] (2)

The logic is straightforward: given that the price has already reached \( b_i(t_i) \), the benefit to \( j \) of dropping out against type \( t_i + dt_i \) instead of type \( t_i \) is \( \theta_j b'_i(t_i) dt_i \); \( j \)'s toehold times the increase in price per share earned by the later exit. The cost is that, with probability \( dt_i/(1 - t_i) \), \( j \) will "win" an auction it would otherwise have lost, suffering a loss equal to the amount bid less the value of the asset conditional on both bidders' being marginal.

It is easy to check that (2) and the corresponding condition for \( b'_j(t_j) \) are sufficient for equilibrium, that is, satisfy global second-order conditions.\(^{21}\) So we have the following lemma.\(^{22}\)

**Lemma 2.** Necessary and sufficient conditions for the bidding strategies \( b_i(t_i) \) and \( b_j(t_j) \) to form a Nash equilibrium are that \( b_i(\cdot) \) and \( b_j(\cdot) \) are increasing functions that satisfy

\[ b'_i(t_i) = \frac{1}{\theta_j} \frac{1}{1 - t_i} \left[ b_i(t_i) - v(t_i, \phi_j(t_i)) \right] \] (3)

and

\[ b'_j(t_j) = \frac{1}{\theta_i} \frac{1}{1 - t_j} \left[ b_j(t_j) - v(\phi_i(t_j), t_j) \right], \] (4)

where \( \phi_i(\cdot) = b_i^{-1}(b_i(\cdot)) \) and \( \phi_j(\cdot) = b_j^{-1}(b_j(\cdot)) \), with boundary conditions given by

\[ b_i(0) = b_j(0) > v(0, 0) \] (5)

and

\[ b_i(1) = b_j(1) = v(1, 1). \] (6)

Equation (3) can be integrated to yield

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\(^{21}\) Assume, for contradiction, that at some bidding level, type \( t'_i \)'s optimal strategy is to deviate to mimic type \( t' \). Observe that at any point a higher type has a greater incentive than a lower type to remain in the bidding (the potential gains from selling out at a higher price are the same and the potential losses from being sold to are less). But the derivation of the first-order condition demonstrates that a type slightly below \( t'_i \) does not wish to stay in the bidding to mimic \( t'_i \) (see n. 20). So \( t' \) prefers to mimic this type than to mimic \( t'_i \), which is a contradiction.

\(^{22}\) Bulow et al. (1995) extend this lemma to a more general setting in which the bidders' valuations of the target company have both private- and common-value components, and they prove existence and uniqueness of equilibrium for the general case.
\[ b_i(t_i) = \frac{1}{\theta_j} (1 - t_i)^{-1/\theta_j} \left[ K - \int_0^1 v(t, \phi_j(t)) (1 - t)^{(1/\theta_j) - 1} dt \right], \]

where \( K \) is a constant of integration. According to boundary condition (6), it is given by

\[ K = \int_0^1 v(t, \phi_j(t)) (1 - t)^{(1/\theta_j) - 1} dt. \]

So we have

\[ b_i(t_i) = \frac{\int_i^1 v(t, \phi_j(t)) (1 - t)^{(1/\theta_j) - 1} dt}{\int_i^1 (1 - t)^{(1/\theta_j) - 1} dt}. \tag{7} \]

Define \( H_k(t_k) \) to be bidder \( k \)'s hazard rate, that is, the instantaneous rate at which bidder \( k \) quits as the price rises, divided by the probability that \( k \) is still present. So \( H_k(t_k) = [1/b_k'(t_k)]/(1 - t_k) \) since types are distributed uniformly. Since \( b_i(t_i) = b_j(\phi_j(t_i)) \), dividing equation (3) by equation (4) yields

\[ \frac{H_i(t_i)}{H_j(\phi_j(t_i))} = \frac{\theta_j}{\theta_i}. \tag{8} \]

Since boundary conditions (5) and (6) imply that \( \phi_j(0) = 0 \) and \( \phi_j(1) = 1 \), the unique solution to (8) is

\[ (1 - t_j)^{\theta_j} = (1 - t_i)^{\theta_i}, \tag{9} \]

that is,

\[ \phi_j(t_i) = 1 - (1 - t_i)^{\theta_j/\theta_i}. \tag{10} \]

Substituting into (7), we get the following proposition.

**Proposition 1.** There exists a unique Nash equilibrium. In it bidder \( i \) remains in the bidding until the price reaches

\[ b_i(t_i) = \frac{\int_i^1 v(t, 1 - (1 - t)^{\theta_j/\theta_i}) (1 - t)^{(1-\theta_j)/\theta_i} dt}{\int_i^1 (1 - t)^{(1-\theta_j)/\theta_i} dt}, \tag{11} \]

and bidder \( j \)'s strategy can be expressed symmetrically.

Note that our equilibrium is unique, in stark contrast to the case without toeholds in which every different weakly increasing function \( \phi_j(t_i) \) yields a distinct equilibrium, \( b_i(t_i) = v(t_i, \phi_j(t_i)) = b_j(\phi_j(t_i)) \)
(see Milgrom 1981).\(^{23}\) The reason is that the toeholds determine a precise relationship for each bidder between its opponent’s hazard rate and the “markup” it will bid over what the company would be worth conditional on its opponent’s being of the lowest remaining type. Without toeholds, these markups are zero, and there is no restriction on the ratio of the hazard rates at any price.\(^{24}\)

The easiest way to calculate bidder \(i\)’s profits is to note, by the envelope theorem, that type \(t_i + dt_i\)’s profits can be computed to first order as though it followed type \(t_i\)’s strategy, in which case it would earn \(t_i\)’s profits, except that the company is worth \([\partial v(t_i, t_j)/\partial t_i]dt_i\) more when it wins against a bidder with signal \(t_j\), so

\[
\frac{d\pi_i(t_i)}{dt_i} = \int_{t_j=0}^{t_i} \frac{\partial v}{\partial t_i}(t_i, t_j) dt_j,
\]

which implies

\[
\pi_i(t_i) = \pi_i(0) + \int_{i=0}^{t_i} \int_{j=0}^{t_j} \frac{\partial v}{\partial t_i}(t_i, t_j) dt_i dt_j, \tag{12}
\]

\[
= \theta_i b_i(0) + \int_{i=0}^{t_i} \int_{j=0}^{t_j} \frac{\partial v}{\partial t_i}(t_i, t_j) dt_i dt_j,
\]

since a bidder with \(t_i = 0\) always sells at \(b_i(0)\).

Bidder \(i\)’s expected profits (after we average across all possible values of its information and simplify) are

\[
\Pi_i = \int_{i=0}^{1} \pi_i(t_i) dt_i = \theta_i b_i(0) + \int_{i=0}^{1} \int_{j=0}^{t_i} (1 - t_i) \frac{\partial v}{\partial t_i}(t_i, t_j) dt_i dt_j. \tag{13}
\]

The expected surplus accruing to all shareholders except the bidders is

\[
\Pi_0 = \int_{i=0}^{1} \int_{j=0}^{1} v(t_i, t_j) dt_i dt_j - \Pi_i - \Pi_j, \tag{14}
\]

and the average sale price is \(\Pi_0/(1 - \theta_i - \theta_j)\).

It is also useful to note that (13) can be written as

\[
\Pi_i = \theta_i b_i(0) + \int_{i=0}^{1} \int_{j=0}^{1} p_i(t_i, t_j) (1 - t_i) \frac{\partial v}{\partial t_i}(t_i, t_j) dt_i dt_j, \tag{15}
\]

\(^{23}\) These Nash equilibria are all perfect Bayesian.

\(^{24}\) More precisely, without toeholds, the two bidders’ optimization conditions are degenerate and so cannot uniquely determine the two equilibrium strategies. Introducing toeholds breaks this degeneracy, giving two distinct optimization conditions that uniquely determine the equilibrium strategies. Introducing private-value components into valuations would also break the degeneracy.
in which \( p_i(t_i, t_j) \) is the probability with which \( i \) wins the company if the bidders’ signals are \( t_i \) and \( t_j \). So substituting \( [p_i(t_i, t_j) + p_j(t_i, t_j)] v(t_i, t_j) \) for \( v(t_i, t_j) \), we can collect terms to rewrite (14) as

\[
\Pi_0 = \int_{t_i=0}^{1} \int_{t_j=0}^{1} \left\{ \left[ v(t_i, t_j) - (1 - t_i) \frac{\partial v}{\partial t_i}(t_i, t_j) \right] p_i(t_i, t_j) + \left[ v(t_i, t_j) - (1 - t_j) \frac{\partial v}{\partial t_j}(t_i, t_j) \right] p_j(t_i, t_j) \right\} dt_j dt_i
\]

\[- \theta_i b_i(0) - \theta_j b_j(0),\]

or

\[
\Pi_0 = E_{t_i,t_j}(MR_{\text{winning bidder}}) - \theta_i b_i(0) - \theta_j b_j(0),
\]

(16)

in which \( MR_i \) is \( i \)'s “marginal revenue” as defined in Section II.

**Linear Example**

As an example, we explicitly compute the case in which the company’s value is just the sum of the bidders’ signals, \( v = t_i + t_j \). Performing the integration in (11), we have

\[
b_i(t_i) = 2 - \frac{1}{1 + \theta_j} (1 - t_i) - \frac{1}{1 + \theta_i} (1 - t_i)^{\theta_i/\theta_j}.
\]

(17)

Hence,

\[
\pi_i(t_i) = \theta_i \left( \frac{\theta_i}{\theta_i + 1} + \frac{\theta_j}{\theta_j + 1} \right)
\]

\[+ t_i - \left( \frac{\theta_j}{\theta_i + \theta_j} \right) [1 - (1 - t_i)^{(\theta_i+\theta_j)/\theta_j}].\]

(18)

So also

\[
\Pi_i = \theta_i \left( \frac{\theta_i}{\theta_i + 1} + \frac{\theta_j}{\theta_j + 1} + \frac{1}{2\theta_i + 4\theta_j} \right),
\]

(19)

\[
\Pi_0 = 1 - (\theta_i + \theta_j) \left( \frac{\theta_i}{\theta_i + 1} + \frac{\theta_j}{\theta_j + 1} \right) - \frac{\theta_i}{2\theta_i + 4\theta_j} - \frac{\theta_j}{4\theta_i + 2\theta_j},
\]

(20)
and the average sale price is

\[ \frac{\theta_j(2\theta_j + \theta_i + 1)}{(\theta_j + 1)(2\theta_j + \theta_i)} + \frac{\theta_i(2\theta_i + \theta_j + 1)}{(\theta_i + 1)(2\theta_i + \theta_j)}. \]

The bidding functions for this example are illustrated in figure 1 for the case in which the toeholds are \( \theta_1 = 0.05 \) and \( \theta_2 = 0.01 \). Observe that the bidder with the larger toehold always bids more than in the symmetric equilibrium without toeholds, whereas the bidder with the smaller toehold bids less than if neither bidder had a toehold except for very low values of its signal. Figure 2 also shows the bidding functions when the toeholds are \( \theta_1 = 0.10 \) and \( \theta_2 = 0.01 \); increasing bidder 1’s toehold makes that bidder bid more aggressively (and increases its expected profits) for all values of its signal.

The next section describes properties of the equilibrium, including those illustrated in the figures, that apply in the general case.
IV. Properties of the Equilibrium

If there were no toeholds, type $t_i$ would bid up to the price $v(t_i, \phi_j(t_i))$ at which it would just be indifferent about winning the auction; but it is immediate from equation (7) that every bidder except the highest possible type, $t_i = 1$, bids beyond this price.\(^{25}\) So except for types $t_i = 1$ and $t_j = 1$, any bidder that narrowly “wins” the auction loses money.

From equation (8), bidder $i$ always quits at a rate $\theta_j/\theta$, times as fast as bidder $j$, so it follows immediately that $i$ “wins” the auction, that is, buys the company, with probability $\theta_i/ (\theta_i + \theta_j)$. Thus \(probabil-\)

\(^{25}\) Of course, this does not mean that bidders necessarily bid more than if there were no toeholds, since the functions $\phi(x)$ are different.
ities of winning the auction are highly sensitive to the relative sizes of bidders’ stakes, and a bidder’s probability of winning is increasing in its stake.

It also follows that increasing a bidder’s stake increases its probability of winning, conditional on whatever information it has (i.e., \( \phi_i(t_i) \) is strictly increasing in \( \theta_i \) for all \( 0 < t_i < 1 \)) and that if \( i \)’s stake is smaller than \( j \)’s, then bidder \( i \) will lose to any bidder \( j \) with equally optimistic, or not too much less optimistic, information than it has (i.e., \( \theta_i < \theta_j \Rightarrow t_i > \phi_j(t_i) \) for all \( 0 < t_i < 1 \)).

Note, in particular, that a bidder with a zero stake has zero probability of winning. To see why, observe that if at any point the lowest possible remaining types of \( i \) and \( j \) were known to be \( t_i \) and \( t_j \), then \( i \), with a zero stake, will bid up to \( v(t_i, t_j) \) and \( j \), with a positive stake, will bid strictly more. So whatever the current lowest types are, there are always more of \( i \)’s types that must quit before any of \( j \)’s types leave.\(^{26}\)

From (11), increasing a bidder’s stake always makes it bid more aggressively. That is, \( \partial b_i / \partial \theta_i > 0 \) for all \( t_i < 1 \).\(^{27}\) This is what we expect: a higher stake makes a bidder more like a seller that wants to set a high price than like a pure buyer that wants to buy low.

Since \( b_i(0) = b_j(0) \) and bidding strategies are continuous, all types of bidder \( j \) with sufficiently pessimistic information also bid more aggressively if \( i \)’s stake is increased. The intuition is that because \( i \) is bidding more aggressively, low types of bidder \( j \) should take the opportunity to bid the price up under it.

However, for higher types of bidder \( j \), it is not clear whether increasing \( i \)’s stake should make \( j \) more or less aggressive: bidder \( j \) also has to take account of the larger winner’s curse of winning against a more aggressive bidder \( i \). In fact, there is no general result about whether raising \( i \)’s stake raises or lowers \( j \)’s bid.\(^{28}\)

Even though raising a bidder’s stake makes some types of its oppo-

\(^{26}\) However, the result that a player with any positive stake always beats a bidder with no toehold (and, more generally, that bidders’ probabilities of winning are in proportion to their toeholds even when the toeholds are arbitrarily small) depends critically on the pure common-values assumption. With any private-value components to valuations, sufficiently small toeholds would have only a small effect on outcomes.

\(^{27}\) Throughout this section we perform comparative statics thinking of firms’ bidding and profit functions as functions of \( \theta_i \) and \( \theta_j \) as well as of \( t_i \) or \( t_j \).

\(^{28}\) It is easy to check for the linear case that

\[ \frac{\partial b_i}{\partial \theta_i} = \frac{1 - t_j}{(1 + \theta_i)^2} \left[ 1 + \frac{(1 + \theta_i)^2 \theta_i}{1 + \theta_j} (1 - t_j)^{(\theta_i - \theta_j) / \theta_j} \log (1 - t_i) \right]. \]

So as a result of an increase in the share of the bidder with the larger toehold, the opponent bids more/less aggressively according to whether its type is below/above some cutoff level. As a result of an increase in the share of the bidder with the smaller toehold, both the weak and strong types of the opponent always bid more aggressively whereas intermediate types bid less aggressively.
nent more aggressive—so results in lower ex post profits for some types of the bidder—*increasing a bidder’s stake always increases its expected profits, whatever its signal*. In fact, increasing a bidder’s toehold increases its expected profits in two ways: it both raises the price \( b_i(0) \) at which the bidding starts and at which the bidder can sell out if it has the lowest possible signal, and also increases the incremental surplus that it earns from any higher signal (since \( \partial \phi_j / \partial \theta_i > 0 \) for all \( 0 < t_i < 1 \)).

*Increasing i’s toehold increases j’s profits if j’s signal is below some critical level*, since when \( j \) has a low signal it is likely to sell and if \( j \) sells it sells for a higher price. (In particular, \( \Pi_j(0) = b_j(0) \theta_j = b_i(0) \theta_i \) is larger.) Conversely, however, it *reduces j’s profits if its signal is above a certain level*, since when \( j \) buys it must pay more. (To check this, recall that \( \partial \phi_j / \partial \theta_i < 0 \), and it is easy to see that \( \partial \pi_j / \partial \theta_i < 0 \) at \( t_j = 1 \) since type \( t_j = 1 \) always buys and always pays more if \( \theta_i \) is larger.) *Overall, increasing i’s toehold reduces the profits of j averaged over all j’s types* (i.e., \( \partial \Pi_j / \partial \theta_i < 0 \)).

The expected price conditional on winning is the same for both bidders (and equals the average sale price) because the relative rates at which the two bidders quit is the same at every price.

Observe that when toeholds are small, \( b_i(0) = b_j(0) \) is of first order in \( \theta_i + \theta_j \), so \( b_i(0) = \theta_j b_j(0) \) is of second order in \( \theta_i + \theta_j \). From (16), we therefore have \( \Pi_0 \approx E_{t_i, t_j} (\text{MR}_{\text{winning bidder}}) \). Furthermore, by our assumption that the bidder with the higher signal has the higher marginal revenue, the expected marginal revenue of the winner is maximized over all possible mechanisms if and only if the bidder with the higher signal always wins the auction, that is, only when toeholds are symmetric. So with sufficiently small toeholds, the non-bidding shareholders’ expected wealth is highest with equal toeholds; the more unequal the toeholds, the more likely it is that the bidder with the lower signal (hence the lower marginal revenue) will win the auction, so the lower the expected wealth of the nonbidding shareholders.\(^{30}\)

\(^{29}\) For small \( \theta_i \) and \( \theta_j \), we have

\[
b_i(0) = b_j(0) \approx \frac{\partial v}{\partial t_i} (0, 0) \theta_i + \frac{\partial v}{\partial t_j} (0, 0) \theta_j.
\]

\(^{30}\) More asymmetric toeholds may increase the expected wealth of the nonbidding shareholders if the bidder with the higher signal does not necessarily have the higher marginal revenue. An example is \( v = t_i^2 + t_j^2 \). The reason is that even a bidder with an arbitrarily tiny toehold has no reason to quit below the value the company would have if its opponent had the lowest possible signal. Getting this value from the bidder with the smaller toehold—which is equally likely to be the bidder with the better or the worse information—yields a higher price for this valuation function than a more symmetric contest in which the price is more likely to be determined by the
The following proposition makes this more precise.

**Proposition 2.** The expected sale price is higher from an (ascending) auction when bidders’ toeholds are in a more equal ratio than when bidders’ toeholds are in a less equal ratio, if the toeholds are sufficiently small. (That is, for any given $0 \leq \lambda_1 < \lambda_2 \leq 1$, there is a $\Theta$ such that the expected sale price with any $\theta_i < \Theta$ and $\theta_j = \lambda_2 \theta_i$ exceeds the expected sale price with $\theta_i$ and $\theta_j = \lambda_1 \theta_i$.)

With larger toeholds the terms $\theta_i b_i(0)$ and $\theta_j b_j(0)$ are nontrivial, so $\Pi_0$ is not just a function of the expected marginal revenue of the winner. However, it remains true that, for any given $\theta_i + \theta_j$, symmetric toeholds are most desirable from the viewpoint of the nonbidding shareholders provided that the lowest possible bid, $b_i(0) = b_j(0)$, is not too much higher for asymmetric than for symmetric toeholds.

Thus it seems likely that the expected sale price will typically be increasing as the relative sizes of the bidders’ toeholds are made more equal, whatever their absolute sizes, and this is confirmed in the linear example from Section III.

*Linear example (cont.).*—In the linear example, $v = t_i + t_j$, the expected price increases as the sizes of the toeholds are made more equal, for any fixed sum of the sizes.$^{31}$

Note that if $\theta_i + \theta_j$ is small, $\Pi_0$ depends, except for high-order terms, only on the ratio $\theta_i : \theta_j$. (This ratio determines the correspondence functions $\phi_i(\cdot)$ and hence determines which bidder wins the company.) It follows that, while toeholds remain small, giving both bidders free shares that proportionately increase their stakes by diluting the remaining shareholders’ holdings has no first-order effect on anyone’s expected wealth (before they know their types); each bidder’s gain from its additional stake is just canceled by its loss from its opponent’s more aggressive behavior. For example, in the linear example, giving away 5 percent of a company in equal shares to two

$^{31}$ Let $\theta_i = \psi x$ and $\theta_j = (1 - \psi) x$. Then

$$\Pi_0 = \frac{1 + 2z}{2 + z} - \left( \frac{1 + 2zx}{1 + x + zx^2} \right) x^2,$$

where $z = \psi (1 - \psi)$. So $\Pi_0$ is increasing in $z$, and $z$ is increasing in $\psi$ for $0 \leq \psi < 1/2$ and decreasing in $\psi$ for $1/2 < \psi \leq 1$. So, for fixed $\theta_i + \theta_j$, $\Pi_0$ always increases as toeholds become more symmetric. In particular, if $\psi = 1/2$ (symmetric toeholds), $\Pi_0 \approx \frac{1}{2} - x^2$ whereas if $\psi = 0$ or $\psi = 1$ (only one bidder has a toehold), $\Pi_0 \approx \frac{1}{2} - x^2$. 

bidder with the worse information. For further discussion of the assumption that the bidder with the higher signal has the higher marginal revenue, see Bulow and Klemperer (1999).
bidders that previously had arbitrarily tiny equal toeholds costs the remaining shareholders less than 0.5 percent of their expected wealth.\footnote{32}

It also follows that diluting the stock by giving free shares to the bidder with the smaller toehold can increase the expected sale price per share, that is, increase the nonbidding shareholders’ wealth.

In summary, even small toeholds can have a large effect on the competition between the bidders. A bidder with a large toehold bids more aggressively and wins the auction with a higher probability. If the bidders’ toeholds are sufficiently asymmetric, the bidder with a smaller toehold can be forced to quit at a very low price and the auction can generate a much lower expected revenue for the nonbidding shareholders.

V. Sealed-Bid Auctions

Since the “winner’s curse” effects we have described mean that the bidder with the larger toehold wins with a high probability and at a low price, it is natural to ask whether the alternative common auction format—the first-price, sealed-bid auction—performs any better from the viewpoint of the nonbidding shareholders.\footnote{33} (Of course, without toeholds, first-price and ascending auctions yield the same expected revenue when symmetric buyers are in symmetric equilibrium.)\footnote{34}

In a first-price (i.e., sealed-bid) auction, each bidder \( k = i, j \) independently makes a single “best and final offer,” \( \tilde{b}_k(t_k) \) per unit, and the highest bidder buys the fraction of the company, \( 1 - \theta_i \), that it does not already own at the share price bid. In the equilibrium of this case, type \( t_j \) of \( j \) will choose to beat all of the opponent’s types below \( t_i \) (by bidding \( \tilde{b}_i(t_i) \)), where \( t_i \) is chosen to maximize \( j \)'s expected revenues

\[
\max_{t_i} \left\{ \int_{t_i}^{t_j} \left[ v(t, t_j) - (1 - \theta_j) \tilde{b}_i(t_i) \right] dt + \theta_j \int_{\tau = t_i}^{1} \tilde{b}_i(t) dt \right\}. \tag{21}
\]

Setting the derivative equal to zero and substituting \( t_j = \tilde{\phi}_j(t_i) \) (i.e.,

\footnote{32} Of course, this result relies on \( v(0, 0) = 0 \). More generally, giving away options with an exercise price equal to the lowest possible value of the company, i.e., \( v(0, 0) \), has the effects described.

\footnote{33} See n. 10 for a discussion of the practical feasibility of the first-price auction.

\footnote{34} By the revenue equivalence theorem due to Myerson (1981) and Riley and Samuelson (1981). (See Klemperer [1999a] for a simple exposition and further discussion.) For other considerations that might affect the shareholders’ choice between different auction types, see the articles in Klemperer (1999b).
letting $\tilde{\phi}_j(\cdot)$ and $\tilde{\phi}_i(\cdot)$ be the equilibrium correspondence functions) yields

$$
\tilde{b}_i'(t_i) = \frac{1}{1 - \theta_j} \cdot \frac{1}{t_i} \left[ v(t_i, \tilde{\phi}_j(t_i)) - \tilde{b}_i(t_i) \right].
$$

(22)

The intuition is that, given that $j$ decides not to beat types of $i$ above $t_i$, bidding even lower to win against $dt_i$ fewer types saves an additional $(1 - \theta_j) \tilde{b}_i'(t_i) dt_i$ in payments when $j$ wins, but the cost is that, with probability $dt_i/t_i$, $j$ loses an auction it would otherwise have won and so forgoes $v(t_i, t_j) - \tilde{b}_i(t_i)$.

Notice that this intuition, and so also the derivative (22), corresponds exactly to our original problem, with the change of variable $\theta_k$ to $1 - \theta_k$ and $t_k$ to $1 - t_k$ except in $v(\cdot, \cdot)$ for $k = i, j$. It follows that the arguments of Section III extend immediately to imply the following lemma.

**Lemma 3.** Necessary and sufficient conditions for the bidding strategies $\tilde{b}_i(t_i)$ and $\tilde{b}_j(t_j)$ to form a Nash equilibrium for the first-price auction are that $\tilde{b}_i(\cdot)$ and $\tilde{b}_j(\cdot)$ are increasing functions with

$$
\tilde{b}_i'(t_i) = \frac{1}{1 - \theta_j} \cdot \frac{1}{t_i} \left[ v(t_i, \tilde{\phi}_j(t_i)) - \tilde{b}_i(t_i) \right]
$$

(23)

and

$$
\tilde{b}_j'(t_j) = \frac{1}{1 - \theta_i} \cdot \frac{1}{t_j} \left[ v(\tilde{\phi}_i(t_j), t_j) - \tilde{b}_j(t_j) \right],
$$

(24)

where $\tilde{\phi}_i(\cdot) = \tilde{b}_i^{-1}(\tilde{b}_j(\cdot))$ and $\tilde{\phi}_j(\cdot) = \tilde{b}_j^{-1}(\tilde{b}_i(\cdot))$, with the boundary conditions given by

$$
\begin{align*}
\tilde{b}_i(0) &= \tilde{b}_i(0) = v(0, 0), \\
\tilde{b}_i(1) &= \tilde{b}_i(1) < v(1, 1).
\end{align*}
$$

(25)

Likewise, we have

$$
\tilde{\phi}_j(t_i) = t_i^{(1-\theta_i)/(1-\theta_j)}
$$

(26)

and the following proposition.

**Proposition 3.** There exists a unique Nash equilibrium of the first-price auction. In it $i$ bids

$$
\tilde{b}_i(t_i) = \int_0^{t_i} v(t, t^{(1-\theta_i)/(1-\theta_j)}) t^{\theta_i/(1-\theta_j)} dt
$$

$$
\int_0^{t_i} t^{\theta_i/(1-\theta_j)} dt
$$

(27)

and bidder $j$’s bid can be expressed symmetrically.
Now (26) implies that $i$ “wins” with probability $(1 - \theta_i)/[(1 - \theta_i) + (1 - \theta_j)]$. If $\theta_i > \theta_j$, this is smaller than the probability $\theta_i/(\theta_i + \theta_j)$ with which $i$ would win the ascending auction, so proposition 4 follows.

**Proposition 4.** The probability that the bidder with the higher signal wins the auction is greater in the first-price auction than in the ascending auction.

Thus the outcomes of first-price auctions are less sensitive to toeholds than the outcomes of ascending auctions, although it remains true that the bidder with the larger toehold has a higher probability of winning.

The intuition is that a bidder with a toehold still has an incentive to bid higher than otherwise: bidding more aggressively is less costly when winning the auction means buying only fraction $1 - \theta$ rather than all of the company. However, this effect is generally small unless $\theta$ is close to one (in which case the bidder has control anyway; our model therefore assumes $\theta < \frac{1}{2}$). Furthermore, and more important, the indirect or “strategic” effect due to the winner’s curse on the opponent is much smaller in first-price than in ascending auctions.\(^{35}\) So the extreme outcome of the ascending auction, that a bidder with a relatively small toehold is almost completely driven out of the bidding, does not arise in the first-price auction.

Because toeholds provide greater incentives for bidding aggressively in ascending auctions than in first-price auctions, ascending auctions yield higher prices on average when toeholds are symmetric.

**Proposition 5.** With symmetric toeholds, the expected sale price is higher in an ascending auction than in a first-price auction.\(^{36}\)

However, when toeholds are very asymmetric, the winner’s curse effect that the bidder with the smaller toehold is forced to quit at a very low value in an ascending auction implies that first-price auctions are likely to perform better.

**Proposition 6.** With asymmetric toeholds, the expected sale price

---

\(^{35}\) In an ascending auction, when bidder $i$ bids more aggressively, bidder $j$ must bid less because, conditional on winning at any price, its revenue is lower. (That is, bidding strategies are “strategic substitutes” in the terminology introduced by Bulow, Geanakoplos, and Klemperer [1985].) In a first-price auction, by contrast, bidder $j$’s response to bidder $i$’s bidding more is ambiguous: when $i$ bids more, $j$ wants to bid less on the grounds that its marginal profit when it wins is lower but more on the grounds that its probability of winning is lower, so increasing its bid is less costly. So the ascending-auction logic—that when $i$ bids a little more $j$ bids a similar amount less, so $i$ bids a similar amount more, so $j$ bids a similar amount less, etc.—does not apply in first-price auctions.

\(^{36}\) This result does not depend on the assumption of pure common values. Singh (1998) obtains this result for the pure private-values case.
is higher in a first-price auction than in an ascending auction if the
toeholds are sufficiently small. (That is, for any $\lambda \neq 1$, the first-price
auction yields a higher expected price for all $\theta_i, \theta_j$ such that $\theta_j = \lambda \theta_i$
$\leq \bar{\theta}$, for some $\theta$.)

A more formal way to understand propositions 5 and 6 is to recall
that the expected sale price equals $\Pi_0 / (1 - \theta_i - \theta_j)$, and $\Pi_0$ can be
written as in (16) for the ascending auction. By an exactly similar
logic, $\Pi_0$ for the first-price auction can also be written as in (16)
except that the term $b_i(0)$ is replaced by the expected price received
by bidder $i$ in a first-price auction if $i$ has the lowest possible signal,
that is, $\int_{t_i = 0} b_i(t_i) dt_i$, and the term $b_j(0)$ is replaced similarly. There
are therefore two differences between a first-price auction and an
ascending auction.

First, the price received by bidder $i$ with signal zero in a first-price
auction ($\int_{t_i = 0} b_i(t_i) dt_i$) is the average bid of a bidder $j$ that does not
know $i$’s signal, whereas in an ascending auction bidder $i$ must drop
out immediately at $b_i(0)$. When toeholds are symmetric, this is the
only distinction between the expressions for $\Pi_0$ for the two types of
auction, so the ascending auction yields higher prices for symmetric
toeholds (proposition 5).

Second, as proposition 4 demonstrates, the first-price auction is
won by the bidder with the higher signal in more cases than in the
ascending auction, so the first-price auction is more often won by
the bidder with the higher marginal revenue and so is likely to have
the higher expected marginal revenue of the winning bidder.37
In the limit, as toeholds become arbitrarily tiny, this is the only dis-
tinction between the expressions for $\Pi_0$ for the two types of auction,
so we expect the first-price auction to yield higher prices for asym-
metric toeholds if the toeholds are not too large (proposition 6).38

If bidders’ toeholds are neither small nor symmetric, the sale price
comparison between the two auction forms is ambiguous, but our
leading example suggests that first-price auctions are likely to be
better in practice if there is much asymmetry in the relative sizes of
the toeholds.

Linear example (cont.).—In the linear example $v = t_i + t_j$, a suffi-
cient condition for the expected price to be higher in a first-price

37 However, this need not be the case, even under our assumption that the bidder
with the higher signal has the higher marginal revenue, because it is not true that
the higher signal wins in the first-price auction in every case in which it wins in the
ascending auction.
38 An example that shows that if the bidder with the higher signal does not always
have the higher marginal revenue then an ascending auction may always yield a
higher expected price than a first-price auction is $v = t_i + t_j$ (see Bulow and Klem-
perer 1999).
auction than in an ascending auction is \( \theta_i < \frac{1}{2} \theta_j \) or \( \theta_i > 8 \theta_j \). If \( \theta_k < 0.1 \), \( k = i, j \), a sufficient condition is \( \theta_i < \frac{1}{4} \theta_j \) or \( \theta_i > 4 \theta_j \).

VI. Selling a Second Toehold

An alternative approach to compensating for the advantage that a bidder with a toehold has is to "level the playing field" by selling shares (or, equivalently, options) to the second bidder so that it has an equal stake.\(^{39}\) Even if these shares are sold very cheaply (so that all types of the second bidder will wish to buy them), the likely higher price from a fairer contest may more than outweigh the cost to the remaining shareholders of diluting their stake. With sufficiently small toeholds, it always pays to subsidize the smaller toeholder in this way.

For example, with the linear value function \( v = t_i + t_j \), if just one of the two bidders has a toehold, say \( \theta \), the expected profits of the nonbidding shareholders are \( \frac{1}{2} - \left[ \frac{\theta^2}{(1 + \theta)} \right] \) (from [20]). The bidder without the toehold makes zero expected profit (whatever its signal), so even if it had the lowest possible signal, it would be prepared to pay \( 2 \theta^2 / (1 + \theta) \), that is, \( \theta b(0) \) when both bidders have a stake of \( \theta \), for a stake of equal size. The expected profits of the nonbidding shareholders would then be \( 2 \theta^2 / (1 + \theta) \) plus the expected profits from the bidding, \( \frac{1}{2} s - \left[ \frac{4 \theta^2}{(1 + \theta)} \right] \), which equals \( \frac{1}{2} s - \left[ \frac{2 \theta^2}{(1 + \theta)} \right] \) in all. This exceeds the expected profits if there were no such sale, \( \frac{1}{2} s - \left[ \frac{\theta^2}{(1 + \theta)} \right] \), for all \( \theta \leq \frac{1}{2} \).

In fact, even if the stake could only be given away free,\(^{40}\) giving away the stake would dominate not doing so for all \( \theta \leq \frac{1}{4} \).\(^{41}\)

VII. Asymmetric Value Functions

Our analysis thus far has assumed that the value function is symmetric in bidders’ signals, that is, that the bidders have equally valuable

\(^{39}\) Selling shares at price \( p \) is equivalent in this context to giving options for the same number of shares at exercise price \( p \).

\(^{40}\) Note that we have set the base price of the stock to be zero if both bidders observe the lowest possible signal. So "given away free" here means selling them at the base price of the stock.

\(^{41}\) Thus selling shares, or giving options, at a price close to the lowest possible value of the company may be acceptable management behavior in a context in which the value function is hard to assess. In fact, selling, or giving, a second toehold is even more desirable than this if it is done through, e.g., issuing new shares that dilute the size of the first bidder’s stake, rather than by just selling a fraction of the nonbidding shareholder’s shares. Dilution is probably more realistic, but it is not needed for our result.
private information about the value of the company. In fact, none of our analysis depends on this assumption. However, if the value function is not symmetric, it is implausible that the bidder with the higher signal will always have the higher marginal revenue, and dropping this assumption requires dropping propositions 2 and 6. (Propositions 1, 3, 4, and 5 are unaffected; they depend neither on the symmetry of the value function nor on any assumption about marginal revenues.)

If the bidders' information is not equally valuable, then the bidder to whose information the value is less sensitive—the bidder, \( k \), with the lower \( \partial v(\cdot, \cdot)/\partial t_{k} \)—will typically have a higher marginal revenue when \( t_{i} = t_{j} \), that is, when each bidder receives a signal that is the same fraction of the way down the distribution of signals that it could have received. Therefore, by contrast with proposition 2, an auction in which the low-information bidder has the larger toehold and so sometimes wins when it has the lower signal may yield higher expected revenue than an auction with symmetric toeholds.\(^{42}\) Similarly, by contrast with proposition 6, if the low-information bidder has the larger toehold, an ascending auction may be preferred to a first-price auction, since the ascending auction gives a greater bias in favor of the larger toeholder's probability of winning. Of course, if the low-information bidder also has the smaller toehold, then an ascending auction will be particularly disastrous.

**VIII. Conclusion**

Toeholds can dramatically influence takeover battles. A bidder with a large toehold will have an incentive to bid aggressively, essentially because every price it quotes is both a bid for the rest of the company and an ask for its own shares. This increased aggressiveness will cause a competitor to alter its strategy as well. A competitor with a smaller toehold that is relatively pessimistic about the value of the company will become more aggressive, counting on the large toeholder to buy it out at a higher price. If the competitor has an optimistic assessment of the company's prospects, though, the large toeholder's aggressive strategy will cause the competitor to become more conservative because of an exacerbated winner's curse.

Because toeholds make a bidder more aggressive, which can make

\(^{42}\) For example, if the value function is linear but twice as sensitive to \( i \)’s signal as to \( j \)’s signal (i.e., \( v = 2t_{i} + t_{j} \)), then in the limit, as all the toeholds become tiny, the expected sale price is maximized when \( j \)'s toehold is approximately three times as large as \( i \)'s toehold.
a competitor more conservative, which can make the bidder still more aggressive, and so on, even small toeholds can have large effects. A toehold can sharply improve a bidder’s chance of winning an auction and raise the bidder’s expected profits at the expense of both other bidders and stockholders.

The strategic consequences that so benefit the toeholder create a problem for a board of directors interested in attaining the highest possible sale price for its investors. The board of a target company may therefore wish to “level the playing field” by selling a toehold to a new bidder, or by changing the rules of the auction.

Appendix

Proofs

Proof of Lemma 1

Let $\bar{B}$ be the lowest price level at or below which, with probability one, at least one bidder has dropped out. It is easy to see that if a low type gets the same expected surplus from two different quitting prices and the lower price is below $\bar{B}$, then a higher type always strictly prefers the higher quitting price. So at least up to $\bar{B}$, higher types quit (weakly) after lower types.

Define the common bidding range as price levels below $\bar{B}$.

Now if $i$ has an “atom” (i.e., an interval of its types drops out at a single price) within the common range, then $j$ cannot have an atom at the same price, since an interval of $j$’s types cannot all prefer to quit simultaneously with $i$’s atom rather than leave either just before or just after.

We next argue that the equilibrium bidding functions $b_i(t_i)$ and $b_j(t_j)$ are single-valued and continuous on the common range; that is, there are no “gaps” (no intervals of prices within the common range within which a bidder drops out with probability zero). The reason is that if $i$ has a gap, then $j$ would do better to raise the price to the top of the gap (thus raising the price $j$ receives for its share) than to drop out during the gap. So $j$ must have a gap that starts no higher than the start of $i$’s gap. Furthermore, unless $i$ has an atom at the start of the gap, $j$ would do better to raise the price to the top of the gap than to drop out just below the start of $i$’s gap; that is, $j$’s gap starts lower than $i$’s. So, since we have already shown that $i$ and $j$ cannot both have atoms at the same price, we obtain a contradiction.  

Similarly, it follows that $b_i(0) = b_j(0)$, since if $b_i(0) > b_j(0)$, then type 0 of bidder $j$ would do strictly better to increase its bid a little.

Now, observe that if $i$ has an atom in the common range, there cannot be a $t_i$ that is willing to drop out just after the atom quits; $t_i$ would either prefer to quit just before the atom (if $t_i$’s value conditional on $i$ being among the types within the atom is less than the current price) or prefer

\footnote{Note that without toeholds, gaps would be feasible, since a bidder that knows it will be the next to drop out is indifferent about the price at which it does so.}
to quit a finite distance later (since \( t_i \)'s lowest possible value conditional on \( i \) being above the atom must otherwise strictly exceed the current price). So since we have already shown that there are no gaps, any atom must be at the top of the common bidding range.

It now follows that \( b_i(0) = b_j(0) > v(0, 0) \) since, if not, then type 0 of bidder \( j \) would do better to raise its bid slightly; raising its bid by \( \epsilon \) gains \( \epsilon \theta_j \) when it still sells (with probability close to one) and loses less than \( \epsilon (1 - \theta_j) \) when it ends up buying (which happens with a probability that can be made arbitrarily small by reducing \( \epsilon \)).

At the top of the common range, assume, without loss of generality, that \( j \) is the player that quits with probability one by or at price \( \bar{B} \). Then, for some \( \tilde{t}_i \), the types (and only the types) \( t_i \geq \tilde{t}_i \) of \( i \) quit at or above \( \bar{B} \) (by the argument in the first paragraph of this proof). Then \( \bar{B} \geq v(\tilde{t}_i, 1) \), so that it is always rational for \( j \) to sell at \( \bar{B} \). But also \( \bar{B} \leq v(\tilde{t}_i, 1) \) (because either type \( t_i = \tilde{t}_i \) is willing to buy at \( \bar{B} \) with probability one; or if type \( \tilde{t}_i \) is not buying with probability one, then \( j \) must have an atom at \( \bar{B} \) and \( \tilde{t}_i \) is bidding \( \bar{B} \), so \( \bar{B} \leq v(\tilde{t}_i, 1) \) otherwise \( \tilde{t}_i \) will quit just before \( j \)'s atom). So \( \bar{B} = v(\tilde{t}_i, 1) \). Now we cannot have \( \tilde{t}_i < 1 \) or \( j \)'s types just below one would prefer quitting just after \( \bar{B} \) to just before \( \bar{B} \); either \( i \) has an atom at \( \bar{B} \), so buying just above \( \bar{B} = v(\tilde{t}_i, 1) \) is profitable, or \( i \) does not have an atom, so raising \( t_i \)'s bid by \( \epsilon \) gains \( \epsilon \theta_j \) when it still sells (with probability close to one, conditional on having reached price \( \bar{B} = v(\tilde{t}_i, 1) \)) and loses less than \( \epsilon (1 - \theta_j) \) when it ends up buying (which happens with a probability that can be made arbitrarily small by reducing \( \epsilon \)). So \( \bar{B} = v(1, 1) \), and it is straightforward that neither player can have an atom at this price (no type below one would wish to win with probability one at this price).\(^{44}\)

Finally, since we showed that there can be no interval within the bidding range within which a bidder quits with probability zero, note that bidders cannot choose mixed strategies. Q.E.D.

**Proof of Proposition 2**

Since the correspondence function \( \phi_i(t_i) \) is independent of \( \theta_i \), for any given ratio \( \theta_i : \theta_j, E_{t_i,t_j}(MR_{\text{winning bidder}}) \) is also independent of \( \theta_i \), for any given ratio and is strictly lower for the ratio \( \lambda_1 \) than the ratio \( \lambda_2 \) by our assumption that \( t_i > t_j \Rightarrow MR_i > MR_j \). But for any \( \lambda_1 \) or \( \lambda_2 \), \( \lim_{\theta_i \to 0} \pi_k(0) = 0 \), \( k = i, j \), so the result follows straightforwardly from (16). Q.E.D.

**Proof of Proposition 5**

From the argument leading up to (14), the expected sale price in the second-price auction is

\(^{44}\) Note that we have shown only that players quit by \( \bar{B} \) with probability one. Strictly speaking, in a Nash equilibrium, the (zero-probability) types \( t_i = 1 \) and \( t_j = 1 \) can quit above \( \bar{B} \) since it is a zero-probability event that the price will reach \( \bar{B} \). (In a perfect Bayesian equilibrium, however, all types including \( t_i = 1 \) and \( t_j = 1 \) must quit by \( \bar{B} \).)
\[
\frac{1}{1 - \theta_i - \theta_j} \left\{ \int_{j_i=0}^{1} \int_{j_j=0}^{1} v(t_i, t_j) dt_i dt_j - \left[ \pi_i(0) + \int_{j_i=0}^{1} \int_{j_j=0}^{\phi_i(t_i)} (1 - t_i) \frac{dv(t_i, t_j)}{dt_i} dt_i dt_j \right] \\
- \left[ \pi_j(0) + \int_{j_i=0}^{1} \int_{j_j=0}^{\phi_j(t_j)} (1 - t_j) \frac{dv(t_i, t_j)}{dt_j} dt_i dt_j \right] \right\}.
\]

By the same logic, the expected sale price in the first-price auction is the same expression, but \(\hat{\phi}_k(\cdot)\) is substituted for \(\phi_k(\cdot)\) and \(\tilde{\pi}_k(0)\) for \(\pi_k(0)\), \(k = i, j\), in which \(\tilde{\pi}_k(0)\) is bidder \(k\)'s surplus when \(k\) has its lowest possible signal. If \(\theta_i = \theta_j = \theta\), then \(\phi_j(t_i) = \hat{\phi}_j(t_i) = t_i\), so the difference between these expressions is

\[
\frac{1}{1 - 2\theta} [\tilde{\pi}_i(0) + \tilde{\pi}_j(0) - \pi_i(0) - \pi_j(0)].
\]

Substituting \(\pi_i(0) = \theta_i \delta_i(0)\) and \(\tilde{\pi}_i(0) = \theta_i \int_{j_i=0}^{1} \delta_j(t_j) dt_j\) (since a bidder with signal zero always sells) yields (after evaluating \(\int_{j_i=0}^{1} \delta_j(t_j) dt_j\) by parts) that this difference is

\[
\frac{1}{1 - 2\theta} \int_{j_i=0}^{1} 2v(t, t) \left[\left[(1 - t) - (1 - t)^{(1-\theta)/(1-\theta)}\right] - \left[t^{\theta/(1-\theta)} - t\right]\right] dt.
\]

This is positive since \(v(t, t)\) is monotonic increasing in \(t\) and the expression in braces has expected value zero and is negative for all \(t \in (0, \hat{t})\) and positive for all \(t \in (\hat{t}, 1)\), for some \(\hat{t}\). Q.E.D.

**Proof of Proposition 6**

For a given \(\lambda\), write \(E(\lambda)\) and \(\tilde{E}(\lambda)\) for the values of \(E_{i,t} (MR_{\text{winning bidder}})\) for the ascending auction and first-price auction, respectively. The term \(E(\lambda)\) is independent of \(\theta\), (since \(\phi(\cdot)\) is independent of \(\theta_\cdot\)), and \(\tilde{E}(\lambda)\) is monotonic continuous decreasing in \(\theta\), with \(\lim_{\theta \to 0} \tilde{E}(\lambda) = E(1)\), since \(\hat{\phi}_j(t_i) = t_i^{(1-\theta)/(1-\theta)}\) is monotonic and continuous in \(\theta\), for every \(t_i\) and \(\lim_{\theta \to 0} \hat{\phi}_j(t_i) = t_i\), for every \(t_i\). Furthermore, by our assumption that \(t_i > t_j \Rightarrow MR_i > MR_j\), \(E(1) > E(\lambda)\) for all \(\lambda \neq 1\). Finally, it is straightforward that \(\lim_{\theta \to 0} \pi_k(0) = \lim_{\theta \to 0} \tilde{\pi}_k(0) = 0\), \(k = i, j\), for all \(\lambda\), so the result follows easily from (16). Q.E.D.

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