A Simple Model for Valuing Default Swaps when both Market and Credit Risk are Correlated

by

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Abstract

This paper provides a simple analytic formula for valuing default swaps with correlated market and credit risk. We illustrate the numerical implementation of this model by inferring the default probability parameters implicit in default swap quotes for twenty two companies over the time period 8/21/00 to 10/31/00. For comparison, with also provide implicit estimates for the standard model (a special case of our approach) where market and credit risk are statistically independent.

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A recent Risk magazine [2000] survey showed that the market for default swaps, as measured by the notional amounts of contracts traded per year, has grown from about 50 billion in 1998 to over 400 billion dollars in 2000. This exponential growth has generated significant interest in the fair valuation of default swaps by both the academic and practitioner communities.

The existing literature investigating the valuation of default swaps (see Hull and White [2000, 2001], Martin, Thompson and Browne [2000], Wei [2001], and the survey paper by Cheng [2001]) gives the impression that simple models for pricing default swaps are only available when credit and market risk are statistically independent. Indeed, it is commonly believed that models incorporating correlated market and credit risk are quite complex, necessarily requiring burdensome recursive numerical procedures. For example, from Hull and White [2000, p. 30] "Like most other approaches, ours assumes that default probabilities, interest rates, and recovery rates are mutually independent. Unfortunately, it does not seem to be possible to relax these assumptions without a considerably more complex model."

In contradiction with this commonly held belief, this paper provides a simple analytic formula for the valuation of default swaps when market and credit risk are correlated. This analytic formula is easy to understand and to compute. This formula is derived in the context of the reduced-form credit risk model from Jarrow [2001] where correlated defaults arise due to the fact that a firm's default intensities depend on common macro-factors. The common macro-factor used in our paper is the spot rate of interest. The spot rate of interest is assumed to follow an extended Vasicek model in the HJM framework.

We illustrate the numerical implementation of this model by inferring the default probability parameters implicit in the term structure of default swap quotes for twenty two different companies over the time period 8/21/00 to 10/31/00. The data used for this investigation was downloaded from Enron's web site. The twenty-two different firms were chosen to stratify various industry groupings: financial, food and beverages, petroleum, airlines, utilities, department stores, and technology. For comparison purposes, the standard model with statistically independent market and credit risk (a special case of our model) is also calibrated to this market data. One can also easily calibrate our simple model with correlated market and credit risk to exactly match the observed default swap quote term structure.

An outline of this paper is as follows. Section I introduces the notation and the reduced form credit risk model. Section II studies the pricing of binary default swaps. Section III provides the empirical specification of the model. The data is discussed in section IV. The spot rate parameter estimation is performed in section V. Section VI provides the default intensity estimates, while section VII concludes the paper.

I. The Model Structure

This section introduces the notation and briefly summarizes the reduced form credit risk model contained in Jarrow [2001]. This model is the basis for valuing default swaps. Trading can take place anytime during the interval $[0,\overline{T}]$. Traded are default-free zero-coupon bonds and risky (defaultable) zero-coupon bonds of all maturities. Markets are assumed to be frictionless with no arbitrage opportunities and complete.

Let $p(t,T)$ represent the time t price of a default-free dollar paid at time T where $0 \le t \le T \le \overline{T}$. The instantaneous forward rate at time t for date T is defined by $f(t,T) = -\partial \log p(t,T) / \partial T$. The spot rate of interest is given by $r(t) = f(t,t)$.

Consider a firm issuing risky debt. Let $v(t,T)$ represent the time t price of a promised dollar to be paid by this firm at time T where $0 \le t \le T \le \overline{T}$. The debt is risky because if the firm defaults prior to time T, then the promised dollar may not be paid. Let τ denote the first time that this firm defaults ($\tau > \overline{T}$ is possible if the firm does not default). The default time is a random variable. We let

$$N(t) = 1_{\{\tau \le t\}} = \begin{cases} 1 & if \quad \tau \le t \\ 0 & otherwise \end{cases} \tag{1}$$

denote the point process indicating whether or not default has occurred prior to time *t*. It is assumed that the point process has a random intensity $\lambda(t)$ where $\lambda(t)\Delta$ gives the approximate probability of default for this firm over the time interval $[t, t+\Delta]$. The intensity process is defined under the risk neutral probability. This statement will become clear below.

If default occurs, we let the zero-coupon bond receive a *fractional recovery* of $\delta(\tau)v(\tau-,T)$ dollars where $0 \le \delta(\tau)$ and $\tau-$ represents an instant before default. Under this recovery rate structure, the debt is worth only a fraction of its pre-default value.

Under the assumption of no arbitrage and complete markets, standard arbitrage pricing theory[1] implies that there exists a unique equivalent probability *Q* such that the present values of the zero-coupon bonds are computed by discounting at the spot rate of interest and then taking an expectation with respect to *Q*. That is,

$$p(t,T) = E_t\left(e^{-\int_t^T r(u)du}\right) \tag{2}$$

and

$$v(t,T:\delta) = E_t\left(\delta(\tau)v(\tau-,T)e^{-\int_t^\tau r(u)du} 1_{(t<\tau\le T)} + 1\, e^{-\int_t^T r(u)du} 1_{(T<\tau)}\right) \tag{3}$$

where $E_t(.)$ is conditional expectation with respect to *Q* at time *t*. As indicated, the risky debt value is composed of two parts. The first part is the present value of the promised payment in default. The second part is the present value of the promised payment if default does not occur.

We next study the pricing of credit derivatives using the approach of Lando [1998]. We provide the fair price for credit derivatives in a perfectly liquid market. We assume that the point process is modeled as a Cox process with an intensity function $\lambda(t, X_t)$ where $\{X_t : t \in [0,T]\}$ is a vector stochastic process representing the state variables underling the evolution of the economy. These macro-economic state variables induce the correlation between market and default risk. For example, if X_t quantifies market risk and $\lambda(t, X_t)$ increases as X_t increases, then as market risk increases, the likelihood of the firm defaulting increases as well.

The cash flows from most credit derivative can be decomposed into three different types:

(i) The first is a random payment [2] Y_T at time *T*, but only if there is no default prior to time T.

(ii) The second is a random payment rate of $y_t dt$ at time *t* for the period *[0,T]*, but only if there is no default prior to the time the payment is received.

(iii) The third is a random payment that occurs only at default of Ψ_τ, zero otherwise. This payment is made only if default occurs during the time period *[0,T]*.

Under appropriate integrability conditions, the present values of these credit risky cash flows are:

$$V_t 1_{\{\tau>t\}} = E_t\left(Y_T 1_{\{\tau>T\}} e^{-\int_t^T r(u)du} \right) = E_t\left(Y_T e^{-\int_t^T [r(u)+\lambda(u)]du} \right) \qquad (4)$$

$$V_t 1_{\{\tau>t\}} = E_t\left(\int_t^T y_s 1_{\{\tau>s\}} e^{-\int_t^s r(u)du} ds \right) = E_t\left(\int_t^T y_s e^{-\int_t^s [r(u)+\lambda(u)]du} ds \right) \qquad (5)$$

$$V_t 1_{\{\tau>t\}} = E_t\left(1_{\{t<\tau\le T\}} \Psi_\tau e^{-\int_t^\tau r(u)du} \right) = E_t\left(\int_t^T \Psi_s \lambda(s) e^{-\int_t^s [r(u)+\lambda(u)]du} ds \right) \qquad (6)$$

The proof is in the appendix. In each of these valuations, the expectation on the far right side is with respect to the state variables (including the spot rate *r(t)*) underlying the stochastic structure of the economy. The randomness due to the bankruptcy process has been removed through the computation of an iterated expectation. What remains is the valuation of a random cash flow based on the state variables using a "risk adjusted" discount rate. The risk adjustment is the expected loss per unit time that equals the intensity $\lambda(u)$ (with a zero recovery rate).

II. Binary Default Swap

This section specializes the above general structure to value a binary default swap (sometimes called credit swaps), see Rooney [1998]. In this credit swap there are two counterparties, called the Protection seller and the Protection buyer. We assume that both counterparties are default-free. Of course, this can easily be relaxed as discussed in Jarrow and Turnbull [1995]. The swap lasts for a fixed period of time *[0,T]*, the swap's maturity. In this swap, the protection seller receives a fixed payment flow of c_T dollars per unit time from the Protection buyer unless a reference credit (a particular firm with intensity $\lambda(t)$) defaults. Default is called a credit event. If default occurs, the Protection seller pays the Protection buyer *$1* on the default date τ. If there is no default, the Protection seller makes no payment to the Protection buyer. The swap terminates after the default event occurs or at the maturity date of the swap. In traded default swaps, in contrast to the structure given, the payments occur at fixed intervals. If default occurs between payment dates, the accrued portion of the payment due at the next payment date is due at the time of default. The continuous payment structure given provides a reasonable approximation to this condition.

The value of the swap to the Protection seller is:

$$V_t 1_{\{t<\tau\}} = E_t\left(\int_t^T c_T 1_{\{\tau>s\}} e^{-\int_t^s r(u)du} ds\right) - E_t\left(1_{\{t<\tau\le T\}} \cdot e^{-\int_t^\tau r(s)ds}\right) \tag{7}$$

Using expressions (5) and (6), we obtain:

$$V_t 1_{\{t<\tau\}} = E_t\left(\int_t^T c_T e^{-\int_t^s [r(u)+\lambda(u)]du} ds\right) - E_t\left(\int_t^T \lambda(s) e^{-\int_t^s [r(u)+\lambda(u)]du} ds\right) \tag{8}$$

Using the risky zero-coupon's bond price, we recognize this as the value of a risky coupon bond of maturity T paying a continuous cash flow of c_T dollars per unit time with a zero recovery rate less the cost of a dollar default insurance on the firm, i.e.

$$V_t = V_t 1_{\{t<\tau\}} = c_T \int_t^T v(t,s:0)ds - E_t\left(\int_t^T \lambda(s) e^{-\int_t^s [r(u)+\lambda(u)]du} ds \right) \tag{9}$$

This simple formula applies when both market and credit risk is correlated. As seen, whether or not we obtain a simple analytic formula for the price of a default swap depends on the evaluation of the second term on the right side of expression (9).

In standard default swaps, the cash payment c_T, called the default swap rate, is determined at time 0 such that $V_0 = 0$, i.e.

$$c_T = \frac{E_0\left(\int_0^T \lambda(s) e^{-\int_0^s [r(u)+\lambda(u)]du} ds \right)}{\int_0^T v(0,s:0)ds} \tag{10}$$

The purpose of the next section of the paper is to generate a parametric version of this model so that expressions (9) and (10) have simple analytic expressions.

III. Empirical Specification

To obtain a simple, but realistic empirical formulation of the above model, we utilize a special case of Jarrow [2001] where the economy is Markov in a single state variable – the spot rate of interest. Although Jarrow includes the cumulative excess return on an equity market index, Janosi, Jarrow, Yildiray [2000] found that this added no additional explanatory power in the pricing of corporate debt.

For the spot rate of interest, we use a single factor model with deterministic volatilities, sometimes called the extended Vasicek model, i.e.

(Spot Rate Evolution)

$$dr(t) = a[\bar{r}(t) - r(t)]dt + \sigma_r dW(t) \tag{7}$$

where $a \neq 0$, $\sigma_r > 0$ are constants, $\bar{r}(t)$ is a deterministic function of t chosen to match the initial zero-coupon bond price curve [3], and $W(t)$ is a standard Brownian motion under Q initialized at $W(0) = 0$. The evolution of the spot rate is given under the risk neutral probability Q.

We assume that the intensity function is almost linear in the spot rate of interest.

(Intensity a Function of the Spot Rate of Interest)

$$\lambda(t) = max[\lambda_0(t) + \lambda_1 r(t),\ 0] \tag{10}$$

where $\lambda_0(t) \geq 0$ is a deterministic function of time and λ_1 is a constant. In this formulation, the (pseudo) probability of default per unit time is assumed to be the maximum of a linear function of the spot rate $r(t)$ and zero. The maximum operator is necessary to keep the intensity function non-negative. For analytic convenience, we will drop the maximum operator in the empirical implementation. This linear approximation implies that negative default rates ($\lambda(t) < 0$) are possible.

Finally, to price risky debt, we assume that the recovery rate is a constant. In fact, in the valuation of default swaps, this constant recovery rate assumption is unnecessary. It is included only to provide expression (12) below for risky debt prices when the recovery rate is non-zero.

(Constant Recovery Rate)

$$\delta(t) = \delta \quad where \quad \delta \text{ is a constant.}$$

Given this structure, Jarrow [2001] obtains closed form solutions for default free and risky zero-coupon bond prices:

$$p(t,T) = e^{-\mu_1(t,T) + \sigma_1^2(t,T)/2} \tag{11}$$

and

$$v(t,T:\delta) = p(t,T)e^{-\int_t^T \lambda_0(s)(1-\delta)ds - \lambda_1(1-\delta)\mu_1(t,T) + \left(2\lambda_1(1-\delta) + \lambda_1^2(1-\delta)^2\right)\sigma_1^2(t,T)/2} \tag{12}$$

where no default has occurred at or prior to time t,

$$\mu_1(t,T) = \int_t^T f(t,u)du + \int_t^T b(u,T)^2 du \big/ 2, \quad \sigma_1^2(t,T) = \int_t^T b(u,T)^2 du,$$

$$\textit{and} \qquad b(u,t) = \sigma_r \left(1 - e^{-a(t-u)}\right) \big/ a$$

For this paper, we are interested in the pricing of binary default swaps.

Substitution of the linear intensity into the default swap's valuation gives the following formula. The proof is in the appendix.

$$V_t 1_{\{\tau > t\}} = c_T \int_t^T v(t,s:0)ds - \int_t^T [\lambda_0(s) + \lambda_1[\mu_0(t,s) - (1+\lambda_1)\sigma_{01}(t,s)]]v(t,s:0)ds \quad (13)$$

where $\mu_0(t,s) = f(t,s) + b(t,s)^2 / 2$ and $\sigma_{01}(t,s) = b(t,s)^2 / 2$

This is the desired analytic expression for the fair value of a default swap, easily computed using only knowledge of the default free term structure evolution and the current valuation of the firm's risky debt prices. If risky debt prices are not available, then the risky debt prices can be estimated using the analytic formula given by expression (12).

Finally, the default swap rate is determined by solving expression (13):

$$c_T = \frac{\int_0^T [\lambda_0(s) + \lambda_1[\mu_0(0,s) - (1+\lambda_1)\sigma_{01}(0,s)]]v(0,s:0)ds}{\int_0^T v(0,s:0)ds} \quad (14)$$

IV. Description of the Data

This section describes the data used to illustrate the numerical implementation of expression (14). The data was downloaded from Enron's web site from 8/21/00 to

10/31/00. We selected 22 different firms chosen to stratify various industry groupings: financial, food and beverages, petroleum, airlines, utilities, department stores, and technology. The 22 firms included in this study are listed in Exhibit 1. For parameter estimation of the spot rate process, daily U.S. Treasury bond, note and bill prices were also downloaded from Bloomberg over the same time period.

V. Estimation of the Spot Rate Process Parameters

To implement the estimation of the default and liquidity discount parameters, we first need to estimate the parameters for the state variable processes (*r(t)*). The inputs to the spot rate process evolution are the forward rate curves (*f(t,T)*) and the spot rate parameters (a, σ_r). We discuss the estimation of these inputs in this section.

For the estimation of the forward rate curves, a two-step procedure is utilized. First, for a given time *t*, the discount bond prices (*p(t,T)* for various *T*) are estimated by solving the following minimization problem:

$$\text{choose} \quad (p(t,T) \text{ for all relevant } T \leq max\{T_i : i \in I_t\})$$

$$\text{to minimize} \sum_{i \in I_t} \left[B_i(t,T_i) - B_i(t,T_i)^{bid} \right]^2 \qquad (15)$$

where I_t is an index set containing the various U.S. Treasury bonds, notes and bills available at time *t*, $B_i(t,T_i)$ is the model price (expression (4)) for the i^{th} bond with maturity T_i as a function of (*p(t,T)*), and $B_i(t,T_i)^{bid}$ is the market bid price for the i^{th} bond with maturity T_i. The discount bond prices' maturity dates T coincide with the maturities of the Treasury bills, and the coupon payment and principal repayment dates for the Treasury notes and bonds.

Step 2 is to fit a continuous forward rate curve to the estimated zero-coupon bond prices (*p(t,T)* for all $T \leq max\{T_i : i \in I\}$). We use the maximum smoothness forward rate curve as developed by Adams and van Deventer [1994] and refined by Janosi and Jarrow

[2000]. Briefly, we choose the unique piecewise, 4th degree polynomial with the left and right end points left "dangling" that minimizes $\int_t^{max\{T_i : i \in I_t\}} \left| \partial^2 f(t,s) / \partial s^2 \right| ds$.

For the spot rate parameters (a, σ_r) estimation, the procedure follows that used in Janosi, Jarrow, Yildiray [2000]. The procedure is based on an explicit formula for the variance of the default-free zero-coupon bond prices derived using expression (7), see Heath, Jarrow and Morton [1992]. For $\Delta = 1/365$ (a day), the expression is:

$$var_t [log(P(t+\Delta,T) / P(t,T)) - r(t)\Delta] = \left(\sigma_r^2 \left(e^{-a(T-t)} - 1\right)^2 / a^2\right)\Delta \qquad (16)$$

First we fix a time to maturity $T-t \in$ *{3 months, 6 months, 1 year, 5 years, 10 years, the longest time to maturity of an outstanding Treasury bond closest to 30 years}.* Then, we compute the sample variance, denoted v_T, using the smoothed forward rate curves previously generated over our sample period. To estimate the parameters (σ_r, a) we run the nonlinear regression

$$v_T = \left(\sigma_r^2 \left(e^{-a(T-t)} - 1\right)^2 / a^2\right)\Delta + e_T \qquad (17)$$

across the bond's time to maturities $T-t \in$ *{ ¼, 1/2, 1, 5, 10, longest time to maturity closest to 30}* where e_T is the error term.

The parameter estimates and their standard errors for this non-linear regression are: $\sigma_r = 0.00593 \quad (0.0021)$ and $a = 0.03450 \quad (0.0003)$.

VI. Default Parameter Estimation

To illustrate the use of expression (14), this section infers the default parameters consistent with the given term structure of default swap quotes obtained from Enron's web page. The default swap rates obtained are for contracts of maturities 1, 2, 3, 4 and 5 years. This gives 5 observations per day to estimate the default parameters in expression

(14). The default parameters to estimate are the deterministic function $\lambda_0(t)$ and the constant λ_1. Exhibit 1 contains the averages of the default swap quotes for the various companies over the observation period. Notice that for all companies, the average swap quote increases with the maturity of the swap. For example, American Airlines' quotes are 149.6944 b.p. for the 1 year swap and 196.222 b.p. for the 5 year swap.

Two formulations of our analytic expression will be estimated. Model 1, the standard model, has default risk independent of interest rate risk. This is the model contained in Martin, Thompson and Browne [2001].[4] The second, model 2, is a simple two-parameter version of expression (14) that incorporates correlated interest rate and default risk. We choose the simplest form of the general model to provide a base case for subsequent empirical investigation into the model's validity.

(Model 1) The Standard Model: $\lambda_0(t)$ is piecewise constant and $\lambda_1 = 0$ where

$$\lambda_0(t) = \begin{cases} \lambda_0^1 & for \quad t \in [0,1) \\ \lambda_0^2 & for \quad t \in [1,2) \\ \lambda_0^3 & for \quad t \in [2,3) \\ \lambda_0^4 & for \quad t \in [3,4) \\ \lambda_0^5 & for \quad t \in [4,5) \end{cases}$$

In this model, the function $\lambda_0(t)$ is calibrated to exactly match the term structure of default swap quotes. Because $\lambda_1 = 0$, default risk and interest rate risk are uncorrelated. Under this structure, expression (14) becomes:

$$c_k = \frac{\sum_{j=1}^{k} \lambda_0^j \int_{j-1}^{j} e^{-\lambda_0^j s} p(0,s)ds}{\sum_{j=1}^{k} \int_{j-1}^{j} e^{-\lambda_0^j s} p(0,s)ds} \text{ for } k = 1,2,\ldots,5$$

We can invert this system for $\lambda_0(t)$. This model gives an exact fit to the term structure of default swap quotes.

(Model 2) Correlated Defaults: $\lambda_0(t) = \lambda_0$ and λ_1 are constants.

$$c_T = \frac{\int_0^T [\lambda_0 + \lambda_1 [\mu_0(0,s) - (1+\lambda_1)\sigma_{01}(0,s)]] v(0,s:0) ds}{\int_0^T v(0,s:0) ds}$$

This is also a special case of our model because the intercept of the intensity process, a deterministic function, is restricted to be a constant. This is done to provide a base case for subsequent investigation.

We can invert this valuation formula for both λ_0 and λ_1. Since this is only a two-parameter model, there will be errors in matching the term structure of default swap quotes. We choose the parameters to minimize the sum of squared error between the theoretical and market quotes.

Exhibit 2 contains the average default intensity parameters for model 1 over the sample period, measured in basis points. These parameter values exactly match the term structure of default swap quotes.

Exhibit 3 contains the average default intensity parameters for model 2 over the sample period, measured in basis points. As noted, for all firms λ_1 is positive, indicating that as interest rates increase the likelihood of default increases as well. The sign of the interest rate coefficient in these intensity functions is consistent with simple economic intuition.

Exhibit 4 contains an illustrative graph of the parameter estimates for American Airlines for both models 1 and 2 over the time period. Exhibit 5 shows the actual swap rate quotes for American Airlines over the time period studied, and the computed swap rate curves for model 2. As seen, the simple two-parameter model is able to match the shape of the default swap term structure reasonably well. This is mildly surprising given that it is only a two-parameter model.

Exhibit 6 contains the average pricing errors and the standard errors for model 2. As indicated, the errors tend to be negative for the short end (years 1 and 2) of the default swap term structure and positive for the long end (years 4 and 5), with the best matching occurring for the 3-year quote. The average pricing errors are –13.93, -7.75, 3.24, 5.83, and 12.61.

Although there is pricing error present in the estimated parameters of Exhibit 3, it is important to note that these errors can be eliminated completely. Indeed, by calibrating the intercept function of the intensity $\lambda_0(t)$ as in the standard model, one can exactly match the default swap term structure using a model with correlated market and credit risk.

VII. Conclusion

This paper provides a simple analytic formula for valuing default swaps with correlated market and credit risk. We illustrate the numerical implementation of this model by inferring the default probability parameters implicit in default swap quotes for twenty two companies over the time period 8/21/00 to 10/31/00. For comparison, with also provide implicit estimates for the standard model (a special case of our approach) where market and credit risk are statistically independent. This simple analytic formula can be calibrated to exactly match the observed default swap term structure.

APPENDIX

Facts about the spot rate *r(t)*

From expression (7), we define

$$X_0 \equiv r(s) = f(t,s) + b(t,s)^2/2 + \int_t^s \rho(v,s)dW(v)$$

where $\rho(v,s) = \sigma_r e^{-a(s-v)}$ and

$$b(t,s) = \int_t^s \rho(t,v)dv = \begin{cases} \sigma_r(1-e^{-a(s-t)})/a & for \quad a \neq 0 \\ \sigma_r(s-t) & for \quad a = 0. \end{cases}$$

A direct computation gives:

$$\mu_0(t,s) \equiv E_t(r(s)) = f(t,s) + b(t,s)^2/2 \qquad \text{and} \qquad \text{(A.1)}$$

$$\sigma_0^2(t,s) \equiv Var_t(r(s)) = \int_t^s \rho(v,s)^2 dv. \qquad \text{(A.2)}$$

Define $X_1 \equiv \int_t^T r(s)ds = \int_t^T f(t,s)ds + \int_t^T b(t,s)^2 ds/2 + \int_t^T \int_t^s \rho(v,s)dW(v)ds$.

Changing the order of integration and a direct computation yields

$$\int_t^T b(t,s)^2 ds/2 = \int_t^T b(v,T)^2 dv/2 \qquad and$$

$$\int_t^T \int_t^s \rho(v,s)dW(v)ds = \int_t^T b(v,T)dW(v)$$

Substitution gives

$$\int_t^T r(s)ds = \int_t^T f(t,s)ds + \int_t^T b(v,T)^2 dv/2 + \int_t^T b(v,T)dW(v).$$

A direct computation gives

$$\mu_1(t,T) \equiv E_t(\int_t^T r(s)ds) = \int_t^T f(t,s)ds + \int_t^T b(v,T)^2\, dv/2 \qquad \text{and} \qquad (A.3)$$

$$\sigma_1^2(t,T) \equiv Var_t(\int_t^T r(s)ds) = \int_t^T b(v,T)^2\, dv. \qquad (A.4)$$

Also,

$$\sigma_{01}(t,s) \equiv E_t(r(s)\int_t^s r(u)du) = \int_t^s \rho(v,s)b(v,s)dv = b(t,s)^2 / 2\,. \qquad (A.5)$$

Last, from Heath, Jarrow, Morton [1992, p.88]:

$$p(s,T) = p(t,T)e^{\int_t^s r(u)du - (1/2)\int_t^s b(u,T)^2 du - \int_t^s b(u,T)dW(u)} \qquad (A.6)$$

It can be shown that

$$cov_t(\int_t^T r(u)du, \int_t^s r(u)du) = \sigma_1^2(t,s) \quad for \quad T > s \qquad (A.7)$$

$$cov_t(\int_t^s b(u,T)dW(u), \int_t^s r(u)du) = \int_t^s b(u,T)b(u,s)du \qquad (A.8)$$

Facts about Normal Distributions

Given *(x, y)* is bivariate normal, we have (see Hogg and Craig [1970, p.114]).

$$E_t\left[e^{Ax+By}\right] = e^{\mu_x A + \mu_y B + \left[\sigma_x^2 A^2 + 2\sigma_{xy}AB + \sigma_y^2 B^2\right]/2} \qquad (A.9)$$

where

$\mu_x \equiv E_t(x), \mu_y \equiv E_t(y), \sigma_x^2 \equiv Var_t(x), \sigma_y^2 \equiv Var_t(y) \quad and \quad \sigma_{xy} \equiv Cov_t(x,y).$

Also,

$$\frac{\partial E_t\left[e^{Ax+By}\right]}{\partial A} = E_t\left[xe^{Ax+By}\right] = e^{\mu_x A + \mu_y B + \left[\sigma_x^2 A^2 + 2\sigma_{xy}AB + \sigma_y^2 B^2\right]/2}[\mu_x + \sigma_x^2 A + \sigma_{xy} B] \qquad (A.10)$$

Proof of General Valuation Formulas

Let $E_t(\cdot)$ denote conditional expectation with respect to the information set generated by $\{X_s, N_s\} \quad for \quad s \le t$ where $\{X_s\}$ is a vector stochastic process representing the state variables.

Let $E_t\{\cdot \mid X_T\}$ denote conditional expectation with respect to the information set generated by $\{X_s, N_s\} \quad for \quad s \le t$ and conditional on the information set $\{X_s\} \quad for \quad s \le T$ as well.

Given the information set $\{X_s\} \quad for \quad all \quad s \le T$, a Cox process N_t behaves like a Poisson process with intensity $\lambda(t, X_t)$.

Thus, $$Q_t(\tau > T \mid X_T) = E_t(1_{\{\tau > T\}} \mid X_T) = e^{-\int_t^T \lambda(s, X_s) ds} \tag{A.11}$$

And, $$Q_t(\tau > T) = E_t(E_t(1_{\{\tau > T\}} \mid X_T)) = E_t(e^{-\int_t^T \lambda(s, X_s) ds}). \tag{A.12}$$

Hence, the probability of default prior to time *T* is:

$$Q_t(\tau \le T) = 1 - Q_t(\tau > T) = 1 - E_t(e^{-\int_t^T \lambda(s, X_s) ds}). \tag{A.13}$$

The conditional density function is:

$$Q_t(\tau \in (s, s+ds] \mid X_T) = dQ_t(\tau \le s \mid X_T)/ds = \lambda(s, X_s) e^{-\int_t^s \lambda(u, X_u) du}. \tag{A.14}$$

Derivation of Expression (4)

Using the facts about Cox processes we have that:

$$E_t(Y_T 1_{\{\tau > T\}} e^{-\int_t^T r(u) du}) = E_t(E_t(Y_T 1_{\{\tau > T\}} e^{-\int_t^T r(u) du} \mid X_T))$$

$$= E_t(Y_T e^{-\int_t^T r(u)du} E_t(1_{\{\tau > T\}} | X_T))$$

$$\left[= E_t(Y_T e^{-\int_t^T r(u)du} Q_t(\tau > T | X_T) + 0 \cdot Q_t(t < \tau \le T | X_T)) \right]$$

$$= E_t(Y_T e^{-\int_t^T r(u)du} e^{-\int_t^T \lambda(u)du})$$

This completes the derivation.

Derivation of Expression (5)

Using the facts about Cox processes we have that:

$$E_t(\int_t^T y_s 1_{\{\tau > s\}} e^{-\int_t^s r(u)du} ds) = E_t(E_t(\int_t^T y_s 1_{\{\tau > s\}} e^{-\int_t^s r(u)du} ds | X_T))$$

$$= E_t(\int_t^T y_s e^{-\int_t^s r(u)du} E_t(1_{\{\tau > s\}} | X_T) ds)$$

$$\left[= E_t(\int_t^T [y_s e^{-\int_t^s r(u)du} Q_t(\tau > s | X_T) + 0 \cdot Q_t(t < \tau \le s | X_T)] ds) \right]$$

$$= E_t(\int_t^T y_s e^{-\int_t^s r(u)du} e^{-\int_t^s \lambda(u)du} ds)$$

This completes the derivation.

Derivation of Expression (6)

Using the facts about Cox processes we have that:

$$E_t(1_{\{t < \tau \le T\}} \Psi_\tau e^{-\int_t^\tau r(u)du}) = E_t(E_t(\int_t^T 1_{\{\tau = s\}} \Psi_s e^{-\int_t^s r(u)du} | X_T) ds)$$

$$= E_t\left(\int_t^T \Psi_s e^{-\int_t^s r(u)du} E_t(1_{\{\tau=s\}} \mid X_T)ds\right)$$

$$= E_t\left(\int_t^T \Psi_s e^{-\int_t^s r(u)du} Q_t(\tau \in (s,s+ds] \mid X_T)ds\right)$$

$$= E_t\left(\int_t^T \Psi_s e^{-\int_t^s r(u)du} \lambda(s) e^{-\int_t^s \lambda(u)du} ds\right).$$

This completes the derivation.

Derivation of Expression (13)

$I_t(T)1_{\{t<\tau\}} = E_t\left(1_{\{t<\tau\le T\}} \cdot e^{-\int_t^\tau r(u)du}\right) = E_t\left(\int_t^T \lambda(s) e^{-\int_t^s [r(u)+\lambda(u)]du} ds\right)$. Substitution of the linear intensity into this equation gives:

$I_t(T)1_{\{t<\tau\}} = E_t\left(\int_t^T [\lambda_0(s)+\lambda_1 r(s)] e^{-\int_t^s [r(u)+\lambda_0(s)+\lambda_1 r(u)]du} ds\right)$. Algebra and interchanging the order of integration and expectation gives:

$$I_t(T)1_{\{t<\tau\}} = \left(\int_t^T \lambda_0(s) E_t(e^{-\int_t^s [r(u)+\lambda_0(s)+\lambda_1 r(u)]du})ds\right) +$$

$$\left(\lambda_1 \int_t^T E_t(r(s) e^{-\int_t^s \lambda_0(u)du-(1+\lambda_1)\int_t^s r(u)du})ds\right)$$

Using expression (12) for $v(t,s:0)$, we get

$$= \left(\int_t^T \lambda_0(s) v(t,s:0)ds\right) + \left(\lambda_1 \int_t^T e^{-\int_t^s \lambda_0(u)du} E_t(r(s) e^{-(1+\lambda_1)\int_t^s r(u)du})ds\right).$$

Using expression (A.9) with $x \equiv r(s) = X_0$, $y \equiv \int_t^s r(u)du = X_1$, $A \equiv 0$ and $B \equiv -(1+\lambda_1)$ and expressions (A.1)-(A.5) gives:

$$= \int_t^T \lambda_0(s)v(t,s:0)ds +$$

$$\lambda_1 \int_t^T e^{-\int_t^s \lambda_0(u)du-(1+\lambda_1)\mu_1(t,s)+[1+2\lambda_1+\lambda_1^2]\sigma_1^2(t,s)/2} [\mu_0(t,s)-(1+\lambda_1)\sigma_{01}(t,s)]ds.$$

Using equation (12) gives the result.

ENDNOTES

[1] See Jarrow and Turnbull [1995]. No arbitrage guarantees the existence, but not the uniqueness of the probability *Q*. Without any additional hypotheses on the economy, the uniqueness of *Q* is equivalent to markets being complete, see Battig and Jarrow [1999].

[2] The random variables Y_T and y_t, and the stochastic process Ψ_t are measurable are measurable with respect to the information set generated by $\{ X_s : 0 \leq s \leq T \}$.

[3] In particular, $\bar{r}(t) = f(0,t) + \left(\partial f(0,t) / \partial t + \sigma_r^2 \left(1 - e^{-2at} \right) / 2a \right) / a$ for $a \neq 0$.

[4] Martin, Thompson and Browne [2001] investigate a procedure for fitting a smoothed intensity function to market data. Although this smoothing procedure could also be used here, we do not focus on this aspect of the estimation.

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Exhibit 1: Default Swap Quotes for Various Firms over 8/21/00 to 10/31/00 measured in basis points.

		Avg c_1	Avg c_2	Avg c_3	Avg c_4	Avg c_5
American Airlines Inc.	AMR1	149.6944	167.0556	175.0278	183.9444	196.2222
Archer-Daniels-Midland Co.	ADM	32.4722	38.2222	43.5000	47.5556	53.5278
C P & L Energy Inc.	CPL	51.3056	57.4167	60.5278	64.0833	68.3889
Chase Manhattan Corp.	CMB	30.3889	35.3889	37.2778	39.0000	41.8333
Coca-Cola Enterprises	CCE	35.3056	40.4167	43.5833	45.0556	49.2778
Delta Air Lines Inc.	DAL	130.1944	138.7222	146.4722	151.2222	161.3333
Dow Chemical Company	DOW	23.5000	29.3333	35.6111	40.6667	44.6389
Eastman Kodak Co.	EK	26.7222	29.1667	31.5833	33.6944	36.6667
First Union Corp.	FTU	32.8333	36.6667	38.7222	41.0278	43.8056
J.P. Morgan & Company	JPM	19.8889	22.8056	26.3056	27.9444	31.0000
K Mart Corp.	KM	353.9167	369.3611	458.2778	471.0833	485.9167
Lyondell Chemical Company	LYO	349.6111	349.6389	367.5000	374.6667	387.7222
Merrill Lynch & Co.	MER	27.0278	30.0556	40.0833	43.6389	47.9722
Phillips Petroleum Co.	P	58.5278	67.0556	72.6667	75.1944	79.5000
Ralston-Ralston Purina Group	RAL	61.4444	69.6667	73.9167	76.4167	81.1111
Sears, Roebuck And Co.	S	60.5833	64.7222	71.4722	73.6111	77.0556
Southwest Airlines	LUV	34.8611	40.5278	42.7778	44.5278	46.3333
TXU Corporation	TXU	83.3611	89.0278	95.1389	103.0278	104.8611
Union Oil Co. Of California	UCL1	59.9444	67.9167	72.3889	74.8889	79.5000
Wal-Mart Stores Inc.	WMT	13.5000	16.2778	17.2778	18.1389	20.4444
Xerox Corp.	XRX	161.2778	169.5278	214.5833	189.8333	227.8056
Texas Instruments Inc.	TXN	40.8889	44.3333	50.2778	52.6944	56.1389

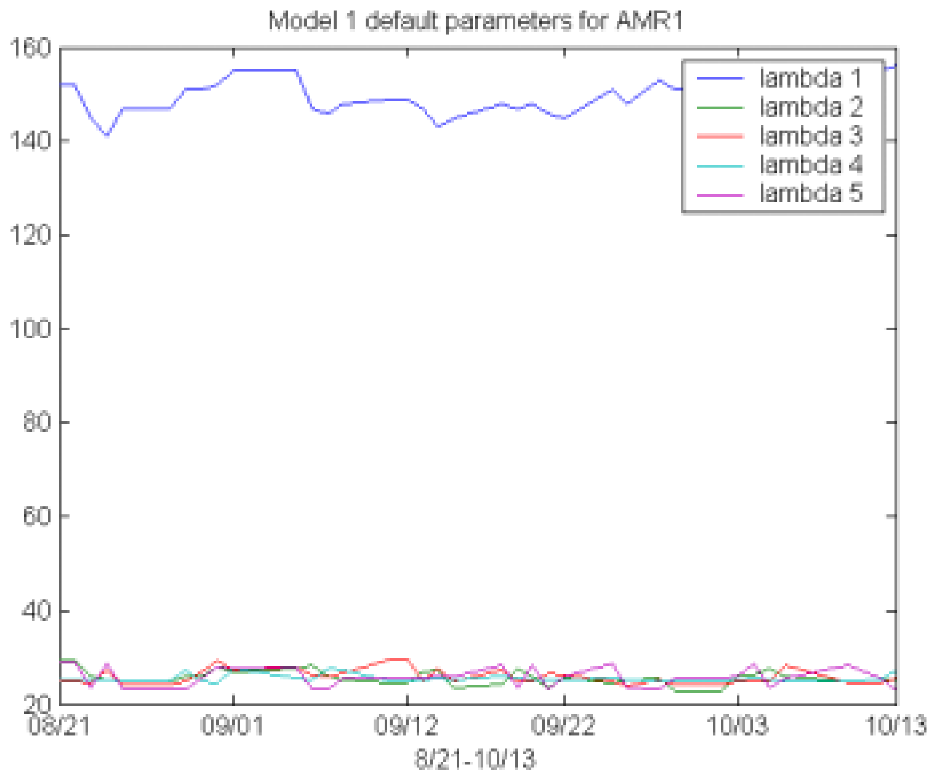
Exhibit 2: Model 1 estimates of $\lambda_0^1,...,\lambda_0^5$ for various firms over the time period 8/21/00 to 10/31/00 measured in basis points.

		Avg λ_0^1	**Avg** λ_0^2	**Avg** λ_0^3	**Avg** λ_0^4	**Avg** λ_0^5	**N**
American Airlines Inc.	AMR1	149.6944	25.8551	25.9323	25.5596	25.7771	36
Archer-Daniels-Midland Co.	ADM	32.4722	24.2311	23.8048	24.3386	23.7516	36
C P & L Energy Inc.	CPL	51.3056	25.9018	26.2306	25.9217	25.2191	36
Chase Manhattan Corp.	CMB	30.3889	24.3501	23.1122	23.5853	23.6264	36
Coca-Cola Enterprises	CCE	35.3056	24.0027	24.9629	24.8912	24.9134	36
Delta Air Lines Inc.	DAL	130.1944	26.0913	25.9644	26.5291	25.9795	36
Dow Chemical Company	DOW	23.5000	21.7930	14.4593	14.4606	14.3817	36
Eastman Kodak Co.	EK	26.7222	19.3607	16.7585	17.6705	17.7911	36
First Union Corp.	FTU	32.8333	23.7851	24.5558	24.7381	24.3545	36
J.P. Morgan & Company	JPM	19.8889	20.9335	5.6304	5.9821	6.3262	36
K Mart Corp.	KM	353.9167	25.9111	26.4591	26.7432	26.2066	36
Lyondell Chemical Company	LYO	349.6111	19.4016	19.2960	19.2386	23.6301	36
Merrill Lynch & Co.	MER	27.0278	20.4044	21.4851	21.3277	20.9874	36
Phillips Petroleum Co.	P	58.5278	26.3940	25.1884	25.2548	25.4656	36
Ralston-Ralston Purina Group	RAL	61.4444	25.9633	25.7631	25.4154	25.3198	36
Sears, Roebuck And Co.	S	60.5833	20.8942	25.1632	25.5317	25.1548	36
Southwest Airlines	LUV	34.8611	24.6734	24.3807	24.7166	24.5128	36
TXU Corporation	TXU	83.3611	19.9234	21.5444	25.9118	23.9295	36
Union Oil Co. Of California	UCL1	59.9444	26.3321	25.1084	26.0188	25.7945	36
Wal-Mart Stores Inc.	WMT	13.5000	13.8871	6.9627	6.9274	7.9734	36
Xerox Corp.	XRX	161.2778	25.3679	26.4067	25.9497	26.2770	36
Texas Instruments Inc.	TXN	40.8889	24.1582	25.5791	24.3188	24.5635	36

Exhibit 3: Model 2 estimates of λ_0, λ_1 for various firms over the time period 8/21/00 to 10/31/00 measured in basis points.

	Avg λ_0	**Avg StdErr$_0$**	**Avg λ_1**	**Avg StdErr$_1$**	**SSE**	**N**
American Airlines Inc.	173.6869	7.0377	11.0057	0.4488	22309.6399	36
Archer-Daniels-Midland Co.	42.8829	3.4159	2.7034	0.2179	4445.6399	36
C P & L Energy Inc.	60.1086	2.6531	3.6905	0.1696	3038.6800	36
Chase Manhattan Corp.	36.6415	1.7476	2.1326	0.1118	1571.4400	36
Coca-Cola Enterprises	42.5605	2.1703	2.6177	0.1389	2123.6799	36
Delta Air Lines Inc.	145.0000	4.8327	9.2301	0.3082	9815.5600	36
Dow Chemical Company	34.6153	3.4997	2.1120	0.2228	5401.7199	36
Eastman Kodak Co.	31.4411	1.6846	1.9607	0.1080	1130.6400	36
First Union Corp.	38.4624	1.7125	2.3262	0.1095	1250.4000	36
J.P. Morgan & Company	25.4682	1.7772	1.8859	0.1138	1247.8400	36
K Mart Corp.	426.0039	24.9005	26.9134	1.5758	250994.0401	36
Lyondell Chemical Company	364.3603	7.8434	23.0640	0.4992	23250.6800	36
Merrill Lynch & Co.	37.6026	3.6423	2.3920	0.2329	4427.4799	36
Phillips Petroleum Co.	70.3085	3.3042	4.3899	0.2111	5338.9999	36
Ralston-Ralston Purina Group	72.2258	3.0590	4.4654	0.1955	4628.8799	36
Sears, Roebuck And Co.	69.2179	2.8086	4.2433	0.1790	3715.8799	36
Southwest Airlines	41.6399	1.8105	2.5913	0.1157	1906.9599	36
TXU Corporation	94.6842	4.9341	6.2449	0.3151	10842.6399	36
Union Oil Co. Of California	70.6460	3.0475	4.4091	0.1948	4535.6399	36
Wal-Mart Stores Inc.	17.0430	1.0653	1.3275	0.0680	577.7998	36
Xerox Corp.	191.8220	12.5238	12.2826	0.8001	64820.0399	36
Texas Instruments Inc.	48.6580	2.5356	3.2647	0.1618	2769.0399	36

Exhibit 4: American Airline Parameter Estimates Across Time



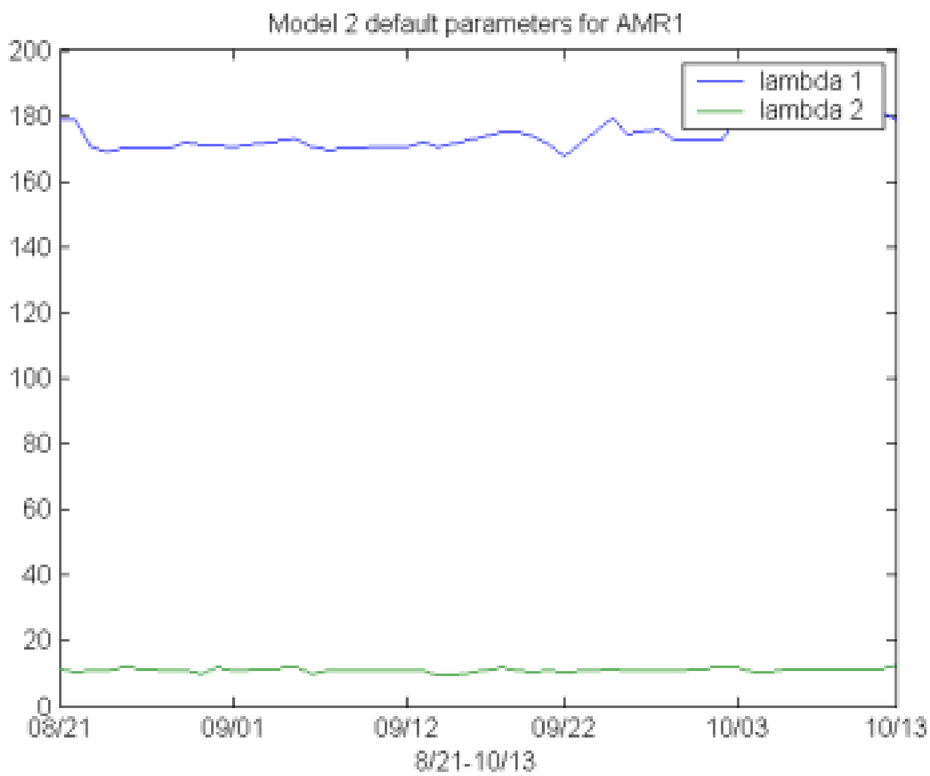
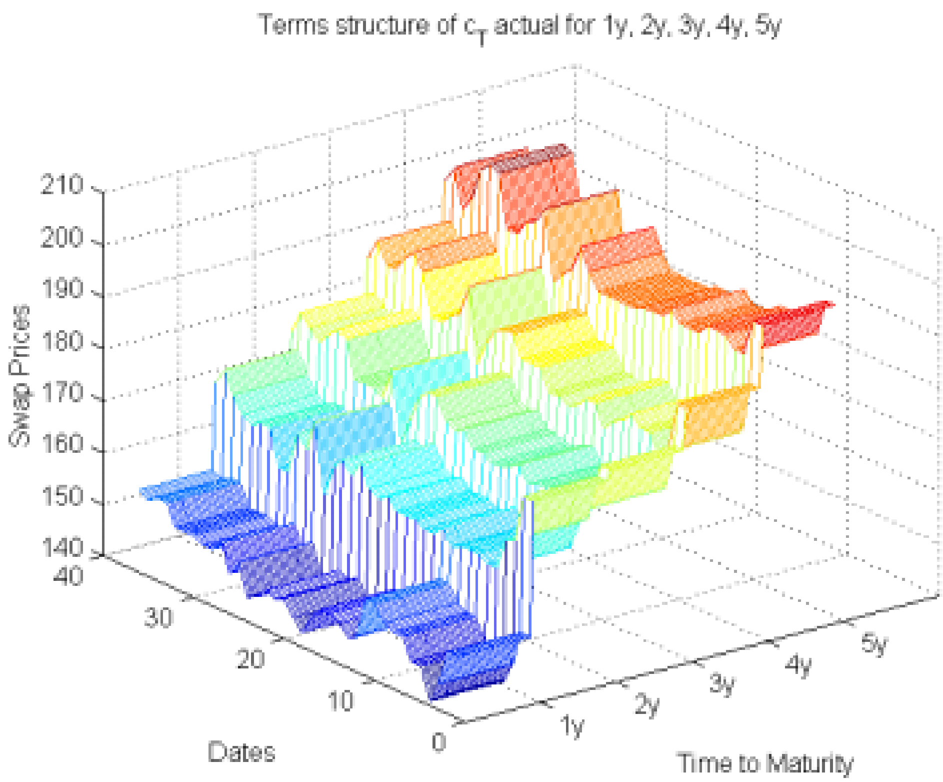


Exhibit 5: American Airlines Default Swap Quotes and the Model 2 Default Swap Rates



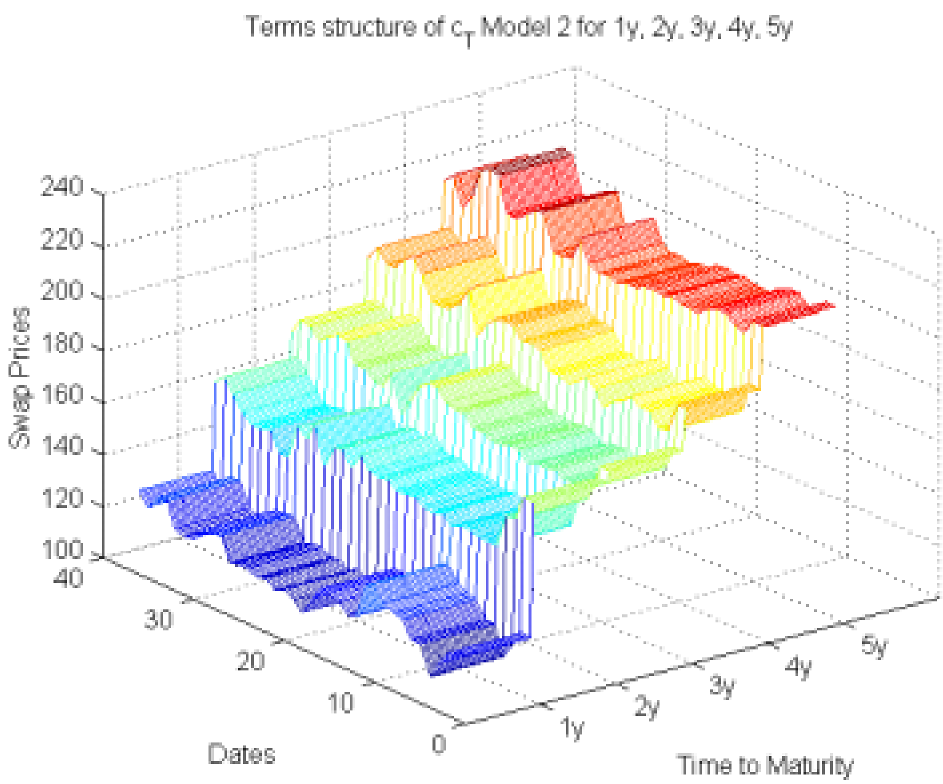


Exhibit 6: Model 2 errors for $c_0,...,c_5$ for various firms over the time period 8/21/00 to 10/31/00 measured in basis points. The average standard error is included underlying each point estimate.

	AvgErr c_1	AvgErr c_2	AvgErr c_3	AvgErr c_4	AvgErr c_5
American Airlines Inc.	-24.6944	-7.3333	0.6389	9.5556	21.8333
	3.1905	2.6952	1.3622	1.7463	1.9744
Archer-Daniels-Midland Co.	-10.5833	-4.8333	0.4444	4.5000	10.4722
	3.4367	3.4311	3.6392	3.6144	4.5645
C P & L Energy Inc.	-9.0389	-2.9278	0.1833	3.7389	8.0444
	1.6684	1.5684	1.5332	1.4003	1.9943
Chase Manhattan Corp.	-6.3889	-1.3889	0.5000	2.2222	5.0556
	1.7071	0.8879	1.0749	0.7646	1.3366
Coca-Cola Enterprises	-7.4222	-2.3111	0.8556	2.3278	6.5500
	2.0033	1.3027	2.0821	1.6938	1.9581
Delta Air Lines Inc.	-15.3944	-6.8667	0.8833	5.6333	15.7444
	6.0568	2.2111	1.8721	3.0971	4.3759
Dow Chemical Company	-11.2500	-5.4167	0.8611	5.9167	9.8889
	4.9149	4.1537	2.7036	2.8619	6.6045
Eastman Kodak Co.	-4.8444	-2.4000	0.0167	2.1278	5.1000
	2.8574	1.5314	0.8423	1.3758	2.4858
First Union Corp.	-5.7778	-1.9444	0.1111	2.4167	5.1944
	1.1786	1.1044	0.9555	0.8133	1.4307
J.P. Morgan & Company	-5.7000	-2.7833	0.7167	2.3556	5.4111
	1.4948	1.4653	1.0303	0.8157	1.6227
K Mart Corp.	-73.7944	-58.3500	30.5667	43.3722	58.2056
	39.6238	37.2448	24.0359	28.8007	31.0385
Lyondell Chemical Company	-16.2167	-16.1889	1.6722	8.8389	21.8944
	19.8447	7.1202	4.9350	5.5078	6.0216
Merrill Lynch & Co.	-10.7278	-7.7000	2.3278	5.8833	10.2167
	2.8506	2.5704	2.0564	1.5583	1.7776
Phillips Petroleum Co.	-12.0611	-3.5333	2.0778	4.6056	8.9111
	1.7077	1.7709	1.5472	2.2309	1.7737
Ralston-Ralston Purina	-11.0667	-2.8444	1.4056	3.9056	8.6000
	2.5067	1.8640	1.7660	1.7563	1.7882

Sears, Roebuck And Co.	-8.9056	-4.7667	1.9833	4.1222	7.5667
	4.9591	4.1491	3.3563	2.9039	2.8451
Southwest Airlines	-6.9444	-1.2778	0.9722	2.7222	4.5278
	2.2094	0.7518	1.2367	1.1782	1.3808
TXU Corporation	-11.7222	-6.0556	0.0556	7.9444	9.7778
	12.9790	8.4247	9.9607	7.1668	10.9853
Union Oil Co. Of California	-10.9833	-3.0111	1.4611	3.9611	8.5722
	2.3472	1.9201	1.7500	1.4965	1.6823
Wal-Mart Stores Inc.	-3.6278	-0.8500	0.1500	1.0111	3.3167
	1.7239	0.8314	0.9941	0.8280	0.8911
Xerox Corp.	-31.3278	-23.0778	21.9778	-2.7722	35.2000
	29.0264	29.8426	43.4229	32.5142	44.3323
Texas Instruments Inc.	-7.9778	-4.5333	1.4111	3.8278	7.2722
	3.6949	4.0469	2.3890	2.4271	2.9460

Averages Across all firms	**AvgErr c_1**	**AvgErr c_2**	**AvgErr c_3**	**AvgErr c_4**	**AvgErr c_5**
	-13.9295	-7.7452	3.2396	5.8280	12.6071
	6.9083	5.4949	5.2066	4.8433	6.1732