Default Parameter Estimation Using Market Prices

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This article presents a new methodology for estimating recovery rates and the (pseudo) default probabilities implicit in both debt and equity prices. In this methodology, recovery rates and default probabilities are correlated and depend on the state of the macroeconomy. This approach makes two contributions: First, the methodology explicitly incorporates equity prices in the estimation procedure. This inclusion allows the separate identification of recovery rates and default probabilities and the use of an expanded and relevant data set. Equity prices may contain a bubble component—which is essential in light of recent experience with Internet stocks. Second, the methodology explicitly incorporates a liquidity premium in the estimation procedure—which is also essential in light of the large observed variability in the yield spread between risky debt and U.S. Treasury securities and the illiquidities present in risky-debt markets.

A value-at-risk measure that successfully integrates market, credit, and liquidity risk is the Holy Grail of a successful risk-management procedure. As I have previously argued (Jarrow 1998), arbitrage-free pricing theory allows this construction, at least conceptually. The remaining obstacles to a successful implementation of this Holy Grail are the selection of a particular parameterization of the general model and the estimation of its parameters.

The available model structures are of two types–structural and reduced form. Structural models are those that endogenize the bankruptcy process by explicitly modeling the assets and liability structure of the company (Merton 1974). Reduced-form models exogenously specify an arbitrage-free evolution for the spread between default-free and credit-risky bonds (Jarrow and Turnbull 1995; Duffie and Singleton 1999).

Structural models have been successfully implemented in professional software.[1] This particular parameterization of the structural approach uses only equity prices and balance sheet data to estimate the bankruptcy process's parameters. The argument is that debt markets are too illiquid and debt prices too noisy to be useful; hence, they should be ignored. Unfortunately, this implementation of the structural approach ignores the possibility of stock-price bubbles (e.g., for Internet stocks) and the misspecification that this omission implies.

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In contrast, the existing literature on implementing reduced-form models concentrates only on debt prices while ignoring equity prices (Jarrow, Lando, and Turnbull 1997); Duffie and Singleton).

The two approaches seem to have partitioned the market data: Structural models use only equity prices, and reduced-form models use only debt prices. This partitioning is artificial and unnecessary. Both markets provide relevant information about a company's default process and parameters, and both should be used. The purpose of this article is to provide a new methodology for implementing reduced-form models that includes both debt and equity prices in the estimation procedure.

In particular, I present a methodology for estimating recovery rates and (pseudo) default probabilities implicit in debt and equity prices. The methodology is quite general; it allows default probabilities and recovery rates to be correlated and dependent on the macroeconomy. Thus, the resulting reduced-form model integrates market and credit risk with correlated defaults.

The article makes two contributions: First, as stated, the methodology explicitly incorporates equity prices in the reduced-form estimation procedure. For a fractional recovery rate, using debt prices alone allows the estimation of only the expected loss—that is, the multiplicative product of the recovery rate times the (pseudo) default probabilities (Duffie and Singleton). The introduction of equity prices enables one to separately estimate these quantities. The procedure used to include equity in the reduced-form model is commonly used in portfolio theory (Duffie 1988). Simply stated, the equity price is viewed as the present value of future dividends and a resale value. The future resale value is consistent with the existence of equity price bubbles (Jarrow and Madan 2000). Given recent market experience with Internet stocks, such an inclusion is necessary for accurate estimation of bankruptcy parameters.

Second, because debt markets are notoriously illiquid, especially in comparison with equity markets, the methodology explicitly incorporates liquidity risk in the reduced-form model and the estimation procedure. Liquidity risk is introduced through the notion of a "convenience yield," a well-studied concept in the literature on commodities pricing that is consistent with an arbitrage-free but incomplete debt market. Liquidity risk introduces an important and necessary additional randomness into the yield spread between risky-bond prices and U.S. Treasury securities. This additional randomness allows for the decomposition of the credit spread into a liquidity-risk component and a credit-risk component. The liquidity-risk adjustment is needed to accurately estimate the bankruptcy parameters from credit spreads.

Model Structure

This section introduces the notation and economic structure of the reduced-form model. The assumption is that markets are frictionless with no arbitrage opportunities. Markets are not assumed to be complete or perfectly liquid, nor are price bubbles excluded.

A probability space underlies the economy in which P represents the "statistical," "objective," or "empirical" probability distribution. I will use the term "statistical" probability distribution. The statistical probability distribution is the probability distribution that standard statistical procedures draw inferences about when using historical market prices. Alternatively stated, it is that probability distribution generating the observed debt and equity prices in the economy.

Trading can take place anytime during the interval $(0, \bar{T})$. Traded are default-free zero-coupon bonds of all maturities, equities, and risky (defaultable) zero-coupon bonds of all maturities. The following notation characterizes these prices and the subsequent estimation procedure.

Let $p(t,T)$ represent the time t price of a default-free dollar paid at time T, where $0 \le t \le T \le \bar{T}$. The default-free forward rates, $f(t,T)$, are implicitly defined by

$$p(t,T) = e^{-\int_t^T f(t,u)\,du}, \tag{1}$$

where u is the variable of integration. The spot rate of interest is given by $r(t) = f(t,t)$.

The notation for prices of the risky zero-coupon debt requires more structure than given so far. Consider a company issuing debt and equity to finance its operations. For the moment, suppose that its debt takes the form of zero-coupon bonds of perhaps different seniorities (in the event of default).

Let $v(t,T\!:\!i)$ represent the time t price of a promised dollar of seniority i to be paid by this company at time T, where $0 \le t \le T \le \bar{T}$. The debt is risky because if the company defaults prior to time T, the promised dollar may not be paid.

Let τ represent the first time that this company defaults ($\tau > \bar{T}$ is possible if the company does not default). The default time, τ, is a random variable. Let

$$N(t) = 1_{(t \ge r)} = \begin{cases} 1 & \text{if } t \ge \tau \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

denote the point process indicating whether or not default has occurred prior to time t. At this stage in the analysis, the point process can be a general stochastic process. Let $\lambda(t)$ represent its random intensity process. The time t intensity process, $\lambda(t)\Delta$, gives the approximate probability of default for this company over the time interval $(t, t + \Delta)$.[2]

Without loss of generality, if default occurs, let the zero-coupon bond of seniority i receive a fractional recovery of $\delta_i(\tau)v(\tau-,T\!:\!i)$ dollars, where $0 \le \delta_i(\tau)$ and $\tau-$ represents an instant before default. After default, the debt is worth only a fraction of its predefault value. The recovery fraction, $\delta_i(t)$, is random. At this point, the fractional recovery rate assumption is without loss of generality because the recovery rate process is completely arbitrary. When a specific parameterization for the recovery rate is imposed for empirical estimation, this assumption becomes restrictive. Note that the recovery rate fraction, $\delta_i(t)$, completely specifies the seniority status of the debt issue. The greater the seniority of the debt issue, the larger the recovery rate—everything else being constant.

This formulation is the standard structure imposed in reduced-form models. Now, consider the formulation of equity prices. For analysis, thinking of equity as the debt issue of "last" seniority is useful. In this analogy, equity pays "coupons," called dividends, and pays a liquidating payoff at time T^* for $0 \le T^* \le \bar{T}$.[3] The time t value of these promised payments equals the value of the equity (per share) and is denoted by $\xi(t)$. This procedure is standard for characterizing equity prices in portfolio theory (see Duffie). The equityholders receive these payments unless the company defaults. If default occurs, the equityholders get a fractional recovery payment on these promises equal to $\delta_e(\tau)\xi(\tau-)$, where $\delta_e(\tau) \equiv 0$.[4]

Now, some notation is needed for these dividend payments. The regular dividends are paid at times 1, 2, . . ., T^* and are denoted by D_t at time t. Assume these dividends are deterministic quantities paid unless the company defaults prior to the dividend-payout date.[5] This formulation implicitly defines T^* as that date to which this deterministic dividend assumption is true. For many stocks, T^* will necessarily be set equal to a year (or less).

The liquidating dividend is paid at time T^* unless default occurs prior to that date. This dividend consists of a random payoff of $L(T^*)$. Let $S(t)$ represent the time t present value of this liquidating dividend conditional upon no default prior to time t.

Finally, some evidence indicates that stock prices contain a "bubble" or "monetary value" component (see Jarrow and Madan). An example is the recent price growth of Internet stocks.[6] So, let $\theta(t)$ represent this time t bubble component in the stock price.

Given this setup, one can easily see that the per share equity value at time t is given by[7]

$$\xi(t) = \begin{cases} S(t) + \theta(t) + \sum_{j \geq t}^{T^*} D_j v(t, j : e) & \text{if } t < \tau \\ 0 & \text{if } t \geq \tau, \end{cases} \tag{3}$$

where $v(t, j{:}e)$ represents a zero-coupon bond of seniority e (equity) issued by this company and Dj is the dividend paid at time j.

The equity value at time t is equal to the present value of the liquidating dividend plus a bubble component plus the present value of the regular dividend payments. The present value of the regular dividend payments is seen to be equivalent to a portfolio of risky zero-coupon bonds of a particular seniority. The seniority is that of equity, with a fractional recovery rate of $\delta_e(t)$. If default occurs, then the value of the equity drops to zero because the fractional recovery rate on the dividends (liquidating and regular) is assumed to be zero. The bond's default parameters are explicitly included within this component of the equity's price.

A special case of Equation 3—the announced dividend model—is worth mentioning because it has been previously used in the option-pricing literature. The dividend at time j, D_j, is zero unless $j = t_x$, where t_x is the next ex-dividend date. Let t_a be the announcement date of the next dividend payment. Then, Equation 3 becomes

$$\xi(t) = \begin{cases} S(t) + \theta(t) + D_{t_x} v(t, t_x{:}e) & \text{if } t \in [t_a, t_x) \text{ and } t < \tau \\ 0 & \text{if } t \in [t_a, t_x) \text{ and } t \leq \tau \end{cases} \tag{4a}$$

and

$$\xi(t) = \begin{cases} S(t) + \theta(t) & \text{if } t \notin [t_a, t_x) \text{ and } t < \tau \\ 0 & \text{if } t \notin [t_a, t_x) \text{ and } t \geq \tau \end{cases}. \tag{4b}$$

The interpretation of Equations 4 is that the dividend is known and deterministic only after it is announced (and prior to its payment). This model is similar to a model used for the valuation of equity options with a known and discrete dividend (see Jarrow and Turnbull 1996). In this example, the date T^* is the same as t_x but only for $t \geq t_a$; otherwise, $T^* = 0$. This example clarifies the robustness of the deterministic dividend assumption and the interaction between the definition of T^* and the specification of the regular dividends, D_t.

Risk-Neutral Valuation

This section presents the valuation formulas used in the estimation of the bankruptcy parameters. Under the assumption of no arbitrage, standard arbitrage pricing theory implies that a probability distribution Q exists such that present values are computed by discounting at the spot rate of interest and then taking an expectation with respect to Q.[8] For example, using this characterization, one can write

$$p(t, T) = E_t\left[e^{-\int_t^T r(u)du}\right], \tag{5}$$

where $E_t(\bullet)$ is conditional expectation with respect to Q at time t.

Equation 5 is the standard risk-neutral pricing relationship satisfied by default-free zero-coupon bonds. Applying this valuation methodology to prices of risky zero-coupon bonds and the liquidating dividend produces

$$v(t, T{:}i) = E_t\left[\delta_i(\tau)v(\tau-, T{:}\,i)e^{-\int_t^\tau r(u)du} 1_{(\tau \leq T)} + 1e^{-\int_t^T r(u)du} 1_{(T < \tau)}\right] \tag{6}$$

and

$$S(t) = E_t\left[L(T^*)e^{-\int_t^{T^*} r(u)du} 1_{(T^* < \tau)}\right]. \tag{7}$$

The risky-debt value is composed of two parts. The first is the present value of the promised payment in case of default. The second is the present value of the promised payment if default does not occur. The present value of the liquidating dividend is similar. The only difference in these two expressions (Equations 6 and 7) is that $L(T^*)$ is random for the liquidating dividend whereas the promised risky-debt payment of $1 is not.

Risk-neutral valuation provides for no analogous expression for the bubble component, $\theta(t)$. The reason is that one cannot write the bubble component as a discounted expectation (see Jarrow and Madan).[9]

Using a result from Duffie and Singleton (Theorem 1), under mild conditions, one can rewrite Equations 6 and 7 as[10]

$$v(t,T:i) = E_t\left(1e^{-\int_t^T\{r(u)+\lambda(u)[1-\delta_i(u)]\}du}\right) \tag{8}$$

and

$$S(t) = E_t\left[L(T^*)e^{-\int_t^{T^*}[r(u)+\lambda(u)]du}\right], \tag{9}$$

where default has not occurred before or at time t and $\lambda(u)$ is the intensity process under risk-neutral measure Q. I call this intensity process the "pseudo probability of default."

The importance of this simplification cannot be overstated. The price of the risky zero-coupon bond can again be written as an expected discounted value, but in this case, the discount factor is the spot rate of interest adjusted for the expected loss in default, $\lambda(t)[1 - d_i(t)]$. A similar statement applies for the present value of the liquidating (random) dividend.

As pointed out by Duffie and Singleton, the pseudo probability of default always appears in this valuation formula for risky debt as part of a multiplicative product. It is always multiplied by the fractional loss in default $[1 - \delta_i(t)]$. Hence, debt prices allow one to estimate only the product of the pseudo probability of default times the fractional loss, not the pseudo probability of default alone. The introduction of the equity valuation process as in Equation 3, in conjunction with Equations 8 and 9, overcomes this difficulty because the fractional loss for equity is known (*a priori*) and equal to 1—that is, $1 - \delta_e(t) = 1$ as $\delta_e(t) = 0$.[11] Thus, a joint estimation of pseudo default probabilities and recovery rates, in which each is identified separately, is possible by using both debt and equity prices. A procedure for this joint estimation is discussed later.

The Liquidity Premium

This section adds the liquidity premium into the preceding model formulation (Equation 8). Liquidity risk is an important consideration in the pricing of risky debt, and its inclusion is motivated by two observations. First, debt prices are difficult to obtain because of the sparsity of secondary-market trading. In fact, at the time this article was written, the most frequent data available were monthly observations—the Wisconsin database (Warga 1999). Compare this lack of frequency with the ready availability of daily transaction data for equity prices. Second, a study by Schwartz (1998) indicated that, even for these monthly bond data, the number of outliers (measured relative to similar debt issues) is significant. One can attribute these outliers to the illiquidity in the market.

Corporate debt issues, analogous to Treasuries, can be used in repurchase agreements (repos) as collateral. Therefore, at times, particular corporate bonds are in short supply, asking prices are high, and special repo rates are low (Rooney 1998). In these cases, one cannot buy the bond at reasonable prices and liquidity causes bond prices to be "too high." Conversely, in times of credit scares and high market volatility, corporate bonds (or particular sovereigns—e.g., Russian bonds) can be sold only at discount prices. In these cases, one cannot sell bonds at reasonable prices and liquidity causes bond prices to be "too low."

Although Duffie and Singleton suggested a modification of Equations 8 to incorporate liquidity risk, they did not give a formal argument justifying its inclusion. This section provides such a formal justification based on a related argument used for convenience yields in Treasury securities that is contained in Jarrow and Turnbull (1997). The justification is consistent with no-arbitrage opportunities but an incomplete debt market.

Consider a market where one cannot synthetically construct a particular credit-risky zero-coupon bond (hereafter called "zero") with price $v_l(t,T:i)$. The subscript l indicates that the market has a liquidity problem. Given an identical credit-risky zero with no liquidity problems and price $v(t,T:i)$, the following no-arbitrage relationships hold:

$v(t,T:i) \le v_l(t,T:i)$ in a shortage so one cannot readily buy, and

$v(t,T:i) \ge v_l(t,T:i)$ in a glut so one cannot readily sell.

The argument is simple. When one cannot synthetically construct the bond on the right side of this expression, the act of arbitrage cannot force equality between the two prices. Thus, a function $\gamma_i(t,T)$ exists such that

$$v_l(t,T:i) = e^{-\gamma_i(t,T)}v(t,T:i). \tag{10}$$

In a shortage, when one cannot readily buy the risky bond, $\gamma_i(t,T) \le 0$ and the function $-\gamma_i(t,T)$ has the interpretation of being a positive convenience yield obtained from holding (storing) the credit-risky zero. In the case of shortages of the risky bond, (special) repo rates are low and storing the bond provides benefits. This case is exactly analogous to positive convenience yields associated with storage of other commodities used in production (such as oil).

When a glut exists and one cannot readily sell the risky bond, $\gamma_i(t,T) \ge 0$ and the function $-\gamma_i(t,T)$ is interpreted as a negative convenience yield obtained from holding (storing) the credit-risky zero. In this case, holding the bond in a portfolio produces a negative externality, which is an implicit storage cost exactly analogous to the negative convenience yields associated with storage of spoilable commodities.

For equity markets, liquidity costs are assumed to be zero. Here again, as for recovery rates, equity forms the base case against which debt's bankruptcy parameters can be estimated.

Model of the Stock-Price Bubble

For simplicity, I model the stock-price bubble component as a random process that is proportional to the present value of the liquidating dividend, as in the following expression:

$$\theta(t) = S(t)\left[e^{\int_0^t \mu_\theta(u)du} - 1\right], \tag{11}$$

where $\mu_\theta(u) \ge 0$ is the continuous return in the stock price resulting from the bubble component.

Combined with Equation 11, Equation 3 can be rewritten as

$$\xi(t) = \begin{cases} S(t)e^{\int_0^t \mu_\theta(u)du} + \sum_{j \ge t}^{T^*} D_j v(t, j:e) & \text{if } t < \tau \\ 0 & \text{if } t \ge \tau \end{cases}. \tag{12}$$

Equation 12 represents a convenient decomposition of the stock price into its underlying components.

Implicit Estimation

For the joint implicit estimation of the recovery rates and the (pseudo) default probabilities, additional structure needs to be imposed on both these quantities. Following Lando (1998), assume that the default process follows a Cox process in which $\lambda(t)$ and $\delta_i(t)$ are predetermined functions of a vector of observable state variables, represented by $\mathbf{X}(t)$ for $0 \le t \le \overline{T}$. The vector $\mathbf{X}(t)$ is a multidimensional stochastic process. The state variables within $\mathbf{X}(t)$ could include the spot interest rate, foreign currencies, GNP measures, or a market index.

Formally,

$$\lambda(t) = \lambda[t, \mathbf{X}(t)] \tag{13a}$$

and

$$\delta_i(t) = \delta_i[t, \mathbf{X}(t)]. \tag{13b}$$

Similarly, the liquidity discount, the bubble component of the stock price, and the present value of the liquidating value of the equity can depend on the same $\mathbf{X}(t)$ state variables. That is,

$$\gamma_i(t,T) = \gamma_i[t, T, \mathbf{X}(t)], \tag{14a}$$

$$S(t) = S[t, \mathbf{X}(t)], \tag{14b}$$

and

$$\mu_\theta(t) = \mu_\theta[t, \mathbf{X}(t)]. \tag{14c}$$

Of course, prior to estimation, these deterministic functions need to be specified.

The estimation is performed at an arbitrary time t by using cross-sectional and time series data. Given is a collection of observable prices of default-free zeros, $p(t,T)$, for various t and T and risky zeros, $v_l(t,T{:}i)$ for various i, t, and T. Also available are an observable equity price $\xi(t)$, observable (predictable) dividends $D_1, \ldots, D_{T^*}$, and observable state variables $\mathbf{X}(t)$. Note that this procedure is conditioned on the fact that the company is not yet in default. With these observables, the left side of the following system of equations is determined:

$$v_l(t,T{:}i) = v_l\{t, T{:}i, \lambda[t,\mathbf{X}(t)], \delta_i[t,\mathbf{X}(t)], \gamma_l[t,T,\mathbf{X}(t)]\} \tag{15a}$$

for various i and T and

$$\xi(t) = \sum_{j \ge t}^{T^*} D_j v\{t, j{:}e, \lambda[t,\mathbf{X}(t)]\} + S[t,\mathbf{X}(t)]e^{\int_0^t \mu_\theta[u,\mathbf{X}(u)]du}. \tag{15b}$$

On the right side of these equations, the dependence of the risky debt and equity prices on the (pseudo) default probabilities (λ), the recovery rate (δ_i), the liquidity premium (γ_i), the bubble component (μ_θ), and the liquidating dividend $[S(t)]$ is made explicit. Notice that the equity prices do not depend on a recovery rate or a liquidity premium.

The two systems of equations (Equations 15a and 15b) can be estimated in three stages. Stage One is to estimate the parameters in the system of equations given by Equation 15a for the risky-debt prices. This system can be estimated cross-sectionally at a particular time t. Here, as long as the number of equations is at least as large as the number of unknowns, the system can be inverted to obtain estimates of the parameters (a sum-of-squared-error-minimizing procedure may be necessary). As indicated previously, however, the recovery rate and the default probability always appear as a product and are inseparable in this system.

In this estimation procedure, the prices of risky zero-coupon bonds were assumed to be observable. But this is not usually the case. Instead, risky *coupon-bearing* bond prices are observable. These procedure can be easily modified, however, to incorporate this difference. This modification is Stage Two in the estimation procedure. There are two basic approaches: One is to first strip out the prices of the zeros from the prices of the coupon bonds before applying the estimation procedure. Various techniques are available for this approach (see Schwartz). The alternative is to apply the joint estimation procedure directly to the prices of the coupon-bearing bond by using the fact that a risky coupon bond is a portfolio of risky zero-coupon bonds. In Equation 15a, the left side would become the observable risky coupon-bearing bond and the right side would become a summation of the relevant zero-coupon bonds weighted by the coupon payments.

Stage Three is to estimate the parameters in Equation 15b for equity prices. For this estimation, the condition is that default has not yet occurred (i.e., $t < \tau$). This single equation can be estimated only by using time-series analysis. The unknowns are the (pseudo) default probability, the bubble component, and the liquidating dividend. Obtaining a solution requires at least as many time-series observations of $\xi(t)$ as there are unknown parameters in $\lambda[t, T, \mathbf{X}(t)]$, $\mu_\theta[t, \mathbf{X}(t)]$, and $S[t, \mathbf{X}(t)]$. Then, given the estimates for the equity price's default parameters, the recovery rates for the various seniority levels of the debt issues can be easily inferred from the debt-price parameters estimated earlier.

An alternative to this three-stage procedure is a single-stage procedure that jointly estimates all of the parameters from the larger system of equations by using coupon bonds and equity prices together. The difference between the two approaches is that the joint estimation procedure constrains the parameters to be identical in the two markets whereas the three-stage procedure does not.

Next, I provide further description of this estimation procedure for a special case of the formulation.

A Practical Empirical Specification

To estimate the system of equations represented by Equations 15, one still needs to specify the various functions in Equations 13 and 14. These functions are specified here, and without loss of generality, we assume that the prices of risky zero-coupon bonds are observable.

For a practical but realistic empirical specification of the reduced-form model, let there be two state variables $\mathbf{X}(t)$ describing the system: (1) the spot rate of interest and (2) a general indicator of the health of the economic system—the cumulative excess return on a market index (as measured from some initial date).[12]

Now, an arbitrage-free evolution for these state variables needs to be specified. First, consider the spot rate of interest, $r(t)$. For illustration purposes, I use a single-factor model with deterministic volatilities that is sometimes called the extended Vasicek model (see Vasicek 1977 and Heath, Jarrow, Morton 1992). The term-structure evolution is described by the evolution of the spot rate of interest under risk-neutral measure Q:

$$dr(t) = a[\bar{r}(t) - r(t)]dt + \sigma_r dW(t), \qquad (16)$$

where

a = a mean-reversion parameter, a constant that is not 0

σ_r = volatility of the spot rate, where $\sigma_r > 0$ is a constant

$\bar{r}(t)$ = a deterministic function of t

$W(t)$ = a standard Brownian motion under Q initialized at $W(0) = 0$.

In Equation 16, the spot rate of interest follows a mean-reverting process under the risk-neutral measure. As shown in Heath, Jarrow, and Morton (1992), to match an arbitrary initial forward-rate curve, one must set

$$\bar{r}(t) = \frac{f(0,t) + [\partial f(0,t)/\partial t + \sigma_r^2(1 - e^{-2at})/2a]}{a}. \qquad (17)$$

Combined with Equation 16, the evolution for the spot rate of interest can be rewritten as

$$r(t) = f(0,t) + \frac{\sigma_r^2(e^{-at} - 1)^2}{2a^2} + \int_0^t \sigma_r e^{-a(t-u)} dW(u). \qquad (18)$$

Note that the spot rate of interest is normally distributed in the extended Vasicek model.

The second state variable is related to a market index, denoted $M(t)$. The evolution for the market index is assumed to satisfy

$$dM(t) = M(t)[r(t)dt + \sigma_m dZ(t)], \qquad (19)$$

where the volatility of the market index, σ_m, is constant and $Z(t)$ is a standard Brownian motion under Q initialized at $Z(0) = 0$ that is correlated with $W(t)$ as $dZ(t)dW(t) = \varphi_{rm}dt$ with φ_{rm} (the correlation between the spot rate and the market index) a constant.[13]

The market index follows a geometric Brownian motion with drift $r(t)$ and volatility σ_m. The drift must be the spot rate of interest under the risk-neutral measure. The evolutions of the market index and the spot rate of interest are correlated, with

$$\operatorname{cor}\left[\frac{dM(t)}{M(t)}, dr(t)\right] = \varphi_m dt. \tag{20}$$

For subsequent use, note the market index process in its integral form:

$$M(t) = M(0)e^{\int_0^t r(u)du - (1/2)\sigma_m^2 t + \sigma_m Z(t)}. \tag{21}$$

Given observation dates 1, 2, 3, . . ., t, Equation 21 can be solved for $Z(t)$ as a function of $Z(t-1)$. This solution is given by

$$\begin{aligned} Z(t) &= Z(t-1) \\ &+ \left[\frac{\log M(t)/M(t-1) - \int_{t-1}^t r(u)du + \int_{t-1}^t (1/2)\sigma_m^2 du}{\sigma_m}\right] \end{aligned} \tag{22}$$

for $t \geq 1$ and $Z(0) = 0$.

One sees here that $Z(t)$ is a measure of the cumulative excess return per unit of risk (above the spot rate of interest) on the market index.[14] $Z(t)$ becomes the second state variable chosen because it is normally distributed (as is the spot rate of interest under Equation 18).

Now, the assumption about the bankruptcy parameters and the recovery rate are imposed:

$$\lambda(t) = \lambda_0 + \lambda_1 r(t) + \lambda_2 Z(t) \tag{23a}$$

and

$$\delta_i(t) = \delta_i, \tag{23b}$$

where λ_0, λ_1, λ_2, and δ_i are constants.

In this formulation, the (pseudo) probability of default is assumed to be a linear function of the state variables $r(t)$ and $Z(t)$. This assumption implies that negative default rates, $\lambda(t) < 0$, are possible. Nonetheless, given the tractability of the subsequent expressions, this step is an acceptable first approximation. Its validity awaits empirical investigation. Also, the fractional recovery rate is assumed to be a constant, which is a first approximation that is easily relaxed.

Given these expressions, Appendix A shows that the prices of the default-free zero-coupon bond and the risky zero-coupon bond can be rewritten as

$$\begin{aligned} p(t,T) &= E_t\left[e^{-\int_t^T r(u)du}\right] \\ &= e^{-\mu_1(t,T) + \sigma_1^2(t,T)/2} \end{aligned} \tag{24}$$

and

$$\begin{aligned} v_l(t,T:i) &= e^{-\gamma_i(t,T)} \\ &\times E_t\left(e^{-\int_t^T \{r(u) + [\lambda_0 + \lambda_1 r(u) + \lambda_2 Z(u)](1-\delta_i)\}du}\right) \\ &= e^{-\gamma_i(t,T)} p(t,T) \\ &\times e^{-\lambda_0(1-\delta_i)(T-t) - \lambda_1(1-\delta_i)\mu_1(t,T) + \left[2\lambda_1(1-\delta_i) + \lambda_1^2(1-\delta_i)^2\right]\delta_1^2(t,T)/2} \\ &\times e^{-\lambda_2(1-\delta_i)Z(t)(T-t) + [1 + \lambda_1(1-\delta_i)]\lambda_2(1-\delta_i)\varphi_{rm}\eta(t,T) + (T-t)^3\lambda_2^2(1-\delta_i)^2/6}, \end{aligned} \tag{25}$$

where no default has occurred at or prior to time t,

$$\mu_1(t,T) = \int_t^T f(t,u)du + \int_t^T \frac{b(u,T)^2 du}{2}, \tag{26a}$$

$$\sigma_1^2(t,T) = \int_t^T b(u,T)^2 du, \tag{26b}$$

$$b(u,t) = \frac{\sigma_r[1 - e^{-a(t-u)}]}{a}, \tag{26c}$$

$$\begin{aligned} \eta(t,T) &= \int_t^T \left\{\int_t^T \left[\int_t^{\min(s,u)} \rho(v,s)dv\right] ds\right\} du \\ &= -\left(\frac{\sigma_r}{a^3}\right)[1 - e^{-a(T-t)}] + \left(\frac{\sigma_r}{a^2}\right) \\ &\times e^{-a(T-t)}(T-t) + \left(\frac{\sigma_r}{2a}\right)(T-t)^2, \end{aligned} \tag{26d}$$

and

$$\rho(v,s) = \sigma_r e^{-a(s-v)}. \tag{26e}$$

To understand these pricing formulas, one must first recognize that the randomness in their values across time occurs for two reasons. The first cause is randomly changing default-free rates. This cause enters through the $\mu_1(t,T)$ term in Equations 24 and 25 and, in particular, through the term involving the current forward rates, $\int_t^T f(t,u)du$. The second reason, which applies only to the risky debt, is the possibility of default, in which case the risky-bond price in Equation 25 drops from $v_l(\tau-,T:i)$ to $v_l(\tau,T:i) = \delta_i v_l(\tau-,T:i)$.

For a better understanding of the randomness arising from changing default-free rates, one can transform these equations to the equation in Jarrow and Turnbull (2000) as follows. Appendix A shows that

$$\mu_1(t,T) = c(t,T) + \frac{b(t,T)r(t)}{\sigma_r}, \tag{27a}$$

where

$$c(t,T) = \int_t^T \left[f(0,u) + \frac{b(0,u)^2}{2} \right] \times du - \frac{b(t,T)\{f(0,t) + [b(0,t)^2/2]\}}{\sigma_r}. \tag{27b}$$

In Equations 27, $c(t,T)$ and $b(t,T)$ are deterministic functions of time. The randomness in Equation 27a is a result of spot interest rate $r(t)$. Substituting Equations 27a and 27b into pricing Equation 24 gives the valuation formula in Jarrow and Turnbull (2000). This substitution also shows that the prices of default-free and risky zero-coupon bonds are Markov in $r(t)$. This Markov structure facilitates computation, and it is an advantage of using the extended Vasicek model.

For understanding the implications of Equation 25 for the yield spread between prices of risky bonds and Treasury prices, the first step is to implicitly define the yield spread at time t for bond T with a particular maturity, $\chi(t,T:i)$, as

$$\frac{v_I(t,T:i)}{p(t,T)} = e^{-\chi(t,T:i)(T-t)}. \tag{28}$$

Using Equation 27a for $\mu_1(t,T)$ and the definition of the yield spread to Treasuries given in Equation 28,

$$\begin{aligned} \chi(t,T:i)(T-t) = {} & \gamma_i(t,T) + \lambda_0(1-\delta_i)(T-t) \\ & + \lambda_1(1-\delta_i)c(t,T) \\ & - \frac{[2\lambda_1(1-\delta_i) + \lambda_1^2(1-\delta_i)^2]\sigma_1^2(t,T)}{2} \\ & + \frac{\lambda_1(1-\delta_i)b(t,T)r(t)}{\sigma_r} \\ & + \lambda_2(1-\delta_i)Z(t)(T-t) \\ & - [1+\lambda_1(1-\delta_i)]\lambda_2(1-\delta_i)\varphi_{rm}\eta(t,T) \\ & - \frac{(T-t)^3\lambda_2^2(1-\delta_i)^2}{6}. \end{aligned} \tag{29}$$

The yield spread consists of a term denoting liquidity risk, $\gamma_i(t,T)$, and all the remaining terms denoting credit risk. The yield spread is random because the liquidity-risk component is random and the credit-risk component contains the spot rate of interest, $r(t)$, and the cumulative excess return on the market index, $Z(t)$, both of which are random.

To complete the empirical formulation, specification is needed of the functional form for the liquidity discount, the bubble component, and the liquidating dividend as given in Equation 14. This is the task to which we now turn.

Assume that

$$L(T^*) = L(t)e^{\int_t^{T^*} r(u)du - (1/2)\int_t^{T^*}\sigma_L^2 du + \int_t^{T^*}\sigma_L dw_L(u)}, \tag{30}$$

where $\sigma_L > 0$ is a constant and $w_L(t)$ is a Brownian motion under the martingale measure Q with $dZ(t)dw_L(t) = \varphi_{mL}dt$ and $dW(t)dw_L(t) = \varphi_{rL}dt$, where φ_{rL} and φ_{mL} are constants.

In Equation 30, $L(t)$ represents the time t liquidation value of the company's assets less liabilities. This liquidation value can be viewed as the market value of a portfolio containing the company's assets and liabilities. This portfolio's value, if held by a default-free entity, evolves through time according to Equation 30. For simplicity, this evolution is assumed to be a geometric Brownian motion under the martingale measure with a drift rate equal to the spot rate, $r(u)$. The value of this portfolio at time T^* will be $L(T^*)$.

In the present case, however, $L(t)$ is held by a company that can default prior to time T^*. If the company defaults, then because of bankruptcy costs (lawyer's fees, lost sales, etc.), the liquidation value declines by the fraction $(1 - \delta_e)$. The implication is that the present value of the liquidation value to an equity holder in the risky company is less than or equal to the present value of the underlying portfolio of assets and liabilities to a default-free agent; that is

$$\begin{aligned} S(t)1_{(\tau > t)} &= E_t\left\{ L(T^*)e^{\int_t^{T^*}[r(u)+\lambda(u)]du} \right\} \\ &\le E_t\left[L(T^*)e^{\int_t^{T^*} r(u)du} \right] \\ &= L(t). \end{aligned} \tag{31}$$

Here, it can be shown (the proof is in Appendix A) that

$$\begin{aligned} S(t) = {} & \frac{L(t)}{p(t,T^*)} \\ & \times e^{-\lambda_1\sigma_1^2(t,T^*) - \lambda_1\sigma_L\varphi_{rL}\int_t^{T^*} b(u,T^*)du - \lambda_2\varphi_{rm}\eta(t,T^*) - \lambda_2\sigma_L\varphi_{mL}(T^*-t)^2/2} \\ & \times v(t,T^*:e). \end{aligned} \tag{32}$$

Unfortunately, this expression for the present value of the liquidating dividend has the unknown $L(t)$ on the right side. Therefore, this form of the present value expression can provide no additional inference about the default parameter process, $\lambda(t)$, because there are more unknowns than observables. This insight motivates the following transformation of Equation 15b.

Let Δ correspond to a discrete change in time. Taking logarithms of Equation 32 and subtracting time $t - \Delta$ from time t gives

$$\log\left[\frac{\xi(t)-\sum_{j\geq t}^{T^*}D_j v(t,j:e)}{\xi(t-\Delta)-\sum_{j\geq t-\Delta}^{T^*}D_j v(t-\Delta,j:e)}\right]$$

$$= \log\left[\frac{L(t)}{L(t-\Delta)}\right] + \int_{t-\Delta}^{t}\mu_\theta(u)du + \log\left[\frac{\psi(t,T^*)}{\psi(t-\Delta,T^*)}\right] + \log\left[\frac{v(t,T^*:e)}{v(t-\Delta,T^*:e)}\right], \quad (33)$$

where

$$\psi(t,T^*) = -\lambda_1\sigma_1^2(t,T^*)-\lambda_1\sigma_L\varphi_{rL}\int_t^{T^*}b(u,T^*)du - \lambda_2\varphi_{rm}\eta(t,T^*)-\frac{\lambda_2\sigma_L\varphi_{mL}(T^*-t)^2}{2}.$$

Next, using the evolution of the liquidation value as given in Equation 30 produces

$$\log\left[\frac{L(t)}{L(t-\Delta)}\right] = \int_{t-\Delta}^{t} r(u)du-\left(\frac{1}{2}\right)\sigma_L^2\Delta + \sigma_L[w_L(t)-w_L(t-\Delta)]. \quad (34)$$

This evolution is under the martingale measure.

Using Girsanov's theorem, we can change the martingale measure to the statistical probability measure. The change transforms the original Brownian motion to a new Brownian motion under the statistical measure and an adjustment for a risk premium,

$$w_L(t) = \hat{w}_L(t) + \int_0^t\Theta_L(u)du, \quad (35)$$

where $\hat{w}_L(t)$ is a Brownian motion under the statistical measure $\hat{Q}$ and $\Theta_L(u)$ is the liquidation value's risk premium.

Using this change of measure produces

$$\log\left[\frac{L(t)}{L(t-\Delta)}\right] = \int_{t-\Delta}^{t}[r(u)+\sigma_L\Theta_L(u)]\,du -\left(\frac{1}{2}\right)\sigma_L^2\Delta+\varepsilon(t-\Delta), \quad (36)$$

where the error terms, $\varepsilon(t-\Delta)\equiv\sigma_L[\hat{w}_L(t) - \hat{w}_L(t-\Delta)]$, for all t are independent identically and normally distributed terms with zero mean and variance of $\sigma_L^2\Delta$.

Substitution of Equation 35 into Equation 33 yields

$$\log\left[\frac{\xi(t)-\sum_{j\geq t}^{T^*}D_j v(t,j:e)}{\xi(t-\Delta)-\sum_{j\geq t-\Delta}^{T^*}D_j v(t-\Delta,j:e)}\right]-\int_{t-\Delta}^{t}r(u)du$$

$$= \int_{t-\Delta}^{t}\left[\sigma_L\Theta_L(u)-\left(\frac{1}{2}\right)\sigma_L^2+\mu_\theta(u)\right]du + \log\left[\frac{\psi(t,T^*)}{\psi(t-\Delta,T^*)}\right] + \log\left[\frac{v(t,T^*:e)}{v(t-\Delta,T^*:e)}\right] + \varepsilon(t-\Delta). \quad (37)$$

For estimation purposes, the excess return on equity can be written as

$$\log\left\{\frac{\xi(t)-\sum_{j\geq t}^{T^*}D_j v(t,j:e)}{\xi(t-\Delta)-\sum_{j\geq t-\Delta}^{T^*}D_j v(t-\Delta,j:e)}\right\}-r(t-\Delta)\Delta$$

$$\approx -\lambda_0\Delta-\lambda_1\left\{\left[\frac{b(t-\Delta,T^*)^2}{2}\right]\Delta-\log\left[\frac{p(t,T^*)}{p(t-\Delta,T^*)}\right]\right\}$$
$$+\lambda_1^2\left[\frac{b(t-\Delta,T^*)^2}{2}\right]\Delta-\lambda_1\sigma_L\varphi_{rL}b(t-\Delta,T^*)\Delta$$
$$-\lambda_2[Z(t)(T^*-t)-Z(t-\Delta)(T^*-t-\Delta)] \quad (38)$$
$$+\frac{\lambda_2^2(T^*-t)^2\Delta}{3}-\lambda_2\sigma_L\varphi_{mL}(T^*-t)\Delta$$
$$+\left\{\sigma_L\gamma_L[t-\Delta,X(t-\Delta)]-\left(\frac{1}{2}\right)\sigma_L^2+\mu_\theta[t-\Delta,X(t-\Delta)]\right\}\Delta$$
$$+\varepsilon(t-\Delta).$$

(The proof is in Appendix A.)

In this nonlinear regression equation for the excess return on the equity, the coefficients give the default parameters. To complete this estimation, a model for the risk premium and the bubble component is required. For example, if one assumes that the risk premium can be approximated by using a capital asset pricing model and that the bubble component can be approximated with the variance of the stock price, then

$$\left\{\sigma_L\gamma_L[t-\Delta,X(t-\Delta)]-\left(\frac{1}{2}\right)\sigma_L^2+\mu_\theta[t-\Delta,X(t-\Delta)]\right\}\Delta = \beta_0\log\left[\frac{M(t)}{M(t-\Delta)}\right]+\beta_1\log\left[\frac{\xi^2(t)}{\xi^2(t-\Delta)}\right], \quad (39)$$

and the system is easily estimated with only two extra parameters, β_0 and β_1, which denote, respectively, the systematic risk of the market portfolio and the risk premium for the bubble component.

Finally, one can model the liquidity discount as a first-order Taylor series approximation for a more general function of the state variables:

$$\gamma_i(t,T) = \gamma_0^i + \gamma_1^i r(t) + \gamma_2^i Z(t). \tag{40}$$

Combining all of these empirical specifications into Equations 15a and 38 gives the following system of equations, which contains both cross-sectional and time-series observations:

$$\begin{aligned} &v_l(t, T:i) \\ &\quad = e^{-\gamma_0^i + \gamma_1^i r(t) + \gamma_2^i Z(t)} p(t, T) \\ &\quad \times e^{-\lambda_0(1-\delta_i)(T-t) - \lambda_1(1-\delta_i)\mu_1(t,T) + \left[2\lambda_1(1-\delta_i) + \lambda_1^2(1-\delta_i)^2\right]\sigma_1^2(t,T)/2} \\ &\quad \times e^{-\lambda_2(1-\delta_i)Z(t)(T-t) + [1+\lambda_1(1-\delta_i)]\lambda_2(1-\delta_i)\varphi_{rm}\eta(t,T) + (T-t)^3\lambda_2^2(1-\delta_i)^2/6} \end{aligned} \tag{41a}$$

for various i, T, and t and

$$\begin{aligned} &\log\left\{\frac{\left[\xi(t) - \sum_{j \geq t}^{T^*} D_j v(t, j:e)\right]}{\left[\xi(t-\Delta) - \sum_{j \geq t-\Delta}^{T^*} D_j v(t-\Delta, j:e)\right]}\right\} - r(t-\Delta)\Delta \\ &\quad \approx -\lambda_0\Delta - \lambda_1\left\{\left[\frac{b(t-\Delta, T^*)^2}{2}\right]\Delta - \log\left[\frac{p(t, T^*)}{p(t-\Delta, T^*)}\right]\right\} \\ &\quad + \lambda_1^2\left[\frac{b(t-\Delta, T^*)^2}{2}\right]\Delta - \lambda_1\sigma_L\varphi_{rL}b(t-\Delta, T^*)\Delta \\ &\quad -\lambda_2[Z(t)(T^*-t) - Z(t-\Delta)(T^*-t-\Delta)] \\ &\quad + \lambda_2^2\frac{(T^*-t)^2\Delta}{3} - \lambda_2\sigma_L\varphi_{mL}(T^*-t)\Delta \\ &\quad + \beta_0\log\left[\frac{M(t)}{M(t-\Delta)}\right] + \beta_1\log\left[\frac{\xi^2(t)}{\xi^2(t-\Delta)}\right] \\ &\quad + \varepsilon(t-\Delta) \end{aligned} \tag{41b}$$

for various t.

Equation 41a is the debt-pricing equation, whereas Equation 41b is the equity-pricing equation. The solution to this system of equations can be obtained via a nonlinear regression. The solution depends on the initial forward rate curve, $f(0, T)$; the term-structure evolution parameters, a and σ_r; the market index parameters, φ_{rm} and σ_m; and the liquidating dividend parameters, φ_{rL}, φ_{mL}, and σ_L. Additional parameters to be estimated are the default process coefficients, λ_0, λ_1, and λ_2; the recovery rate, δ_i; the liquidity discount coefficients, γ_0^i, γ_1^i, and γ_2^i; and the bubble/risk premium coefficients, β_0 and β_1. This system of equations needs to be estimated through the use of both cross-sectional and time-series data.

Conclusion

I presented a new procedure for implicit estimation of a liquidity premium, the recovery rate, and the (pseudo) default probabilities using debt and equity prices. This new procedure is quite general. It allows the default process to be correlated across companies and to depend on the state of the macroeconomy. It allows debt markets to be illiquid and equity markets to contain bubbles. Its empirical evaluation, however, awaits subsequent research.

Although the procedure formally estimates the pseudo or risk-neutral default intensities, if the reduced-form model is properly specified, there is good reason to believe that the statistical and risk-neutral default intensity functions are equal—which they will be if default risk is idiosyncratic after properly conditioning on macroeconomic variables (see Jarrow, Lando, and Yu 1999). This hypothesis of idiosyncratic default risk is intuitively plausible; its validation awaits subsequent research. The methodology presented here is consistent with this properly conditioned reduced-form model.

Appendix A. Proofs

In this appendix, I provide the derivations of Equations 24, 25, 27a, and 27b and lay out the computation of the equity model, Equation 38.

Derivation of Equations 24 and 25. From Equation 21,

$$r(s) = f(t,s) + \frac{b(t,s)^2}{2} + \int_t^s \rho(v,s)\,dW(v), \tag{A1}$$

where

$$\rho(v,s) = \sigma_r e^{-a(s-v)}$$

and

$$b(t,s) = \int_t^s \rho(t,v)dv = \sigma_r \frac{1-e^{-a(s-t)}}{a}.$$

Define

$$X_1 \equiv \int_t^T r(s)\,ds = \int_t^T f(t,s)\,ds + \int_t^T \frac{b(t,s)^2 ds}{2} + \int_t^T\int_t^s \rho(v,s)\,dW(v)\,ds. \tag{A2}$$

After changing the order of integration, a direct computation yields

$$\int_t^T \frac{b(t,s)^2 ds}{2} = \int_t^T \frac{b(v,T)^2 dv}{2} \tag{A3}$$

and

$$\int_t^T\int_t^s \rho(v,s)\,dW(v)\,ds = \int_t^T b(v,T)\,dW(v). \tag{A4}$$

Substitution gives

$$\int_t^T r(s)ds = \int_t^T f(t,s)ds + \int_t^T \frac{b(v,T)^2 dv}{2} + \int_t^T b(v,T)\,dW(v). \tag{A5}$$

A direct computation gives

$$\mu_1(t,T) \equiv E_t\left[\int_t^T r(s)\,ds\right] = \int_t^T f(t,s)\,ds + \int_t^T \frac{b(v,T)^2 dv}{2} \tag{A6}$$

and

$$\begin{aligned}\sigma_1^2(t, T) &\equiv \operatorname{var}_t\left[\int_t^T r(s)ds\right] \\ &= \int_t^T b(v, T)^2 dv.\end{aligned} \tag{A7}$$

Define $X_2 \equiv \int_t^T Z(u)du$. Following Parzen (1962, p. 81),

$$\begin{aligned}\mu_2(t, T) \equiv E_t(X_2) &= \int_t^T E_t[Z(u)]\,du \\ &= \int_t^T Z(t)\,du \\ &= Z(t)(T-t);\end{aligned}$$

$$\begin{aligned}\sigma_2^2(t, T) \equiv \operatorname{var}_t(X_2) &= 2\int_t^T\int_t^v (u-t)du\,dv \\ &= \frac{(T-t)^3}{3};\end{aligned} \tag{A8}$$

$$\begin{aligned}\sigma_{12}(t, T) \equiv \operatorname{cov}_t(X_1, X_2) &= E_t(X_1X_2) - E_t(X_1)E_t(X_2) \\ &= E_t\left[\int_t^T r(s)\,ds\int_t^T Z(s)\,ds\right] - \left[\int_t^T f(t, s)\,ds + \int_t^T \frac{b(v, T)^2 dv}{2}\right]Z(t)(T-t) \\ &= \int_t^T\int_t^T E_t[r(s)Z(u)]dsdu - \left[\int_t^T f(t, s)ds + \int_t^T \frac{b(v, T)^2 dv}{2}\right]Z(t)[T-t].\end{aligned}$$

But

$$\begin{aligned}E_t[r(s)Z(u)] &= E_t\left\{\left[f(t, s) + \frac{b(t, s)^2}{2} + \int_t^s \rho(v, s)dW(v)\right]\left[\int_t^u dz(v) + Z(t)\right]\right\} \\ &= \varphi_{rm}\int_0^{\min(s, u)} \rho(v, s)\,dv + \left[f(t, s) + \frac{b(t, s)^2}{2}\right]Z(t).\end{aligned} \tag{A9}$$

Now,

$$\begin{aligned}\int_t^T\int_t^T\left[f(t, s) + \frac{b(t, s)^2}{2}\right]Z(t)\,duds &= Z(t)(T-t)\int_t^T\left[f(t, s) + \frac{b(t, s)^2}{2}\right]ds \\ &= Z(t)(T-t)\left[\int_t^T f(t, s)ds + \int_t^T \frac{b(v, T)^2 dv}{2}\right].\end{aligned} \tag{A10}$$

Substitution and simplification yield

$$\sigma_{12}(t, T) = \varphi_{rm}\left(\int_t^T\left\{\int_t^T\left[\int_t^{\min(s, u)} \rho(v, s)dv\right]ds\right\}du\right). \tag{A11}$$

Given that (X_1, X_2) is bivariate normal, we have (see Hogg and Craig 1970)

$$E_t\left(e^{AX_1 + BX_2}\right) = e^{\mu_1 A + \mu_2 B + (\sigma_1^2 A^2 + 2\sigma_{12} AB + \sigma_2^2 B^2)/2} \tag{A12}$$

where A and B are arbitrary constants.

$$\mu_1 \equiv E_t(X_1),$$

$$\mu_2 \equiv E_t(X_2),$$

$$\sigma_1^2 \equiv \text{var}_t(X_1),$$

$$\sigma_2^2 \equiv \text{var}_t(X_2),$$

and

$$\sigma_{12} \equiv \text{cov}_t(X_1, X_2).$$

Then, for Equation 24, using Equation A12, we get

$$\begin{aligned} E_t\left\{e^{-[1+\lambda_1(1-\delta_i)]\int_t^T r(u)du}\right\} &= E_t\left(e^{AX_1}\right) \\ &= e^{\mu_1 A + \sigma_1^2 A^2/2}, \end{aligned} \tag{A13}$$

where

$$A = -[1 + \lambda_1(1 - \delta_i)].$$

This gives the desired result.

Next, for Equation 25,

$$E_t\left\{e^{-[1+\lambda_1(1-\delta_i)]\int_t^T r(u)du - \lambda_2(1-\delta_i)\int_t^T Z(u)du}\right\} = E_t\left(e^{AX_1 + BX_2}\right), \tag{A14}$$

where

$$A = -[1 + \lambda_1(1 - \delta_i)]$$

and

$$B = -\lambda_2(1 - \delta_i).$$

Equation A12 gives the desired result.

Derivation of Equation 27a and 27b. Under the spot rate model of Equation 18, it can be shown that the arbitrage-free forward-rate process is given by

$$f(t, u) = f(0, u) + \int_0^t \alpha(v, u)dv + \int_0^t \rho(v, u)dW(v), \tag{A15}$$

where

$$\alpha(v, u) = \rho(v, u)\int_v^u \rho(v, s)ds.$$

We are interested in evaluating the following integral of forward rates:

$$\begin{aligned} \int_t^T f(t, u)\,du &= \int_t^T f(0, u)\,du + \int_t^T \left[\int_0^t \alpha(v, u)\,dv\right]du \\ &\quad + \int_t^T \left[\int_0^t \rho(v, u)dW(v)\right]du. \end{aligned} \tag{A16}$$

Given the definitions of $\alpha(v,u)$, $\rho(v,u)$, and $b(v,u)$, the following facts can be proven by direct computation:

$$\int_t^u \alpha(v,u)dv = \frac{b(t,u)^2}{2} \tag{A17}$$

and

$$\int_t^T \left[\int_0^t \rho(v,u)\,dW(v)\right]du = b(t,T)\int_0^t \rho(v,t)\,dW(v). \tag{A18}$$

Using the first of these facts, we can show that

$$\begin{aligned}\int_t^T \left[\int_0^t \alpha(v,u)dv\right]du &= \int_t^T \left[\int_0^u \alpha(v,u)dv\right]du - \int_t^T \left[\int_t^u \alpha(v,u)dv\right]du \\ &= \int_t^T \left[\frac{b(0,u)^2}{2}\right]du - \int_t^T \left[\frac{b(t,u)^2}{2}\right]du.\end{aligned} \tag{A19}$$

But we know from Equation 18 that

$$r(t) - f(0,t) - \frac{b(0,t)^2}{2} = \int_0^t \rho(v,t)\,dW(v). \tag{A20}$$

Using this observation and Equation A18, we have

$$\int_t^T \left[\int_0^t \rho(v,u)dW(v)\right]du = \frac{b(t,T)[r(t) - f(0,t) - b(0,t)^2/2]}{\sigma_r}. \tag{A21}$$

Direct substitution of these observations produces

$$\begin{aligned}\int_t^T f(t,u)du &= \int_t^T f(0,u)du + \int_t^T \left[\frac{b(0,u)^2}{2}\right]du - \int_t^T \left[\frac{b(t,u)^2}{2}\right]du \\ &\quad + \frac{b(t,T)[r(t) - f(0,t) - b(0,t)^2/2]}{\sigma_r}.\end{aligned} \tag{A22}$$

Substitution of this integral into the definition of $\mu_1(t,T)$, together with the fact that $\int_t^T b(t,u)^2 du/2 = \int_t^T b(u,T)^2 du/2$, gives Equations 27a and 27b.

Equity Model Computations. From Equations 9 and 30, we have

$$S_t = E_t\left\{L(T^*)e^{-\int_t^{T^*}[r(u)+\lambda(u)]du}\right\} = L(t)e^{-(1/2)\int_t^{T^*}\sigma_L^2 du}E_t\left[e^{-\int_t^{T^*}\lambda(u)du + \int_t^{T^*}\sigma_L dw_L(u)}\right]. \tag{A23}$$

Using Equation 23 gives

$$S_t = L(t)e^{-(1/2)\int_t^{T^*}\sigma_L^2 du - \int_t^{T^*}\lambda_0 du}E_t\left[e^{-\int_t^{T^*}\lambda_1 r(u)du - \int_t^{T^*}\lambda_2 Z(u)du + \int_t^{T^*}\sigma_L dw_L(u)}\right]. \tag{A24}$$

To evaluate the expectation, we use Equation A12 with the following identifications:

$$A \equiv 1,$$

$$x \equiv -\lambda_1\int_t^{T^*} r(u)du - \lambda_2\int_t^{T^*} Z(u)du = \lambda_1 X_1 - \lambda_2 X_2,$$

$$B \equiv 1,$$

$$y \equiv \int_t^{T^*} \sigma_L dw_L(u).$$

The expectation is $e^{\mu_x + \mu_y + (1/2)\sigma_x^2 + \sigma_{xy} + (1/2)\sigma_y^2}$, where

$$\mu_x = -\lambda_1\mu_1(t, T^*) - \lambda_2 Z(t)(T^* - t),$$

$$\mu_y = 0,$$

$$\sigma_x^2 = \lambda_1^2\sigma_1^2(t, T^*) + 2\lambda_1\lambda_2\sigma_{12}(t, T^*) + \frac{\lambda_2^2(T^* - t)^3}{3},$$

$$\sigma_y^2 = \sigma_L^2(T^* - t),$$

and

$$\sigma_{xy} = \text{cov}_t\left[-\lambda_1\int_t^{T^*} r(u)du - \lambda_2\int_t^{T^*} Z(u)du, \int_t^{T^*}\sigma_L dw_L(u)\right]$$

$$= -\lambda_1\sigma_L\text{cov}_t\left[\int_t^{T^*} r(u)du, \int_t^{T^*} dw_L(u)\right] - \lambda_2\sigma_L\text{cov}_t\left[\int_t^{T^*} Z(u)du, \int_t^{T^*} dw_L(u)\right].$$

But

$$\text{cov}_t\left[\int_t^{T^*} r(u)du, \int_t^{T^*} dw_L(u)\right] = \text{cov}_t\left[\int_t^{T^*} b(u, T^*)dW(u), \int_t^{T^*} dw_L(u)\right]$$

$$= \varphi_{rL}\int_t^{T^*} b(u, T^*)du. \quad \text{(A25)}$$

Next,

$$\text{cov}_t\left[\int_t^{T^*} Z(u)du, \int_t^{T^*} dw_L(u)\right] = E_t\left[\int_t^{T^*} Z(u)du \int_t^{T^*} dw_L(u)\right], \quad \text{(A26)}$$

because $E_t\left[\int_t^{T^*} dw_L(u)\right] = 0$. But

$$E_t\left[\int_t^{T^*} Z(u)du \int_t^{T^*} dw_L(u)\right] = E_t\left\{\int_t^{T^*}[Z-(u) - Z(t) + Z(t)]du \int_t^{T^*} dw_L(u)\right\}$$

$$= E_t\left\{\int_t^{T^*} E_u\left[\int_t^{u} dZ(v)\int_t^{T^*} dw_L(v)\right]du\right\} \quad \text{(A27)}$$

$$= E_t\left\{\int_t^{T^*}\left[\int_t^{u} dZ(v)\int_t^{u} dw_L(v)\right]du\right\},$$

because $E_u\left[\int_u^{T^*} dw_L(u)\right] = 0$.

Finally,

$$E_t\left\{\int_t^{T^*}\left[\int_t^u dZ(v)\int_t^u dw_L(v)\right]du\right\} = \left\{\int_t^{T^*} E_t\left[\int_t^u dZ(v)\int_t^u dw_L(v)\right]du\right\}$$

$$= \int_t^{T^*}\varphi_{mL}(u-t)\,du. \tag{A28}$$

Thus,

$$\text{cov}_t\left[\int_t^{T^*} Z(u)du, \int_t^{T^*} dw_L(u)\right] = \frac{\varphi_{mL}(T^*-t)^2}{2}. \tag{A29}$$

Substitution of these results into the expression for $S(t)$ gives

$$S_t = L(t)e^{-\lambda_0(T^*-t)-\lambda_1\mu_1(t,T^*)-\lambda_2 Z(t)(T^*-t)+(1/2)\lambda_1^2\sigma_1^2(t,T^*)+\lambda_1\lambda_2\sigma_{12}(t,T^*)+\lambda_2^2(T^*-t)^3/6}$$

$$\times e^{-\lambda_1\sigma_L\varphi_{rL}\int_t^{T^*} b(u,T^*)du-\lambda_2\sigma_L\varphi_{mL}(T^*-t)^2/2+(1/2)\sigma_L^2(T^*-t)}. \tag{A30}$$

From Equation 25, we have

$$\frac{v(t,T^*:E)}{p(t,T^*)}e^{-\lambda_1\sigma_1^2(t,T^*)-\lambda_2\sigma_{12}(t,T^*)} = e^{-\lambda_0(T^*-t)-\lambda_1\mu_1(t,T^*)-\lambda_2 Z(t)(T^*-t)+(1/2)\lambda_1^2\sigma_1^2(t,T^*)+\lambda_1\lambda_2\sigma_{12}(t,T^*)+\lambda_2^2(T^*-t)^3/6}. \tag{A31}$$

Substitution gives

$$S_t = L(t)\frac{v(t,T^*:E)}{p(t,T^*)}e^{-\lambda_1\sigma_1^2(t,T^*)-\lambda_2\sigma_{12}(t,T^*)-\lambda_1\sigma_L\varphi_{rL}\int_t^{T^*} b(u,T^*)du-\lambda_2\sigma_L\varphi_{mL}(T^*-t)^2/2}. \tag{A32}$$

Next, we derive the expression for the excess return on equity. From Equation 12 and Equation A30, if we take natural logarithms and then the difference from time $t-\Delta$ to time t, we obtain

$$\begin{aligned}\log\left[\frac{\xi_t-\sum_{j\geq t}^{T^*}D_j v(t,j:e)}{\xi_{t-\Delta}-\sum_{j\geq t-\Delta}^{T^*}D_j v(t-\Delta,j:e)}\right] &= \log\left(\frac{L_t}{L_{t-\Delta}}\right)+\int_{t-\Delta}^{t}\mu_\theta(u)du-\lambda_0[(T^*-t)-(T^*-t-\Delta)]\\ &-\lambda_1[\mu_1(t,T^*)-\mu_1(t-\Delta,T^*)]-\lambda_2[Z(t)(T^*-t)-Z(t-\Delta)(T^*-t-\Delta)]\\ &+\left(\frac{1}{2}\right)\lambda_1^2[\sigma_1^2(t,T^*)-\sigma_1^2(t-\Delta,T^*)]+\lambda_1\lambda_2[\sigma_{12}(t,T^*)-\sigma_{12}(t-\Delta,T*)]\\ &+\lambda_2^2\left[\frac{(T^*-t)^3-(T^*-t-\Delta)^3}{6}\right]-\lambda_1\sigma_L\varphi_{rL}\left[\int_t^{T^*}b(u,T^*)du-\int_{t-\Delta}^{T^*}b(u,T^*)du\right]\\ &-\lambda_2\sigma_L\varphi_{mL}\left[\frac{(T^*-t)^2-(T^*-t-\Delta)^2}{2}\right].\end{aligned} \tag{A33}$$

In the following identifications, terms of order Δ^p for $p\geq 2$ are omitted:

$$\lambda_0[(T^*-t)-(T^*-t-\Delta)] = \lambda_0\Delta. \tag{A34}$$

Next, we have

$$\lambda_1[\mu_1(t,T^*)-\mu_1(t-\Delta,T^*)] = \lambda_1\left\{-\log\left[\frac{p(t,T^*)}{p(t-\Delta,T^*)}\right]+\int_{t-\Delta}^{t}\frac{b(v,T^*)^2 dv}{2}\right\}$$

$$\approx \lambda_1\left\{-\log\left[\frac{p(t,T^*)}{p(t-\Delta,T^*)}\right]+\frac{b(t-\Delta,T^*)^2\Delta}{2}\right\}; \tag{A35}$$

$$\left(\frac{1}{2}\right)\lambda_1^2[\sigma_1^2(t,T^*)-\sigma_1^2(t-\Delta,T^*)] = \left(\frac{1}{2}\right)\lambda_1^2\int_{t-\Delta}^{t} b(v,T^*)^2 dv$$

$$\approx \frac{\lambda_1^2 b(t-\Delta,T^*)^2\Delta}{2}; \tag{A36}$$

$$\lambda_1\lambda_2[\sigma_{12}(t,T^*)-\sigma_{12}(t-\Delta,T^*)] = \lambda_1\lambda_2\varphi_{rm}\left[\int_t^T\int_t^T\int_t^{\min(s,u)}\rho(v,s)\,dvdsdu-\int_{t-\Delta}^{T}\int_{t-\Delta}^{T}\int_{t-\Delta}^{\min(s,u)}\rho(v,s)\,dvdsdu\right]$$

$$\le \lambda_1\lambda_2\varphi_{rm}\left[\int_t^T\int_t^T\int_{t-\Delta}^{\min(s,u)}\rho(v,s)\,dvdsdu-\int_{t-\Delta}^{T}\int_{t-\Delta}^{T}\int_{t-\Delta}^{\min(s,u)}\rho(v,s)\,dvdsdu\right] \tag{A37}$$

$$= \lambda_1\lambda_2\varphi_{rm}\int_{t-\Delta}^{t}\int_{t-\Delta}^{t}\int_{t-\Delta}^{\min(s,u)}\rho(v,s)\,dvdsdu \le \lambda_1\lambda_2\varphi_{rm}\int_{t-\Delta}^{t}\int_{t-\Delta}^{t}\int_{t-\Delta}^{t}\rho(v,s)\,dvdsdu$$

$$\approx \lambda_1\lambda_2\varphi_{rm}\rho(t-\Delta,t)\Delta^3;$$

$$\lambda_2^2\left[\frac{(T^*-t)^3-(T^*-t-\Delta)^3}{6}\right] = \lambda_2^2\left[\frac{2(T^*-t)^2\Delta-(T^*-t)^2\Delta^2-\Delta^3}{6}\right]$$

$$\approx \lambda_2^2\left[\frac{(T^*-t)^2\Delta}{3}\right]; \tag{A38}$$

$$\lambda_1\sigma_L\varphi_{rL}\left[\int_t^{T^*} b(u,T^*)\,du-\int_{t-\Delta}^{T^*} b(u,T^*)\,du\right] = \lambda_1\sigma_L\varphi_{rL}\int_{t-\Delta}^{t} b(u,T^*)\,du$$

$$\approx \lambda_1\sigma_L\varphi_{rL}\,b(t-\Delta,T^*)\Delta; \tag{A39}$$

$$\lambda_2\sigma_L\varphi_{mL}\left[\frac{(T^*-t)^2-(T^*-t-\Delta)^2}{2}\right] = \lambda_2\sigma_L\varphi_{mL}\left[\frac{2(T^*-t)\Delta-\Delta^2}{2}\right]$$

$$\approx \lambda_2\sigma_L\varphi_{mL}(T^*-t)\Delta. \tag{A40}$$

Combined, these identifications produce

$$\log\left[\frac{\xi_t-\sum_{j\ge t}^{T^*}D_j v(t,j:e)}{\xi_{t-\Delta}-\sum_{j\ge t-\Delta}^{T^*}D_j v(t-\Delta,j:e)}\right] = \log\left(\frac{L_t}{L_{t-\Delta}}\right)+\int_{t-\Delta}^{t}\mu_\theta(u)du-\lambda_0\Delta$$

$$-\lambda_1\left\{-\log\left[\frac{p(t,T^*)}{p(t-\Delta,T^*)}\right]+b(t-\Delta,T^*)^2\frac{\Delta}{2}\right\}$$

$$-\lambda_2[Z(t)(T^*-t)-Z(t-\Delta)(T^*-t-\Delta)] \tag{A41}$$

$$+\lambda_1^2 b(t-\Delta,T^*)^2\frac{\Delta}{2}+\lambda_2^2(T^*-t)^2\frac{\Delta}{3}-\lambda_1\sigma_L\varphi_{rL}\,b(t-\Delta,T^*)\Delta$$

$$-\lambda_2\sigma_L\varphi_{mL}(T^*-t)\Delta.$$

which completes the derivation of Equation 38.

Notes

1. See Jarrow and Turnbull (2000) for a review.
2. The intensity process is defined under the risk-neutral probability. This statement will become clear after the next section, "Risk-Neutral Valuation."
3. A convenient approach is to think of the liquidating payoff as the present value of all future dividends paid over the time period (T^*, ∞).
4. In fact, as the subsequent analysis will show, what is really being assumed here is that $\delta_e(\tau)$ is the minimal recovery rate. Under this interpretation, all the subsequent recovery rates will be relative to $\delta_e(\tau)$.
5. As will be seen later, if the future dividends are random, they are included within the $S(t)$ component—that is, the time t present value of the liquidating dividend conditional upon no default prior to time t.
6. For example, *Money Magazine* in April 1999 gave Yahoo's P/E as 1,176.6 (p. 169).
7. Equation 3 is a simple no-arbitrage restriction that the present value of the sum of multiple cash flows equal the sum of the present values of the cash flows.
8. See Jarrow and Turnbull (1995). "No arbitrage" guarantees the existence but not the uniqueness of a probability measure Q. Without any additional hypotheses about the economy, the uniqueness of Q is equivalent to markets being complete (see Battig and Jarrow 1999). In incomplete markets, equilibrium (requiring additional hypotheses) guarantees the uniqueness of Q. The uniqueness of Q is essential for estimation.
9. This insight implies that the techniques for inferring the asset's volatility, by using Merton's model of risky debt, are misspecified in the presence of bubbles.
10. The mild condition is that the value of the debt and equity not jump at the time of default. Given the fractional recovery rate process, this assumption is reasonable.
11. In the event that $\delta_e(\tau)$ is not zero, the estimated fractional loss for equity will be the ratio of $[1 - \delta_i(\tau)]/[1 - \delta_e(\tau)]$.
12. Higher-dimensional systems can be easily accommodated, but this extension is left to subsequent research.
13. The assumption that $M(t)$ earns the riskless return under Q implies that the economy is also arbitrage free with respect to inclusion of an additional traded asset, the market index.
14. The variable $Z(t)$ can be estimated using past observations of $M(t)$.

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Default Parameter Estimation using Market Prices

Corrigendum

by Robert Jarrow[1]

[1] Thanks are expressed to Clive Saunders for pointing out these typos and errors.

Given are the corrected equations. For most of the equations given, some signs were reversed.

$$\log\left(\frac{\left[\xi(t)-\sum_{j\geq t}^{T^*} D_j v(t,j:e)\right]}{\left[\xi(t-\Delta)-\sum_{j\geq t-\Delta}^{T^*} D_j v(t-\Delta,j:e)\right]}\right)-r(t-\Delta)\Delta \tag{38}$$

$$\begin{aligned}
&\approx +\lambda_0\Delta+\lambda_1\left(\left(\frac{b(t-\Delta,T^*)^2}{2}\right)\Delta+\log\left(\frac{p(t,T^*)}{p(t-\Delta,T^*)}\right)\right)-\lambda_1^2\left(\frac{b(t-\Delta,T^*)^2}{2}\right)\Delta\\
&\quad+\lambda_1\sigma_L\varphi_{rL}b(t-\Delta,T^*)\Delta-\lambda_2[Z(t)(T^*-t)-Z(t-\Delta)(T^*-t+\Delta)]\\
&\quad-\lambda_2^2(T^*-t)^2\Delta/2+\lambda_2\sigma_L\varphi_{mL}(T^*-t)\Delta\\
&\quad+[\sigma_L\gamma_L(t-\Delta,X(t-\Delta))-(1/2)\sigma_L^2+\mu_\theta(t-\Delta,X(t-\Delta))]\Delta\\
&\quad+\varepsilon(t-\Delta).
\end{aligned}$$

$$\begin{aligned}
v_l(t,T:i)&=e^{-\gamma_0^i-\gamma_1^i r(t)-\gamma_2^i Z(t)}p(t,T)\bullet\\
&\quad e^{-\lambda_0(1-\delta_i)(T-t)-\lambda_1(1-\delta_i)\mu_1(t,T)+\left(2\lambda_1(1-\delta_i)+\lambda_1{}^2(1-\delta_i)^2\right)\sigma_1^2(t,T)/2}\bullet\\
&\quad e^{-\lambda_2(1-\delta_i)Z(t)(T-t)+\left(1+\lambda_1(1-\delta_i)\right)\lambda_2(1-\delta_i)\varphi_{rm}\eta(t,T)+[T-t]^3\lambda_2^2(1-\delta_i)^2/6}
\end{aligned} \tag{41.a}$$

$$\log\left(\frac{\left[\xi(t)-\sum_{j\geq t}^{T^*} D_j v(t,j:e)\right]}{\left[\xi(t-\Delta)-\sum_{j\geq t-\Delta}^{T^*} D_j v(t-\Delta,j:e)\right]}\right)-r(t-\Delta)\Delta \approx +\lambda_0\Delta \tag{41.b}$$

$$\begin{aligned}
&+\lambda_1\left(\left(\frac{b(t-\Delta,T^*)^2}{2}\right)\Delta+\log\left(\frac{p(t,T^*)}{p(t-\Delta,T^*)}\right)\right)-\lambda_1^2\left(\frac{b(t-\Delta,T^*)^2}{2}\right)\Delta\\
&+\lambda_1\sigma_L\varphi_{rL}b(t-\Delta,T^*)\Delta-\lambda_2[Z(t)(T^*-t)-Z(t-\Delta)(T^*-t+\Delta)]\\
&-\lambda_2^2(T^*-t)^2\Delta/2+\lambda_2\sigma_L\varphi_{mL}(T^*-t)\Delta\\
&+\beta_0\log(M(t)/M(t-\Delta))+\beta_1\log(\xi^2(t)/\xi^2(t-\Delta))\\
&+\varepsilon(t-\Delta)
\end{aligned}$$

$$\begin{aligned}
&E_t(r(s)Z(u))=\\
&\quad E_t\left[\left(f(t,s)+b(t,s)^2/2+\int_t^s \rho(v,s)dW(v)\right)\left(\int_t^u dZ(v)+Z(t)\right)\right]=\\
&\quad\varphi_{rm}\int_t^{\min(s,u)}\rho(v,s)dv+\left[f(t,s)+b(t,s)^2/2\right]Z(t).
\end{aligned} \tag{A9}$$

$$\int_t^T \left(\int_0^t \rho(v,u)dW(v) \right) du = \frac{b(t,T)}{\sigma_r} \int_0^t \rho(v,t)dW(v). \tag{A18}$$

$$S_t = L(t)e^{-\lambda_0(T^*-t)-\lambda_1\mu_1(t,T^*)-\lambda_2 Z(t)(T^*-t)+(1/2)\lambda_1^2\sigma_1^2(t,T^*)+\lambda_1\lambda_2\sigma_{12}(t,T^*)+\lambda_2^2(T^*-t)^3/6} \bullet$$
$$e^{-\lambda_1\sigma_L\varphi_{rL}\int_t^{T^*} b(u,T^*)du-\lambda_2\sigma_L\varphi_{mL}(T^*-t)^2/2} \tag{A30}$$

$$\begin{aligned} log\left(\frac{\xi_t - \sum_{j\geq t}^{T^*} D_j v(t,j:e)}{\xi_{t-\Delta} - \sum_{j\geq t-\Delta}^{T^*} D_j v(t-\Delta,j:e)} \right) &= log\left(\frac{L_t}{L_{t-\Delta}}\right) + \int_{t-\Delta}^{t} \mu_\theta(u)du - \lambda_0[(T^*-t)-(T^*-t+\Delta)] \\ & - \lambda_1[\mu_1(t,T^*) - \mu_1(t-\Delta,T^*)] - \lambda_2[Z(t)(T^*-t) - Z(t-\Delta)(T^*-t+\Delta)] \\ & + (1/2)\lambda_1^2[\sigma_1^2(t,T^*) - \sigma_1^2(t-\Delta,T^*)] + \lambda_1\lambda_2[\sigma_{12}(t,T^*) - \sigma_{12}(t-\Delta,T^*)] \\ & + \lambda_2^2[(T^*-t)^3 - (T^*-t+\Delta)^3]/6 - \lambda_1\sigma_L\varphi_{rL}[\int_t^{T^*} b(u,T^*)du - \int_{t-\Delta}^{T^*} b(u,T^*)du] \\ & - \lambda_2\sigma_L\varphi_{mL}[(T^*-t)^2 - (T^*-t+\Delta)^2]/2. \end{aligned} \tag{A33}$$

$$\lambda_0[(T^*-t)-(T^*-t+\Delta)] = -\lambda_0\Delta \tag{A34}$$

$$\begin{aligned} \lambda_1[\mu_1(t,T^*) - \mu_1(t-\Delta,T^*)] &= \lambda_1[-log\left(\frac{p(t,T^*)}{p(t-\Delta,T^*)}\right) - \int_{t-\Delta}^{t} b(v,T^*)^2 dv/2] \\ &\approx \lambda_1[-log\left(\frac{p(t,T^*)}{p(t-\Delta,T^*)}\right) - b(t-\Delta,T^*)^2\Delta/2]. \end{aligned} \tag{A35}$$

$$\begin{aligned} (1/2)\lambda_1^2[\sigma_1^2(t,T^*) - \sigma_1^2(t-\Delta,T^*)] &= -(1/2)\lambda_1^2 \int_{t-\Delta}^{t} b(v,T^*)^2 dv \\ &\approx -b(t-\Delta,T^*)^2\Delta/2. \end{aligned} \tag{A36}$$

$$\begin{aligned} \lambda_2^2[(T^*-t)^3 - (T^*-t+\Delta)^3]/6 &= \lambda_2^2[-3(T^*-t)^2\Delta - 3(T^*-t)\Delta^2 - \Delta^3]/6 \\ &\approx -\lambda_2^2(T^*-t)^2\Delta/2. \end{aligned} \tag{A38}$$

$$\lambda_1\sigma_L\varphi_{rL}[\int_t^{T^*} b(u,T^*)du - \int_{t-\Delta}^{T^*} b(u,T^*)du] = -\lambda_1\sigma_L\varphi_{rL}\int_{t-\Delta}^{t} b(u,T^*)du$$
$$\approx -\lambda_1\sigma_L\varphi_{rL} b(t-\Delta,T^*)\Delta. \tag{A39}$$

$$\lambda_2\sigma_L\varphi_{mL}[(T^*-t)^2-(T^*-t+\Delta)^2]/2 = \lambda_2\sigma_L\varphi_{mL}[-2(T^*-t)\Delta-\Delta^2]/2$$
$$\approx -\lambda_2\sigma_L\varphi_{mL}(T^*-t)\Delta. \tag{A40}$$

$$\begin{aligned}
log\left(\frac{\xi_t-\sum_{j\geq t}^{T^*} D_j v(t,j:e)}{\xi_{t-\Delta}-\sum_{j\geq t-\Delta}^{T^*} D_j v(t-\Delta,j:e)}\right) &= log\left(\frac{L_t}{L_{t-\Delta}}\right)+\int_{t-\Delta}^{t}\mu_\theta(u)du+\lambda_0\Delta \\
&+\lambda_1[+log\left(\frac{p(t,T^*)}{p(t-\Delta,T^*)}\right)+b(t-\Delta,T^*)^2\Delta/2] \\
&-\lambda_2[Z(t)(T^*-t)-Z(t-\Delta)(T^*-t+\Delta)] \\
&-\lambda_1^2 b(t-\Delta,T^*)^2\Delta/2-\lambda_2^2(T^*-t)^2\Delta/2+\lambda_1\sigma_L\varphi_{rL}b(t-\Delta,T^*)\Delta \\
&+\lambda_2\sigma_L\varphi_{mL}(T^*-t)\Delta.
\end{aligned} \tag{A41}$$