Prices and Spreads in Sequential Markets with Information Imperfections

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Abstract

Two modes of information imperfections—information asymmetry about future cash flows and demand uncertainty—are investigated in a sequential market with two distinguishing characteristics. The first is that market makers have a shorter horizon than that of investors: they are in business to make money from trading with investors rather than from holding the stock until its liquidation value is realized. The second is that both informed and uninformed investors optimize over the quantities of the risky asset they wish to hold. A pooling equilibrium is constructed to resemble the setup of traditional sequential trade models where informed investors take advantage of their private information while hiding among the uninformed investors. The first issue investigated is the statistical behavior of transaction prices. In contrast to the implication of traditional sequential trade models, the transaction price process is not a martingale with respect to the information set of the market makers (i.e., the “public” information set). Rather, the first difference of the transaction price process is positively autocorrelated. The second issue investigated is the interaction between information asymmetry and demand uncertainty (or uncertainty about the composition of the investor population) that was introduced in Saar (2000). Demand uncertainty creates a situation where the introduction of information asymmetry can decrease rather than increase the spread. Hence, with more than one information imperfection in the market, there is no longer a monotone relation between information asymmetry and the spread.
Prices and Spreads in Sequential Markets with Information Imperfections

The role played by market makers in markets with information imperfections has been investigated at length in the market microstructure literature. Two basic assumptions are common to most papers in this area. The first is that market makers have the same horizon as investors. Market makers are assumed to be present in the market at all times to accommodate investors who arrive and want to trade, but their preferences are very similar to those of investors in that they are defined over their wealth when the firm is liquidated after the end of trading. Rather than searching for the price that equates the investors’ supply and demand of shares, these market makers use the order flow only to learn information about the future dividends of the stock. The most prevalent formulation includes market makers with risk neutral preferences and unconstrained resources and so their presence in the market forces prices to be the conditional expected values of the stock’s payoffs. This set up is shared by traditional sequential trade models (Glosten and Milgrom, 1985; Easley and O’Hara, 1987, 1991, 1992; Diamond and Verrecchia, 1987) and by models that follow the Kyle (1985) framework.\(^1\) The second assumption commonly used has to do with the nature of the information imperfection. Most papers assume that there are investors in the market with private information about the future cash flows of the firm. Since they have more information than the rest of the market, their existence creates the problem of “information asymmetry.”\(^2\) Papers in this literature share two main results about prices and spreads: (i) the spread (or price impact) is monotonically increasing in the extent of information asymmetry, and (ii) the transaction price process is a martingale with respect to the information set of the market makers.

A few papers consider market makers with objective functions and constraints defined over the trading period rather than over the end of the economy, but these papers do not investigate information imperfections. For example, Garman (1976) examines the market

\(^1\) For a survey of market microstructure theory see O’Hara (1995).

\(^2\) Another formulation is that of “diverse information,” where the information set of one investor is neither equal nor finer than the information set of another investor. Still, the information in the hands of investors is about the future cash flows of the firm.
power of a market maker who maximizes expected profit per unit time subject to a constraint that his inventory does not drift. Amihud and Mendelson (1980) investigate inventory control by a market maker with a similar objective function but with two bounds on his feasible inventory position. A few recent papers also emphasize the existence of information in financial markets unrelated to the future cash flows of the firm. Saar (2000) shows how demand uncertainty, which is formalized as uncertainty about the composition of the investor population (their preferences and endowments), can create informational effects in prices similar to those described by models that utilize information asymmetry about future cash flows. The expertise of market makers in assessing the demand for the stock is shown to affect prices, volume and the welfare of investors. Lyons (1997) and Cao and Lyons (1999) generate private information in the foreign exchange multiple-dealer setting. Each dealer has sole knowledge of his customers’ orders, and this inventory information gives rise to speculative trading and thus affects prices in the market.3

The goal of this paper is to investigate how information imperfections affect prices in a sequential market where market makers care about supply and demand during the trading period rather than the realization of dividends after trading is over. This seems a more realistic depiction of the manner in which market makers actually price: they constantly search for the price that approximately equates the flow of shares bought and sold by investors, and are generally not interested in holding the stock as an investment vehicle to benefit from its future prospects. As Mayer (1988) notes, “In general, NYSE specialists do not take a view of where a stock is going over time. They are in business not to maximize the value of their inventory but to maximize the turnover of their capital” (p. 211). In particular, I want to examine whether the results of traditional sequential trade models about prices and spreads remain intact when market makers behave in this fashion and when there are multiple sources of information imperfections in the market. Researchers have used spreads

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3 Another related paper is by Kraus and Smith (1989) who stress that uncertainty about future prices can reflect the beliefs, preferences and endowments of the participants in the economy. They refer to this uncertainty as “market created risk” to emphasize that its source is the investors themselves rather than the future cash flows of a firm.
as a proxy for information asymmetry and have tried to identify information effects in prices using estimation procedures based on the martingale property. Are these uses robust to a different, and perhaps more realistic, description of market makers’ preferences and a more general model of information? To examine these issues, I first look at how the introduction of information asymmetry about future cash flows affects the statistical properties of transaction prices. Then, I introduce demand uncertainty to the market and investigate how it interacts with information asymmetry to affect spreads.

The basic setup of the model is similar to the one used in Saar (2000). There are two assets in the economy, a risky stock and a riskless bond. Investors arrive to the market one at a time according to an orderly point process. There are two types of investors with different sets of endowments who maximize constant absolute risk aversion (CARA) expected utility of their final wealth, taking market prices as given. There are also competitive, risk neutral market makers in the economy who maximize expected profit per unit time from trading with the investors. To be able to deliver shares to the investors, market makers have to maintain a balanced inventory. This is formalized by requiring that they equate the expected flow of shares bought and sold. This formulation of the market makers is similar to the one used in Garman (1976) and Brock and Kleidon (1992). Competition among market makers eliminates the market power component of the spread and simplifies the investigation of information imperfections.\footnote{Saar (2000) explores both the monopolist and competitive market maker settings and shows how competition indeed drives the market power spread component to zero. The results that I present here also hold in a market where trading is facilitated by a monopolist market maker rather than many competitive market makers.}

Information asymmetry about future cash flows is introduced by assuming that a fraction of the investor population observes a signal (either good or bad) about the mean of the distribution of the liquidation payoff of the stock. To facilitate the comparison with the results of traditional sequential trade models, I focus on the pooling equilibrium explored in that literature whereby informed investors buy if they observe a good signal and sell if they observe a bad signal. In addition, informed investors choose to “hide” by submitting
orders for the same number of shares as the orders submitted by the uninformed investors. Like in the traditional sequential trade models, market makers condition on the information content of the arriving orders and this creates a spread between the bid and the ask prices. In contrast to these models, however, the transaction price process is not a martingale with respect to the information set of the market makers or the “public” information set. Rather, there is a term in the price process that adjusts to information with a lag, and hence the first difference of the transaction price process is positively autocorrelated.

This result is driven by two elements that are absent from traditional sequential trade models. The first element is that uninformed investors optimize over the quantity of the stock they wish to hold. When submitting an order, an uninformed investor knows that she has no private information and this fact is reflected in her optimal order size. The second element driving this result is that market makers in this paper are performing two tasks. On the one hand, they have rational expectations and so are conditioning on the new information in each incoming order when they set prices. On the other hand, they balance expected supply and demand and so must take into account that the order size of an arriving uninformed investor (and hence the order size chosen by an informed investor as well) reflects an information set that does not condition on her order. Hence, the prices that market makers set cannot fully reflect the new information and stay a bit “behind.” Transaction prices in the model can be decomposed into a risk premium term that reflects the updated information set of the market makers and a risk neutral term that is determined by the beliefs of the uninformed investors. This risk neutral term is updated with a lag, creating the positive autocorrelation of transaction price differences.

The manner in which information imperfections affect prices is further examined by introducing demand uncertainty into the economy alongside information asymmetry about future cash flows. I follow Saar (2000) in modeling demand uncertainty by assuming that the relative population weights of investors who belong to different types (i.e., who have different endowments) are unknown. Market makers have a prior over the population weights of the
different types and they use the order flow to learn about the true distribution. While Saar (2000) shows how uncertainty about the population of investors affects prices, I focus on the question of how information asymmetry about future cash flows and demand uncertainty interact in a unified framework. More specifically, traditional sequential trade models show how the spread is increasing in the extent of information asymmetry. This is also the situation in the model employed here when information asymmetry is the only information imperfection in the market. The introduction of demand uncertainty changes the picture.

When demand uncertainty in the market is sufficiently high, the introduction of information asymmetry about future cash flows decreases the spread. When demand uncertainty is low, information asymmetry increases the overall spread but decreases the magnitude of the component of the spread that is due to demand uncertainty. The intuition behind these results is that market makers learn about the population of investors from observing orders. Without information asymmetry, a sell order implies that the arriving investor has a large endowment. Market makers update their beliefs about the distribution of investors and are willing to accommodate the sell order only at a lower price that reflects the information that there are more investors in the market with large endowments. Like in traditional sequential trade models, the spread is created by this process of updating beliefs. When there is information asymmetry about future cash flows in the market, an investor with a small endowment may nonetheless sell shares if she observes a bad signal. When market makers see a sell order, they are not sure whether it is motivated by a low signal or by a large endowment. Hence, there is no longer full revelation of the endowment of an arriving investor, and market makers are unable to revise their beliefs to the same extent. Market makers therefore only partially adjust prices to reflect endowment information and the spread component that arises due to demand uncertainty decreases.

If demand uncertainty is sufficiently high, the inability of market makers to fully infer the type of an arriving investor is the dominant effect and the spread decreases despite the addition of an information asymmetry component to the spread. This result shows that the
relationship between information asymmetry about future cash flows and spreads may not be monotone as predicted by traditional sequential trade models. Furthermore, Saar (2000) shows that econometric spread decomposition procedures that are used in the literature to identify the adverse selection component of the spread are also picking up information about demand. Hence, the information component identified by these procedures (and not only the spread as a whole) may decrease as information asymmetry is introduced into the environment. Identifying the extent of information asymmetry about a stock may therefore require employing various controls for the effects of demand uncertainty.

The rest of this paper proceeds as follows. Section 1 describes the economy and Section 2 introduces the information imperfections. In particular, Section 2.2 investigates information asymmetry about future cash flows and derives the statistical properties of the transaction price process, and Section 2.3 adds demand uncertainty and investigates the properties of the spread with both modes of information imperfections. Section 3 discusses the results and offers concluding remarks.

1 Economy

1.1 Assets

There are two assets in the economy. The first asset is a stock that pays \( \tilde{u} \) dollars at time \( T' \).\(^5\) In addition to uncertainty about the liquidation dividend \( \tilde{u} \), there is uncertainty about the mean of the distribution from which \( \tilde{u} \) is drawn, \( \tilde{\theta} \), where:

\[
\tilde{\theta} = \begin{cases} 
\theta_H & \text{with probability } \rho \\
\theta_L & \text{with probability } 1 - \rho 
\end{cases}
\]

Nature draws \( \theta \in \{\theta_H, \theta_L\} \) before the market opens at time zero, but \( \theta \) need not be known to investors. I will assume that each investor treats \( \tilde{u} \) as being normally distributed with a mean that is equal to the expected value of \( \theta \) given the investor’s information set and variance \( \sigma^2 \). This is done to simplify the analysis since learning about \( \theta \) will only affect the

\(^5\)The risky asset is referred to as a stock only for exposition. The model can be applied to other financial assets as well.
mean and not the variance of the distribution of $\bar{u}$. Equivalently, we could assume that $\bar{u}$ is normally distributed and calculate each investor’s conditional distribution of $\bar{u}$ as a Bernoulli mixture of normal random variables. The resulting distribution would be normal with the same mean as before but with a variance that is also affected by the learning process. The intuition behind the results developed in this paper is similar under both formulations, and so for simplicity I adopt the first approach. The second asset is a riskless bond that pays $R$ dollars at time $T'$. The market for the stock opens at time zero and trading can take place until time $T$, where $T < T'$.

### 1.2 Investors

Investors arrive to the market according to an orderly point process $G(t)$ with a non-zero intensity function (or arrival rate) $g(t)$ that is conditionally independent of the distribution of order sizes. This family of processes includes many familiar stochastic processes such as the Poisson and non-homogeneous Poisson processes. Like in traditional sequential trade models, the sequential arrival of investors to the market is taken as a primitive of the economic system. I do not explicitly consider reasons such as transaction costs, information costs, and decision making costs that can cause investors to arrive one at a time.

The economy is populated by two types of investors, where an investor’s type is indexed by the subscript $i \in \{1, 2\}$. The relative population weight of type 1 is $q$ and the relative population weight of type 2 is $1-q$. Given the arrival of an investor, the probability that she belongs to a certain type is equal to that type’s relative population weight. Each investor has endowments of the stock ($\bar{X}_t$) and the riskless bond ($\bar{M}_t$). All investors maximize constant absolute risk aversion expected utility of their wealth at time $T'$ (the end of the economy when the liquidation payoffs of the assets are realized) with coefficient of risk aversion $\alpha$.\(^6\)

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\(^6\)An orderly point process allows for only one arrival at any point in time. For a standard definition of orderly point processes see Cox and Isham (1980).

\(^7\)The results in this paper are not materially affected by using different coefficients of risk aversion for the two types of investors. Negative exponential utility functions and normally distributed liquidation payoffs are used in this paper for mathematical tractability. This framework has known drawbacks (e.g., unlimited liability) and results in prices that can be negative for some parametric specifications. In what follows, I
Some investors have access to information about \( \theta \). An investor is called informed (denoted by a superscript \( I \)) if she observes \( \theta \) and uninformed (denoted by a superscript \( U \)) otherwise. The relative weight of informed investors in the population is \( \nu \) and the relative weight of the uninformed investors is \( 1-\nu \). Given the arrival of an investor, the probability that she is informed (uninformed) is equal to the relative weight of the informed (uninformed) investors in the population. Access to information is independent of an investor’s type, so that an investor observes \( \theta \) with probability \( \nu \) irrespective of her type.

All investors behave competitively in the sense that they take market prices as given. An investor who traded to re-balance her portfolio does not return to the market. Under these assumptions, the optimal demand for the stock of an investor belonging to type \( i \in \{1, 2\} \) with information set \( j \in \{I, U\} \) who arrives to the market at time \( t \), \( D_{i,t}^j \), solves the following problem:

\[
\max_{D_{i,t}^j} E \left[ -e^{\alpha W_{i,t'}} | \mathcal{I}_t^j \right]
\]

\[\text{s.t. } R M_{i,t}^j + \tilde{u} D_{i,t}^j = W_{i,t'} \]

\[M_{i,t}^j + P_t D_{i,t}^j = \bar{M}_i + P_t \bar{X}_i \]

where \( \mathcal{I}_t^j \) is the information set of the investor, \( P_t \) is the price at which the investor can transact in the stock, and the price of the riskless bond is set to unity. The solution to this problem is well known and the optimal demand is:\(^8\)

\[D_{i,t}^j = \frac{E \left[ \theta | \mathcal{I}_t^j \right] - R P_t}{\alpha \sigma^2} \]

### 1.3 Market Makers

In formulating the market makers’ objectives, I try to capture the essence of their activity. Bagehot (1971) writes that “it is well known that market makers of all kinds make surprisingly little use of fundamental information. Instead, they observe the relative pressure of

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\(^8\)See, for example, Grossman and Stiglitz (1980).
buy and sell orders and attempt to find a price that equilibrate these pressures.” (p. 14).\(^9\) Mayer (1988) also notes that market makers are not interested in taking a position in the stock based on long-term forecasts. Rather, they are constantly searching for the prices at which the flows of shares bought and sold are approximately equal. They are making money by opening a spread, and their constant search for a market clearing price keeps them in business since their inventory does not drift without bound. This behavior sets them apart from investors who trade infrequently to re-balance a portfolio of investments they hold for prolonged periods of time.

Market makers in this paper are not investors—they are not interested in holding the stock until the liquidation payoff is realized. Market makers are in the business of making money by providing trading services to investors who arrive to the market and demand liquidity. I adopt a specification of a market maker’s objective that is similar to the one used by Garman (1976) and Brock and Kleidon (1992). Each market maker in the economy maximizes expected profit per unit time subject to the constraint that his inventory does not drift.\(^10\) In other words, a market maker sets prices at every instant of time by maximizing the expected revenue from selling shares to investors minus the expected cost of buying shares from investors subject to the constraint that the number of shares bought and sold is the same on average. Garman notes that since a market maker exchanges cash for stock and vice versa, these two must be reduced to identical units of measurement. Hence, this formulation of the problem is intuitively appealing since it “forces” the market maker to take profit in cash by permitting no drift in his inventory position. This formulation seems more realistic as an approximation of actual market maker behavior than the traditional formulation in which market makers are similar to investors in that they care about the liquidation payoff of the stock.

Let \(\mathcal{M}_t\) be the information set of a market maker at time \(t\). Let \(P_{i,t}^j\) be the price the

\(^9\) Walter Bagehot was a pseudonym used by Jack L. Treynor.

\(^10\) Amihud and Mendelson (1980) use a similar specification but impose bounds on the allowable inventory position of the market maker instead of requiring a zero drift.
market maker is quoting to an investor who belongs to type $i \in \{1, 2\}$ with information set $j \in \{I, U\}$ who arrives to the market at time $t$ and submits an order $X_{i,t}^{j}$, where $X_{i,t}^{j} > 0$ ($X_{i,t}^{j} < 0$) is interpreted as a buy (sell) order. The general formulation of the market maker’s problem is:

$$\max_{\{P_{1,t}^{U}, P_{1,t}^{I}, P_{2,t}^{U}, P_{2,t}^{I}\}} E \left[ \Pi_{t} \mid \mathcal{M}_{t} \right]$$

s.t. $E \left[ Z_{t} \mid \mathcal{M}_{t} \right] = 0$ \hspace{1cm} (5)

where,

$$\Pi_{t} = q\nu P_{1,t}^{I} X_{1,t}^{I} + q(1 - \nu)P_{1,t}^{U} X_{1,t}^{U} + (1 - q)\nu P_{2,t}^{I} X_{2,t}^{I} + (1 - q)(1 - \nu)P_{2,t}^{U} X_{2,t}^{U}$$ \hspace{1cm} (7)

$$Z_{t} = q\nu X_{1,t}^{I} + q(1 - \nu)X_{1,t}^{U} + (1 - q)\nu X_{2,t}^{I} + (1 - q)(1 - \nu)X_{2,t}^{U}$$ \hspace{1cm} (8)

The constraint that the inventory position of a market maker should not drift with time deserves some attention. Conceptually, buying shares at a very low price and selling shares at a very high price could give the market maker unbounded profits. At these prices, however, it may be that the number of shares bought is much smaller than the number of shares sold, and the market maker could not “deliver” the shares. To have a steady state, a market maker should try to buy shares at a lower price and sell at a higher price, but set these prices in such a way as to balance the number of shares he buys and sells. The market maker here is doing exactly that and hence can be viewed as a personification of the intuition of a secondary market. Without the firm issuing additional securities, prices in a secondary market must adjust such that the number of shares bought and sold is the same. Market makers are needed since investors arrive one at a time, but the market makers basically attempt to balance the expected flow of shares they buy and sell every moment. It is this constraint that gives the market makers’ prices the “flavor” of equating supply and demand. While this form of inventory management may seem a bit restrictive, it is very simple and very useful for the task at hand.

My goal is not to examine how market makers manage their inventories. In Stoll (1978) and Ho and Stoll (1981), inventory control is driven by risk aversion of a market maker
who seeks to maintain an optimal portfolio position. In Amihud and Mendelson (1980), the
market maker must not let the level of inventory get above or below certain bounds. Here, I
abstract from specific characteristics of market makers such as their degree of risk aversion
or wealth constraints since my interest is not in evaluating how these characteristics affect
pricing. Rather, I want to investigate the influence of different modes of information on prices
in a sequential secondary market. Therefore, the market maker’s problem satisfies the simple
requirement that on average, inventory will not have a drift and so supply and demand of
shares in the market will be approximately the same. In fact, this can be viewed as a market
clearing condition in expectations. While this setting abstracts from many constraints that
market makers have in the real world, it contains the essence of the situation described by
Mayer (1988) and Bagehot (1971), and provides a powerful tool for investigating different
modes of information.

Garman (1976) and Saar (2000) show how a monopolist market maker sets prices such
that the price at which investors buy shares is higher than the price at which they sell shares
(i.e., there is a spread) and the market maker has positive expected profit. In the same
framework that is used here, Saar (2000) also shows that competition among market makers
will drive the expected profit of market makers (and the market power spread component)
to zero. The intuition is that of a Bertrand competition, but the expected market clearing
constraint also plays a role. If a single market maker deviates by posting a better price on
one side of the market, he attracts all the order flow only from that side and violates the
no drift constraint. Hence, he has to improve prices on both sides of the market in order
to capture the entire order flow and make positive expected profit. Each market maker
attempts to improve prices by a small amount (i.e., increasing the bid and decreasing the
ask) until his expected profit and the spread in the market are driven to zero.\footnote{In Dennert (1993), increasing the number of market makers increases rather than decreases the spread. The driving force behind his result is that an informed investor trades a certain quantity with each market maker while an uninformed investor trades the same quantity with only one market maker. As a result, the total order size of an informed investor is much larger than that of an uninformed investor. Here, on the other hand, the total order size of an informed investor is identical to that of an uninformed investor, and the order is equally divided among all market makers. Hence, increasing the number of market makers does
 increase the spread.}
focus here is on the manner in which information imperfections affect prices, I abstract from the issue of market power by assuming that there is a large number (with measure one) of competing market makers who set prices such that they make zero expected profit.\textsuperscript{12} Hence, all market makers quote the same price and execute an equal share of the orders that arrive to the market.

2 Modes of Information

2.1 Full Information

I will first establish the benchmark “full information” case in which the market makers’ information set includes all parameters of the asset (including $\theta$), all characteristics of the investor population and past order flow. In general, I assume that market makers do not have special information about the asset that is unknown to investors. Full information therefore means that all investors in the market know the realization of $\theta$, and $\nu$ disappears from (7) and (8). When an investor arrives to the market, she submits an order for $X_{t,t} = D_{t,t}^I - X_i$ shares (where $D_{t,t}^L = D_{t,t}^U$). I will construct an equilibrium using the usual methodology. First, I will conjecture that the market makers can identify the type of an arriving investor from her order size (or direction), and derive the prices they will set. Second, I will use these prices to find the optimal orders of the investors in terms of the primitives of the economy, and show that the orders of the two types are indeed different (confirming the conjecture).

Looking at the market makers’ problem in (5)–(8), we can use the constraint in (6) to write $X_{1,t}$ in terms of $X_{2,t}$, and then plug it into the objective function. Then, imposing zero

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\textsuperscript{12}I also assume that the horizon of trading is short enough so that most market makers are able to meet their obligations and do not go bankrupt. Garman (1976) discusses the possibility of bankruptcy of a market maker who starts the trading period with a certain amount of cash and cannot borrow. Due to the stochastic arrival of buy and sell orders, a market maker with a zero expected profit constraint will hit any boundary we can impose on his cash position with probability one. Since this issue is not my main concern here, I assume that at most a finite number of market makers go bankrupt before the end of trading. Alternatively, one could assume that market makers have unlimited borrowing ability as in traditional sequential trade models.
expected profit results in the following equation:

\[(1 - q)X_{2,t}(P_{2,t} - P_{1,t}) = 0\]  

(9)

Hence, in every economy with trading (i.e., \(X_{2,t} \neq 0\) and \(X_{1,t} \neq 0\)), it must be the case that \(P_{1,t} = P_{2,t} = P_t\) and all market makers are setting a single price to clear every order that arrives to the market.

We can plug the optimal demands of the two types of investors into (6):

\[E[Z_t] = q \left( \frac{\theta - RP_t}{\alpha \sigma^2} - X_1 \right) + (1 - q) \left( \frac{\theta - RP_t}{\alpha \sigma^2} - X_2 \right) = 0\]  

(10)

and solve for the equilibrium price:

\[P_t = \frac{\theta}{R} - \frac{X^* \alpha \sigma^2}{R} \quad \forall t\]  

(11)

where \(X^* = qX_1 + (1 - q)X_2\). This price can then be used to find the optimal orders of the two types of investors in terms of the primitives of the economy:

\[X_{1,t} = -(1 - q)(\bar{X}_1 - \bar{X}_2), \quad X_{2,t} = q(\bar{X}_1 - \bar{X}_2) \quad \forall t\]  

(12)

Inspection of (12) reveals that in any economy with trading (i.e., \(\bar{X}_1 \neq \bar{X}_2\)), investors will submit orders such that \(X_{1,t} \neq X_{2,t}\), and hence the conjecture that the quantity of shares is sufficient to identify the investor type is verified. The economy in which a type 1 investor is a seller (buyer) and a type 2 investor is a buyer (seller) is the one that satisfies \(\bar{X}_1 - \bar{X}_2 > 0\) (\(\bar{X}_1 - \bar{X}_2 < 0\)).

The equilibrium price set by the market makers has the usual structure from asset pricing models: a risk neutral component, the mean of the stock’s payoff distribution divided by the risk free rate, and a risk premium that depends on the relative population weight \(q\). This price also preserves an important notion of optimality in that it is equal to the competitive equilibrium price in an economy that is similar in all respects except that all investors arrive to the market at the same time. The no inventory drift constraint under full information is therefore equivalent to the market clearing condition of the competitive equilibrium and no
market-maker-specific characteristics affect the market price. Hence, market makers are just a conduit through which shares are transferred from sellers to buyers and the market price depends solely on the characteristics of the investor population.

2.2 Information Asymmetry

Information asymmetry is introduced into the market by assuming that only informed investors observe $\theta$, the mean of the stock’s payoff distribution. One of the important features of the market is that trading is anonymous. The only information market makers have about an arriving investor is contained in the order the investor submits (i.e., number of shares and direction). There is no independent verification of an investor’s type or information set.\(^{13}\) Market makers therefore seek to learn from the order flow about $\theta$. I am interested in focusing on a situation similar to that described in traditional sequential trade models, where (i) informed investors pool with the uninformed investors, and (ii) an informed investor who observes a good signal always buys the stock while an informed investor who observes a bad signal always sells the stock. Hence, I will also construct an equilibrium where informed investors buy (sell) if they observe $\theta_H$ ($\theta_L$) and use the same size of orders as the uninformed investors to avoid fully revealing their information.\(^ {14}\)

The traditional sequential trade model literature defines the transaction price as the equilibrium price that reflects all information available to the public and to the market maker including the information contained in the transaction itself (i.e., semi-strong price efficiency). In these models, the price that the market maker sets to execute a transaction for a single quantity of shares, $Q$, is just the conditional expectation of the liquidation payoff when the information set includes the arrival of the order, $P_Q = \frac{E[\pi|Q]}{R}$. Hence, at any point in time there is only one equilibrium price—the price of the most recent transaction. Looking

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\(^{13}\)For the situation where market makers may have information about traders beyond that contained in the orders see Battalio and Holden (1997).

\(^{14}\)In Easley and O’Hara (1987), informed investors can choose between small and large trade sizes. There are two equilibria in their framework: one where informed and uninformed investors use both trade sizes and another where informed investors use only the large trade size while uninformed investors use both trade sizes.
forward, the market maker can determine in advance what prices he will set for different incoming orders. Since a buy order \((B)\) contains different information than a sell order \((S)\), the equilibrium prices conditional on these orders will be different, or \(P_B = \frac{E[u|B]}{R} \neq \frac{E[u|S]}{R} = P_S\). Note, however, that these two prices never exist at the same time—each is conditional on a different information set. Nonetheless, if the market maker is asked what will be the prices at which he will execute an incoming buy order or an incoming sell order, he could quote these two prices. Hence, the quote describes potential equilibrium prices.

A similar situation takes place in this paper. Market makers set a price by solving the problem in (5)–(8) subject to a zero expected profit constraint and conditional on a single information set. This information set will now include the information contained in the incoming order in addition to all available public information. For convenience, I slightly change the notation of the market makers’ information set by writing down separately the incoming order and all other information that is available to the market makers. Hence, \(\mathcal{M}_{t,X_1} (\mathcal{M}_{t,X_2})\) is the information set of the market makers when the investor who arrives to the market submits an order for \(X_{1,t}\) (\(X_{2,t}\)) shares. The reason that there are only two possible order sizes is the requirement that informed investors pool with the uninformed investors, and these two sizes will be equal to the optimal orders of the two types of uninformed investors.

The optimal order sizes of the uninformed investors can be found from (4) using the information set of the uninformed investors, \(\mathcal{I}_t^U\):

\[
X_{1,t} = \frac{\theta_t - RP_t}{\alpha \sigma^2} - \bar{X}_1, \quad X_{2,t} = \frac{\theta_t - RP_t}{\alpha \sigma^2} - \bar{X}_2
\]

(13)

where \(\theta_t = \rho_t \theta_H + (1 - \rho_t) \theta_L\) and \(\rho_t = P(\theta = \theta_H | \mathcal{I}_t^U)\). The uninformed investors, like the market makers, are learning from the order flow about \(\theta\). They start the day with a prior, \(\rho\), and use Bayes rule to update their beliefs. It should be intuitively clear that unlike the market makers, uninformed investors do not condition on the contemporaneous transaction when they arrive to the market. Market makers who see an order do not know if it comes from an informed or from an uninformed investor. They therefore set a price that takes into
account the probability that this order comes from an informed investor. An uninformed investor who arrives to the market, on the other hand, knows for sure that she has no information and therefore does not condition on the size of her order. Furthermore, she does not condition on the price set by the market makers since her information set is in fact finer. In other words, while she needs to take the price as given to determine how many shares to trade, she does not use the price to update her beliefs about $\theta$ since she knows that her order does not convey any information and hence her beliefs should not change.

To simplify the exposition, I assume that $\bar{X}_1 - \bar{X}_2 > 0$ without loss of generality since we could always rename the two types of investors. From (12), this is the condition for type 1 investors to be sellers and type 2 investors to be buyers in the full information economy. I will construct the equilibrium in three steps. First, I will write down the trading strategies of investors assuming that informed investors pool with the uninformed investors. Second, I will solve for the price that the market makers will set for each type of order that can arrive. Third, I will derive the conditions under which informed investors who face these prices indeed want to pool with the uninformed investors. These will be the conditions for the existence of the pooling equilibrium.

Consider the following trading strategies of the investors whereby (i) informed investors pool with uninformed investors, and (ii) informed investors buy (sell) if they observe good (bad) signals:

\[
\begin{array}{ccc}
\text{Uninformed} & \text{Informed} \\
\text{Investors} & & \\
X_{1,t} & X_{1,t} & \text{if } \theta = \theta_L \\
X_{2,t} & X_{2,t} & \text{if } \theta = \theta_H \\
\end{array}
\]

(14)

Assume for now that it is indeed optimal for the informed investors to pool with the uninformed investors. In order to price an incoming order for $X_{1,t}$ shares, market makers plug the trading strategies of the informed and uninformed investors into (6) and use $M_{t,X_1}$ as
their conditioning information set. Note that whether informed investors buy or sell depends on $\theta$. Market makers start the day with a prior on $\theta$ and use the order flow to update their beliefs. Since both $q$ and $\nu$ are known, (6) can be written with the conditional probability $\rho_{1,t} = P(\theta = \theta_{H} \mid M_{t,X_{1}})$ as follows:

$$ E[Z_{t} \mid M_{t,X_{1}}] = q\nu[\rho_{1,t}X_{2,t} + (1 - \rho_{1,t})X_{1,t}] + $$

$$ q(1 - \nu)X_{1,t} + (1 - q)\nu[\rho_{1,t}X_{2,t} + (1 - \rho_{1,t})X_{1,t}] + (1 - q)(1 - \nu)X_{2,t} = 0 \quad (15) $$

Since the same information set is used in the objective function and in the expected market clearing constraint, we can perform the same manipulation as in (9) and get a single equilibrium price conditional on the arrival of an order for $X_{1,t}$ shares. Plugging (15) into (5) and setting the expected profit equal to zero yields:

$$ [\nu(1 - \rho_{1,t}) + (1 - \nu)q]X_{1,t}(P_{1,t}^{U} - P_{2,t}^{U}) = 0 \quad (16) $$

Hence, in every economy with trading (i.e., $X_{2,t} \neq 0$ and $X_{1,t} \neq 0$), it must be the case that $P_{1,t}^{U} = P_{2,t}^{U} = P_{1,t}^{e}$, where the notation $P_{1,t}^{e}$ is used for the single equilibrium price when the market makers’ information set is $M_{t,X_{1}}$.

We can find $P_{1,t}^{e}$ by plugging the optimal orders into (15) as follows:

$$ \left(\frac{\theta_{t} - R\rho_{1,t}}{\alpha\sigma^{2}} - \bar{X}_{1}\right)[\nu(1 - \rho_{1,t}) + (1 - \nu)q] + $$

$$ \left(\frac{\theta_{t} - R\rho_{1,t}}{\alpha\sigma^{2}} - \bar{X}_{2}\right)[\nu\rho_{1,t} + (1 - \nu)(1 - q)] = 0 \quad (17) $$

Solving for $P_{1,t}^{e}$ yields:

$$ P_{1,t}^{e} = \frac{\theta_{t} - \alpha\sigma^{2}X_{1}^{U}}{R} \quad (18) $$

where $X_{1}^{U} = a_{1,t}X_{1} + (1 - a_{1,t})X_{2}$ and $a_{1,t} = \nu(1 - \rho_{1,t}) + (1 - \nu)q$. A different price will prevail when the conditioning information set contains the arrival of an order for $X_{2,t}$ shares, $M_{t,X_{2}}$. Like in traditional sequential trade models, these two prices can never exist at the same time, but they can be given the quote interpretation if $X_{1,t}$ is an order to sell shares and $X_{2,t}$ is
an order to buy shares. Market makers set the price of an incoming order for $X_{2,t}$ shares by solving (15) but replacing $\rho_{1,t}$ with $\rho_{2,t} = P (\theta = \theta_H \mid \mathcal{M}_{t,X_2})$. The resulting price is:

$$P_{2,t}^\infty = \frac{\theta_t - \alpha \sigma^2 X_{2,t}^\nu}{R}$$  \hspace{1cm} (19)

where $X_{2,t}^\nu = a_{2,t} \bar{X}_1 + (1 - a_{2,t}) \bar{X}_2$ and $a_{2,t} = \nu (1 - \rho_{2,t}) + (1 - \nu) q$. It is straightforward to show that at these prices, $X_{1,t}$ is a sell order and $X_{2,t}$ is a buy order. To complete the construction of the equilibrium, we have the following proposition:

**Proposition 1** It is optimal for investors to carry out the trading strategies in (14) when the market makers post the quote:

$$P_t = \begin{cases} 
\frac{\theta_t - \alpha \sigma^2 X_{2,t}^\nu}{R} & \text{for all orders } X = X_{2,t} \\
\infty & \text{for all orders } X \neq X_{2,t}, X > 0 \\
\frac{\theta_t - \alpha \sigma^2 X_{1,t}^\nu}{R} & \text{for all orders } X = X_{1,t} \\
0 & \text{for all orders } X \neq X_{1,t}, X < 0
\end{cases}$$  \hspace{1cm} (20)

when the following two conditions hold:

$$2(1 - \rho_t)(\theta_H - \theta_L) + \alpha \sigma^2 (X_{2,t}^\nu + \bar{X}_2 - 2 \bar{X}_1) \geq \max \left\{ C \left[ -2(1 - \rho_t)(\theta_H - \theta_L) - \alpha \sigma^2 (X_{1,t}^\nu - \bar{X}_1) \right], 0 \right\}$$  \hspace{1cm} (21)

$$2\rho_t(\theta_H - \theta_L) - \alpha \sigma^2 (X_{1,t}^\nu + \bar{X}_1 - 2 \bar{X}_2) \geq \max \left\{ \frac{1}{C} \left[ -2\rho_t(\theta_H - \theta_L) + \alpha \sigma^2 (X_{2,t}^\nu - \bar{X}_2) \right], 0 \right\}$$  \hspace{1cm} (22)

where $C = \frac{\nu \rho_{1,t} + (1 - \nu)(1 - q)}{\nu(1 - \rho_{2,t}) + (1 - \nu) q}$.

Proofs of all the propositions are provided in the Appendix. Conditions (21) and (22) guarantee that the participation and incentive compatibility constraints of the pooling equilibrium are satisfied. These conditions hold when there is a sufficient difference between good and bad information (i.e., $\theta_H$ is sufficiently larger than $\theta_L$) and when not all private information has been incorporated into the prices (i.e., $\rho_t$ is not too close to either zero or one). Also, these conditions are more likely to hold when $\sigma^2$ is not too high.

When prices incorporate most of the private information, there may come a time when these conditions are no longer satisfied because $\theta_t$ is too close to either $\theta_H$ or $\theta_L$. In such
a case, other equilibria are possible. For example, it can be shown that there is a pooling equilibrium where each informed investor submits the order of an uninformed investor of the same type, regardless of her private information. This happens when pooling with the other investor type becomes too costly and trading on the private information is not profitable enough. In this case, the order flow stops being informative about \( \theta \) and the market makers quote a single price that does not change. This equilibrium is not as interesting since no one is trading on private information.\(^\text{15}\) In the rest of this section, I want to discuss the more interesting equilibrium, where the order flow conveys information and prices adjust accordingly.

The market makers’ price schedule in (20) shows that they execute buys (of size \( X_{2,t} \)) and sells (of size \( X_{1,t} \)) using two different equilibrium prices, \( P_{2,t}^e \) and \( P_{1,t}^e \). Hence, the introduction of information asymmetry creates a “spread,” which is defined here to mean the difference between the potential equilibrium prices for buying and selling.

**Proposition 2** In an economy with information asymmetry and where conditions (21) and (22) hold,

\[
S_{t}^{IA} = P_{2,t}^{e} - P_{1,t}^{e} = \frac{\alpha\sigma^2(X_{1} - \bar{X}_{2})\nu(\rho_{2,t} - \rho_{1,t})}{R} = \frac{\alpha\sigma^2(X_{1} - \bar{X}_{2})\nu^2\rho_{t}(1 - \rho_{t})}{R\alpha(1 - \alpha)} > 0
\]

where \( \rho_{t} = P(\theta = \theta_{H} | \mathcal{M}_{t}) \) and \( a_{t} = \nu(1 - \rho_{t}) + (1 - \nu)q \). The spread is monotonically increasing in the relative weight of the informed investors, \( \nu \).

In this model, like in the traditional sequential trade models, the spread is the difference between the price at which market makers are willing to buy shares and the price at which they are willing to sell shares. Each price is conditional on a different information set and so as transaction prices, they can never really exist at the same time. With rational

\(^{15}\)The normal-exponential framework and the manner in which market makers set prices do not allow for a fully-revealing, separating equilibrium (since order sizes in equilibrium are independent of the private information). See Glosten (1989) for an example of a separating equilibrium in a trading model with risk averse investors and information asymmetry.
expectations, the market makers can be viewed as saying: “if a buy order will arrive, we will execute it at the price $P_{1,t}^c$, and if a sell order will arrive, we will execute it at the price $P_{2,t}^c$.” This declaration constitutes the quote. The spread is increasing in the extent of informed trading ($\nu$) and with the riskiness of the stock ($\sigma^2$). This result is similar in nature to the spread result of traditional sequential trade models. The difference between the predictions of the two types of models arises when one considers the statistical properties of transaction prices.

In traditional sequential trade models, the transaction price process is a martingale. The reason is that prices are simply conditional expected values of the future liquidation payoff. Here, on the other hand, prices are set by market makers who do not behave like investors (i.e., do not have the same horizon as investors). Instead, market makers are intermediaries who provide liquidity services for a fee. When they are competitive and risk neutral, this fee is zero, but the intuition remains the same. To carry out their business, they need to balance on average the number of shares bought and sold. As I will now show, this constrains their ability to quote prices that reflect all available information.

Denote the transaction price process by $\{P_{1,t_n}^c\}_{n=1}^N$, where $t_n$ stands for the time of the $n$th transaction. Note that the information set of the market makers (or the uninformed investors) at any time between two transactions is the same as their information set immediately following the most recent transaction. After the transaction at $t_{n-1}$ (and before the one at $t_n$), market makers set two rational expectations prices, or a quote, for the next transaction. These prices are:

$$\text{Ask} \equiv P_{2,t_{n-1},t_n}^c < t \leq t_n = \theta_{t_n} - \frac{\alpha \sigma^2 X_{2,t_{n-1}}^\nu}{R}, \quad \text{Bid} \equiv P_{1,t_{n-1},t_n}^c < t \leq t_n = \theta_{t_{n-1}} - \frac{\alpha \sigma^2 X_{1,t_{n-1}}^\nu}{R}.$$

The first term of these prices is the expectation of $\theta$ using the posterior beliefs at $t_{n-1}$, which are the prior beliefs at $t_n$. This term is in the quote since an uninformed investor optimizes using her information set without conditioning on her own order. The second term involves a weighted average of the endowments of type 1 and type 2 investors using the posterior beliefs about $\theta$ at $t_n$, where the posterior takes into account the arrival of the next order
(either a buy or a sell). Market makers are conditioning on the arriving order to get a better estimate of the risk premium, which is determined by the supply and demand for the risky asset and therefore depends on whether informed investors are buying or selling.

When an order arrives at \( t_n \), it is executed at the price that reflects the nature of the order, and the transaction price can be written as:

\[
P^{tr}(t_n) = \frac{\theta_{t_{n-1}}}{R} - \frac{\alpha \sigma^2 X_{t_{n-1}}^\nu}{R}
\]

The posterior of the market makers’ beliefs, \( \rho_{t_n} \), becomes the new prior. The next uninformed investor who arrives to the market does not know whether the previous order originated from an uninformed or from an informed investor, and hence she updates her information set and solves for the optimal demand that reflects her information set at \( t_n \). When setting the new quote, market makers are going one step further and condition on both the information at time \( t_n \) and the order that will arrive at \( t_{n+1} \):

\[
\text{Ask} \equiv P_{2, t_n < t \leq t_{n+1}}^e = \frac{\theta_{t_n}}{R} - \frac{\alpha \sigma^2 X_{t_{n-1}}^\nu}{R}, \quad \text{Bid} \equiv P_{1, t_n < t \leq t_{n+1}}^e = \frac{\theta_{t_n}}{R} - \frac{\alpha \sigma^2 X_{t_{n-1}}^\nu}{R}
\]

and the next transaction will therefore execute at the price:

\[
P^{tr}(t_{n+1}) = \frac{\theta_{t_n}}{R} - \frac{\alpha \sigma^2 X_{t_{n+1}}^\nu}{R}
\]

The first term of the transaction price changed from \( \frac{\theta_{t_{n-1}}}{R} \) to \( \frac{\theta_{t_n}}{R} \) due to the updating of beliefs by the uninformed investors. The second term changed from \( \frac{\alpha \sigma^2 X_{t_{n-1}}^\nu}{R} \) to \( \frac{\alpha \sigma^2 X_{t_{n+1}}^\nu}{R} \) due to the updating of beliefs by the market makers. It is always the case that the first term of the transaction price reflects an information set that is “lagging” behind the information set that is used for the second term. More generally, we have the following result:

**Proposition 3** In an economy with information asymmetry and where conditions (21) and (22) hold,

\[
E \left[ P^{tr}(t_{n+1}) - P^{tr}(t_n) \mid \mathcal{M}_{t_n} \right] = \\
\left\{ \begin{array}{ll}
-\frac{\sqrt{\rho_{n-1}}(1-\rho_{n-1})\theta_H\theta_L}{2\theta_H\theta_L} & \text{if the } n\text{th transaction was a sell} \\
-\frac{\sqrt{\rho_{n-1}}(1-\rho_{n-1})\theta_H\theta_L}{2\theta_H\theta_L} & \text{if the } n\text{th transaction was a buy}
\end{array} \right.
\]

(25)

21
Proposition 3 shows that knowing the sign of the most recent transaction can tell us something about the next transaction price. This effect disappears if there are no informed investors in the market (i.e., $\nu = 0$) or if there is no uncertainty about the signal (i.e., $\rho_t = 0$ or $\rho_t = 1$).

The progression of the transaction price process depends on both the lagged information effect of Proposition 3 and the spread effect of Proposition 2. In traditional sequential trade models, the spread result is sufficient to guarantee that the execution price of a sell (buy) is lower (higher) than the price of the previous transaction. Here, I need to impose a condition that states that the direct price impact of the transaction (Proposition 2) is stronger than the lagged information effect of the previous transaction (Proposition 3).

\[
\alpha \sigma^2 (\bar{X}_1 - \bar{X}_2) \nu \frac{\rho_{t+1} (1 - \rho_{t+1})}{a_{t+1}} > (\theta_H - \theta_L) \frac{\rho_{t-1} (1 - \rho_{t-1})}{1 - a_{t-1}} \Rightarrow P_{1,t} - P_{t} \quad (26)
\]

\[
\alpha \sigma^2 (\bar{X}_1 - \bar{X}_2) \nu \frac{\rho_{t+1} (1 - \rho_{t+1})}{a_{t+1}} > (\theta_H - \theta_L) \frac{\rho_{t-1} (1 - \rho_{t-1})}{1 - a_{t-1}} \Rightarrow P_{2,t} - P_{t} > 0 \quad (27)
\]

When we combine the conditions from (26) and (27)—that prices go down on sells and up on buys—with Proposition 3, we get the result that the first difference of the transaction price process is positively autocorrelated. The reason is that if the transaction at time $t_n$ is a sell, (26) guarantees that $P_{t} - P_{t-1} < 0$, while Proposition 3 shows that $E [P_{t} - P_{t-1} | M_t] < 0$. Similarly, if the transaction at $t_n$ is a buy, (27) guarantees that $P_{t} - P_{t-1} > 0$, while Proposition 3 shows that $E [P_{t} - P_{t-1} | M_t] > 0$. So, each transaction generates two effects. A buyer order will have a positive contemporaneous price impact (for the same reason that asks are higher than bids) and a delayed effect that will contribute a positive price impact to the next transaction price irrespective of the nature of the next transaction. Similarly, a sell order will have a negative contemporaneous price impact and a delayed effect that will contribute a negative price impact to the next transaction price irrespective of the nature of that transaction.

\footnote{If we want to allow for the possibility that prices do not change with each transaction, the strict inequalities in (26) and (27) can be replaced with weak inequalities. The only difference down the road is that instead of positive autocorrelation of price differences we can talk about non-negative autocorrelation.}
What is the intuition behind this result? All investors in the market submit orders to buy or sell using the quantities of shares that are optimal for the uninformed investors. An uninformed investor who arrives to the market and needs to calculate her demand does not condition on her order or the market price since she knows that her order is not motivated by private information. Market makers who observe an order do not know if it comes from an informed or from an uninformed investor. They therefore set a price that takes into account the probability that this order comes from an informed investor. In other words, the information set of the uninformed investors is $\mathcal{I}^U_t = \mathcal{M}_t$ while the one used by the market makers is $\mathcal{M}_{t,X_t}$, so that the updating process of the market makers’ beliefs is one step ahead of the updating process of the uninformed investors. Since market makers serve as a conduit through which investors overcome the intertemporal problem of finding a trading counterpart, they are not completely free to set whatever price they wish. In order to equate supply and demand, they have to set a price that takes into account that the arriving uninformed investor uses the information set $\mathcal{I}^U_t$. The next uninformed investor who will arrive to the market, however, does not know if the previous transaction was initiated by an informed or by an uninformed investor. She conditions on the previous transaction and her expectation of the mean of the stock’s payoff is taken using $\mathcal{I}^U_{t,X_t}$. Hence, the first term of the price is always updated with a lag.

Since the market makers’ information set at each point in time is equivalent to the information set of all uninformed investors except for the one who submits an order exactly at that time, the transaction price process is not a martingale with respect to the “public” information set. Still, no one in the economy is able to arbitrage away this effect. I return to discuss this issue and why it should be true even if we consider a more flexible model in Section 3.
2.3 Demand Uncertainty and Information Asymmetry

In principle, uncertainty about demand in the market can arise from uncertainty about many different elements in the economic environment (endowments, preferences, information sets, private investment opportunities, and so on). Saar (2000) defines the concept of “demand uncertainty” more narrowly to describe uncertainty about the distribution of preferences and endowments of investors in the market. While such uncertainty can arise in different market structures, Saar (2000) relates it to sequential markets where different investors arrive to trade at different points in time. The postulate is that when not all investors are in the market all the time, no market observer has knowledge of the overall demand for the asset. In other words, market makers are uncertain about the preference and endowments of investors who have not yet arrived to trade. Relating sequential arrival to demand uncertainty naturally leads to the issue of learning, where each arrival of an investor causes a revision in the market’s beliefs about the composition of the investor population. Saar (2000) shows that these revisions create spreads and affect volume, volatility, and the welfare of investors in the market.

Demand uncertainty can also be viewed from the perspective of information differentials since before arriving to the market, each investor has a piece of information only she knows—her own demand (or equivalently, her preferences and endowments). When she arrives to the market, her order reveals her demand and prices adjust as market makers update their beliefs about the distribution of preferences and endowments among investors. Demand uncertainty therefore gives rise to informational effects in prices where this information is not about the future cash flows of the firm but about the investors who demand the stock.

I follow Saar (2000) in modeling uncertainty about the composition of the investor population by assuming that the market makers’ information set does not include \( q \), the parameter that represents the distribution of types in the population. Since all investors have the same coefficient of risk aversion, the distribution over types in this paper is simply a distribution over the endowments of investors. From the point of view of the market makers, nature
draws a realization of $q$ from a certain distribution before the market opens. Market makers have a prior on $q$ at time zero denoted by $f^0(q)$. The prior distribution can be rather general but its support should be in $[0, 1]$. The prior can be interpreted as the experience market makers develop for assessing demand. Each time an investor arrives and submits an order, market makers use Bayes rule to update their beliefs about the composition of the investor population. To simplify the exposition, I assume that the distribution of types in the population and the distribution of the mean of the future cash flows are independent.

Assume for now that there is no information asymmetry about $\theta$. We can construct an equilibrium as before by conjecturing (and subsequently verifying) that the market makers can identify the types of investors from their orders. The expected market clearing condition in (6) can be used to price an incoming order of size $X_{1,t}$ as follows:

$$
E[Z_t \mid \mathcal{M}_{t, X_1}] = q_{1,t} \left( \frac{\theta - RP_{1,t}^e}{\alpha_1 \sigma^2} \right) + (1 - q_{1,t}) \left( \frac{\theta - RP_{1,t}^e}{\alpha_2 \sigma^2} \right) = 0
$$

where $q_{1,t} = E[q \mid \mathcal{M}_{t, X_1}]$. The only difference of the above from (10) is that here the population parameter $q$ is replaced by its conditional expected value given the arrival of an order of a type 1 investor. Solving for $P_{1,t}^e$ we get,

$$
P_{1,t} = \frac{\theta - \alpha \sigma^2 \hat{X}_{1,t}}{R}
$$

where, $\hat{X}_{1,t} = q_{1,t} \bar{X}_1 + (1 - q_{1,t}) \bar{X}_2$. A similar analysis can be carried out for the price that will prevail if an order of a type 2 investor arrives. The conditioning information set in this case is $\mathcal{M}_{t, X_2}$, and the resulting price is:

$$
P_{2,t}^e = \frac{\theta - \alpha \sigma^2 \hat{X}_{2,t}}{R}
$$

where, $\hat{X}_{2,t} = q_{2,t} \bar{X}_1 + (1 - q_{2,t}) \bar{X}_2$ and $q_{2,t} = E[q \mid \mathcal{M}_{t, X_2}]$.

Like in traditional sequential trade models, at any point in time there is only one equilibrium price—the price of the most recent transaction. However, it is straightforward to show that these prices imply that $X_{1,t}$ is a sell order and $X_{2,t}$ is a buy order, and therefore $P_{1,t}^e$ and $P_{2,t}^e$ can be given the quote interpretation. Since investors are price takers, their orders
reveal to the market makers their types and market makers are able to execute each order at the appropriate price.

The arrival of an investor belonging to type $i \in \{1, 2\}$ causes market makers to believe that there are more investors of type $i$ in the market. In particular, the following relationships hold:\textsuperscript{17}

\begin{align*}
E[q | \mathcal{M}_t, X_t] &= E_t[q] + \frac{V_t[q]}{E_t[q]} > E_t[q] \\
E[q | \mathcal{M}_t, X_t^x] &= E_t[q] - \frac{V_t[q]}{1 - E_t[q]} < E_t[q]
\end{align*}

(30) \hspace{1cm} (31)

where $E_t[q] = E[q | \mathcal{M}_t]$, and $V_t[q] = E[q^2 | \mathcal{M}_t] - (E_t[q])^2$. These relationships can be used to show the following result:\textsuperscript{18}

**Proposition 4** In the economy with demand uncertainty, there exists a positive spread between the bid and ask prices:

\begin{align*}
S^e_t DU &= P^e_{2,t} - P^e_{1,t} = \frac{\alpha \sigma^2 (\bar{X}_1 - \bar{X}_2)(q_{1,t} - q_{2,t})}{R} \\
&= \frac{\alpha \sigma^2 (\bar{X}_1 - \bar{X}_2)V_t[q]}{RE_t[q](1 - E_t[q])} > 0
\end{align*}

(32) \hspace{1cm} (33)

The spread is monotonically increasing in the extent of demand uncertainty ($V_t[q]$).

Market makers attempt to learn about the population of investors from the arriving orders. This creates a situation where prices move with each order and a spread is created since buy and sell orders have price impacts in opposite directions. Note that market makers use only the first two moments of their beliefs about $q$ when they set prices. Hence, the extent of demand uncertainty in the market can be represented by the variance of the market makers’ beliefs about $q$. The larger the variance, the more uncertainty there is about the population parameter. When $V_t[q]$ is equal to zero, we are back in the case of full information. While the above proposition states that the spread increases with the extent of uncertainty about the composition of the population of investors, traditional sequential trade models focus on

\textsuperscript{17}For a proof see Proposition 1 in Saar (2000).

\textsuperscript{18}This proposition is a simplified version of Proposition 2 and Proposition 4 in Saar (2000).
the spread that is created as a result of information asymmetry about future cash flows. The pooling equilibrium introduced in Section 2.2 can be used to investigate the interaction of information asymmetry and demand uncertainty in a unified framework.

Let both q and θ be unknown. The pooling equilibrium is constructed as before by assuming that investors follow the strategies in (14), finding the prices that market makers set, and solving for the conditions that make the pooling equilibrium’s trading strategies optimal for investors who face these prices. We can use the assumption that q and θ are drawn from independent distributions to simplify the expected market clearing condition and find the price that market makers will set for executing an arriving order for X_{1,t} shares:

\[ E[Z_t | \mathcal{M}_{t,X_1}] = \hat{q}_{1,t}\nu[\hat{\rho}_{1,t}X_{2,t} + (1 - \hat{\rho}_{1,t})X_{1,t}] + \hat{q}_{1,t}(1 - \nu)X_{1,t} \]

\[ + (1 - q)\nu[\hat{\rho}_{1,t}X_{2,t} + (1 - \hat{\rho}_{1,t})X_{1,t}] + (1 - \hat{q}_{1,t})(1 - \nu) \]

\[ = \left( \frac{\theta_t - R\hat{P}^{*}_{1,t}}{\alpha \sigma^2} - \hat{X}_1 \right) \left( \nu(1 - \hat{\rho}_{1,t}) + (1 - \nu)\hat{q}_{1,t} \right) \]

\[ + \left( \frac{\theta_t - R\hat{P}^{*}_{1,t}}{\alpha \sigma^2} - \hat{X}_2 \right) \left( \nu\hat{\rho}_{1,t} + (1 - \nu)(1 - \hat{q}_{1,t}) \right) \]

\[ = 0 \]

where \( \hat{q}_{1,t} = E[q | \mathcal{M}_{t,X_1}] \) and \( \hat{\rho}_{1,t} = P(\theta = \theta_H | \mathcal{M}_{t,X_1}) \). I am using the notation \( \hat{q}_{1,t} \) instead of \( q_{1,t} \) and \( \hat{\rho}_{1,t} \) instead of \( \rho_{1,t} \) to emphasize that these posteriors are taken with respect to the joint distribution of q and θ.

The price that solves this condition is:

\[ \hat{P}_{1,t} = \frac{\theta_t - \alpha \sigma^2 \hat{X}_{1,t}}{R} \]

(34)

where \( \hat{X}_{1,t} = \hat{a}_{1,t}\hat{X}_1 + (1 - \hat{a}_{1,t})\hat{X}_2 \) and \( \hat{a}_{1,t} = \nu(1 - \hat{\rho}_{1,t}) + (1 - \nu)\hat{q}_{1,t} \). Similarly, market makers set the price of an incoming order for \( X_{2,t} \) shares by solving the above expected market clearing equation with \( \mathcal{M}_{t,X_2} \) as the conditioning information set. The resulting price is:

\[ \hat{P}_{2,t} = \frac{\theta_t - \alpha \sigma^2 \hat{X}_{2,t}}{R} \]

(35)

where \( \hat{X}_{2,t} = \hat{a}_{2,t}\hat{X}_1 + (1 - \hat{a}_{2,t})\hat{X}_2 \) and \( \hat{a}_{2,t} = \nu(1 - \hat{\rho}_{2,t}) + (1 - \nu)\hat{q}_{2,t} \). It is straightforward to show that at these prices, \( X_{1,t} \) is indeed a sell order and \( X_{2,t} \) is indeed a buy order. Consider the
following quote posted by the market makers at any time \( t \) prior to the arrival of an order:

\[
P_t = \begin{cases} 
P_{2,t}^e & \text{for all orders } X = X_{2,t} \\
\infty & \text{for all orders } X \neq X_{2,t}, X > 0 \\
P_{1,t}^e & \text{for all orders } X = X_{1,t} \\
0 & \text{for all orders } X \neq X_{1,t}, X < 0 
\end{cases} \tag{36}
\]

The conditions in (21) and (22) (with simple modifications) still guarantee the existence of
the pooling equilibrium.\(^{19}\)

In order to investigate the spread in the market and its relation to the information
asymmetry spread in (23) and the demand uncertainty spread in (32), we need the following
proposition:

**Proposition 5** *In the pooling equilibrium with information asymmetry and demand uncer-
\[\hat{q}_{1,t} = E[q | \mathcal{M}_{t,X_1}] = \hat{q}_t + \frac{(1 - \nu)V_t[\hat{q}]}{\nu(1 - \hat{\rho}_t) + (1 - \nu)\hat{q}_t} > \hat{q}_t \tag{37}\]
\[\hat{q}_{2,t} = E[q | \mathcal{M}_{t,X_2}] = \hat{q}_t - \frac{(1 - \nu)V_t[\hat{q}]}{\nu\hat{\rho}_t + (1 - \nu)(1 - \hat{q}_t)} < \hat{q}_t \tag{38}\]
\[\hat{\rho}_{1,t} = P(\theta = \theta_H | \mathcal{M}_{t,X_1}) = \hat{\rho}_t \left[ \frac{(1 - \nu)\hat{q}_t}{\nu(1 - \hat{\rho}_t) + (1 - \nu)\hat{q}_t} \right] < \hat{\rho}_t \tag{39}\]
\[\hat{\rho}_{2,t} = P(\theta = \theta_H | \mathcal{M}_{t,X_2}) = \hat{\rho}_t \left[ \frac{\nu + (1 - \nu)(1 - \hat{q}_t)}{\nu\hat{\rho}_t + (1 - \nu)(1 - \hat{q}_t)} \right] > \hat{\rho}_t \tag{40}\]

where \( \hat{\rho}_t = E[q | \mathcal{M}_t] \), \( V_t[\hat{q}] = E[q^2 | \mathcal{M}_t] - \hat{q}_t^2 \), and \( \hat{\rho}_t = P(\theta = \theta_H | \mathcal{M}_t) \).

Proposition 5 describes the evolution of the market makers’ beliefs in an economy with
information asymmetry and demand uncertainty. It shows that the arrival of \( X_{1,t} \), a sell
order, causes market makers to believe that there are more type 1 investors in the population
and that there is a lower likelihood of good information (i.e., \( \theta = \theta_H \)). Similarly, the arrival
of \( X_{2,t} \), a buy order, causes market makers to believe that there are more type 2 investors
in the population and that there is a higher likelihood of good information. We can now
describe the spread between the price at which market makers are willing to buy and the
price at which they are willing to sell as follows:

\(^{19}\)The modifications are replacing \( C = \frac{\nu \rho_{1,t} (1 - \nu)(1 - \omega)}{\nu (1 - \omega)(1 - \nu) q} \setting, \ X_{1,t}^v, \ \text{and} \ X_{2,t}^v \) by \( \tilde{C} = \frac{\nu \hat{\rho}_{1,t} (1 - \nu)(1 - \omega)}{\nu (1 - \omega)(1 - \nu) \hat{q}_t} \setting, \ \hat{X}_{1,t}^v, \ \text{and} \ \hat{X}_{2,t}^v \), respectively.
Proposition 6 In the pooling equilibrium with information asymmetry and demand uncertainty, there exists a positive spread between the bid and ask prices:

\[ S_{1,t}^{IA} + DU = P_{2,t}^e - P_{1,t}^e = \frac{\alpha \sigma^2 (\bar{X}_1 - \bar{X}_2) [\nu (\hat{\rho}_{2,t} - \hat{\rho}_{1,t}) + (1 - \nu)(\hat{q}_{1,t} - \hat{q}_{2,t})]}{R} \]

(41)

\[ = \frac{\alpha \sigma^2 (\bar{X}_1 - \bar{X}_2) [\nu^2 \hat{\rho}_t (1 - \hat{\rho}_t) + (1 - \nu)^2 V_t[\hat{q}] - V_t[\hat{q}]]}{R \hat{a}_t (1 - \hat{a}_t)} > 0 \]  

(42)

where \( \hat{a}_t = \nu (1 - \hat{\rho}_t) + (1 - \nu)\hat{q}_t \).

The spread expression in (41) is comprised of a weighted average of two terms. The first term characterizes the extent of information asymmetry and is the difference between the posterior of good information conditional on a buy order and the posterior of good information conditional on a sell order, \( \hat{\rho}_{2,t} - \hat{\rho}_{1,t} \). The second term characterizes the extent of demand uncertainty and is the difference between the posterior of the population parameter \( q \) conditional on a sell order and the posterior of \( q \) conditional on a buy order, \( \hat{q}_{1,t} - \hat{q}_{2,t} \). The weight on the information asymmetry term is the relative population weight of the informed investors while the weight on the demand uncertainty term is the relative population weight of the uninformed investors. Furthermore, eliminating demand uncertainty by setting \( V_t[\hat{q}] = 0 \) reduces (42) to the information asymmetry spread in (24), and eliminating information asymmetry by setting \( \nu = 0 \) reduces (42) to the demand uncertainty spread in (33).

While both demand uncertainty and information asymmetry about future cash flows give rise to a spread when each information imperfection is alone in the market, the interesting question is whether one kind of information imperfection influences the manner in which another kind of information imperfection affects market prices. Say we start with an economy in which there is information asymmetry but no demand uncertainty. Then we introduce demand uncertainty (so that \( \hat{\rho}_t = \rho_t \)) and look at the effect on the quote. The question that we ask is whether the addition of demand uncertainty increases or decreases the spread.

\[ S_{1,t}^{IA} - S_{1,t}^{IA} + DU = \frac{\alpha \sigma^2 (\bar{X}_1 - \bar{X}_2)}{R} [\nu [(\rho_{2,t} - \hat{\rho}_{2,t}) + (\hat{\rho}_{1,t} - \rho_{1,t})] - (1 - \nu)(\hat{q}_{1,t} - \hat{q}_{2,t})] \]

(43)
Note that the term multiplied by $\nu$ in the above expression compares $\hat{\rho}_{1,t}$ and $\hat{\rho}_{2,t}$, which are functions of $\hat{q}_t$, to $\rho_{1,t}$ and $\rho_{2,t}$, which are functions of $q$. Since the market makers’ beliefs about the expected value of $q$ can be above or below the true population parameter, these two sets of posteriors cannot be easily compared.

Nonetheless, assume that the market makers’ beliefs are in fact centered on the true population parameter. Then, the first term of (43) is equal to zero and the second term is:

$$-\frac{\alpha \sigma^2(\bar{X}_1 - \bar{X}_2)(1 - \nu)(\hat{q}_{1,t} - \hat{q}_{2,t})}{R} = -\frac{\alpha \sigma^2(\bar{X}_1 - \bar{X}_2)(1 - \nu)^2 V_t[\hat{q}]}{R \hat{a}_t(1 - \hat{a}_t)} < 0$$

Hence, $S^{{\text{IA+DU}}}_t$ is greater than $S^{{\text{IA}}}_t$. The introduction of demand uncertainty when the market makers’ expectations of $q$ are close to the population parameter (irrespective of the variance) increases the spread. While the magnitude of the demand uncertainty component in the presence of information asymmetry may be different than without it, the spread is still monotonically increasing in the extent of demand uncertainty ($V_t[\hat{q}]$).

Going in the other direction, say we start with an economy in which there is demand uncertainty but no information asymmetry. Then we introduce information asymmetry (so that $\hat{q}_t = E_t[\hat{q}]$ and $V_t[\hat{q}] = V_t[\hat{q}]$) and look at the effect on the spread.

$$S^{{\text{DU}}}_t - S^{{\text{IA+DU}}}_t = \frac{\alpha \sigma^2(\bar{X}_1 - \bar{X}_2)(q_{1,t} - q_{2,t})}{R} - \frac{\alpha \sigma^2(\bar{X}_1 - \bar{X}_2) [\nu(\hat{\rho}_{2,t} - \hat{\rho}_{1,t}) + (1 - \nu)(\hat{q}_{1,t} - \hat{q}_{2,t})]}{R}$$

$$= -\frac{\alpha \sigma^2(\bar{X}_1 - \bar{X}_2) \nu}{R} \left[ \frac{\nu \hat{\rho}_t (1 - \hat{\rho}_t)}{\hat{a}_t (1 - \hat{a}_t)} \right]$$

$$V_t[\hat{q}] \left[ \frac{1}{\hat{q}_t (1 - \hat{q}_t)} + (1 - \nu) \left( \frac{1 - \hat{\rho}_t}{\hat{q}_t \hat{a}_t} + \frac{\hat{\rho}_t}{(1 - \hat{q}_t)(1 - \hat{a}_t)} \right) \right] \right)$$

While the first term inside the brackets is positive, the second term is negative and includes $V_t[\hat{q}]$ that is absent from the first term. Hence, the higher the variance of the market makers’ beliefs about $q$, the more negative the term in the brackets and the larger is $S^{{\text{DU}}}_t$ relative to $S^{{\text{IA+DU}}}_t$. This means that when there is a sufficient amount of demand uncertainty in the market, introducing information asymmetry about future cash flows can decrease rather than increase the spread.
What is the intuition behind this result? Market makers learn about the population of investor from observing orders. Without information asymmetry, a sell order for $X_{1,t}$ shares implies that a type 1 investor arrived to the market. We can think about it as if the private information of an investor is her own type, and the order she submits results in full revelation of her private information. Market makers then update their beliefs about the distribution of investors so that their expectation of $q$ increases, and hence prices decrease to reflect the information that there are more investors in the market with large endowments.

What creates the spread is this process of updating the expectation of $q$. When we introduce information asymmetry about future cash flows, a type 2 investor will sell $X_{1,t}$ shares if she observes a bad signal, $\theta_L$. Otherwise, she acts like the type 2 uninformed investor and buys shares. When market makers now see an order for $X_{1,t}$ shares, they are not sure if a high signal had occurred, in which case the order must come from a type 1 investor, or if a low signal had occurred, in which case the order can be coming from a type 2 investor. If there is a chance that the order is coming from a type 2 investor, market makers must update their beliefs about the population of investors to reflect the possibility that there may be more investors with small rather than large endowments.

Information asymmetry creates a situation in which there is no longer full revelation of the type of an arriving investor, and hence market makers cannot adjust their beliefs to the extent that they could without information asymmetry. For example, we can use (30) to look at the difference between the posterior and the prior expected values of $q$ when only demand uncertainty is present:

$$q_{1,t} - E_t[q] = E_t[q] + \frac{V_t[q]}{E_t[q]} - E_t[q] = \frac{V_t[q]}{E_t[q]}$$

The larger the uncertainty about the investor population (i.e., $V_t[q]$), the larger is the difference between the prior and the posterior and the larger the spread. We can use (37) to find the difference between the prior and the posterior in an economy with both demand
uncertainty and information asymmetry:

\[ \hat{q}_{1,t} - \hat{q}_t = \frac{(1 - \nu)V_t[\hat{q}]}{\nu(1 - \hat{\rho}_t) + (1 - \nu)\hat{q}_t} \]

Now, say we start with an economy in which there is only demand uncertainty and then introduce information asymmetry. In this case, \( \hat{q}_t = E_t[q] \) and \( V_t[q] = V_t[q] \). Then, a simple manipulation shows that:

\[ (q_{1,t} - E_t[q]) - (\hat{q}_{1,t} - \hat{q}_t) = \frac{\nu(1 - \hat{\rho}_{1,t})}{\hat{d}_t\hat{q}_t} > 0 \]

Hence, the difference between the prior and posterior of the population parameter \( q \) is smaller in the presence of information asymmetry, and hence the spread component resulting from demand uncertainty is smaller.

It can still be the case that the total spread increases after the introduction of information asymmetry due to the information asymmetry component of the spread. However, (44) shows that if demand uncertainty is high enough, the inability of market makers to fully infer the investor’s type is the dominating effect and the spread will decrease when information asymmetry is introduced. Hence, the total spread in the market is no longer a monotonically increasing function of information asymmetry.\(^{20}\)

3 Discussion and Conclusions

This paper investigates two modes of information imperfections—information asymmetry about future cash flows and demand uncertainty—in a sequential market with two important properties: (i) market makers focus on making money from facilitating trading rather than from holding the stock, and (ii) both informed and uninformed investors optimize over the quantities of the stock they wish to hold. The focus on market makers who care about supply and demand (rather than the liquidation payoff of the stock) and the joint investigation of

\(^{20}\)In Glosten (1989), the monopolist market maker sets a “zero-quantity spread” that need not be a monotone function of the precision of the signal of the informed traders. However, the spread or price impact of larger trades is always monotonically increasing in the extent of information asymmetry.
information asymmetry and demand uncertainty set this paper apart from the traditional sequential trade model literature and give rise to interesting implications.

The first implication that contrasts with the results of traditional sequential trade models is that in the presence of information asymmetry about future cash flows, the transaction prices process is not a martingale with respect to the information set of the market makers (i.e., the “public” information set). Rather, the first difference of the transaction price process is positively autocorrelated. What exactly generates this different result? First, investors optimize over the quantity of the stock they wish to hold, in contrast to the assumption of a single trade size employed by most of the sequential trade models. In addition to being less restrictive, the formulation adopted here focuses our attention on the fact that the information set of an uninformed investor who arrives to the market is not equal to the market makers’ information set. Second, market makers are looking to set a price that incorporates the new information in incoming orders but also balances expected supply and demand. Hence, they must take into account that the order sizes of the uninformed investors reflect an information set different than theirs. The end result is that some portion of the price is updated with a lag, and hence transaction price differences are positively autocorrelated.

Why isn’t someone arbitraging away this effect? Doesn’t it constitute a profit opportunity? When the entire process of trading is specified, like in the model I use here, these questions cannot be answered by invoking some ethereal traders who could trade infinite quantities until this effect disappears. Therefore, consider the behavior of the uninformed investors in the economy. They use every piece of information in their possession to come up with their optimal demands. Note, however, that they are constrained from coming back to the market like in traditional sequential trade models. Nonetheless, allowing them to return to the market should not change the result drastically since in formulating their dynamic program, they would still use an information set that is finer than the one used by the market makers. Furthermore, the bid-ask spread that arises endogenously in this market will deter
an attempt to eliminate the positive autocorrelation. For example, assume that uninformed investors can reverse their trades at will. The price at which an uninformed investor buys is adjusted upward to reflect her arrival and therefore is higher than it should have been from her point of view (since she knows that her order contains no information). However, reversing her trade will create a negative price impact and hence this round trip entails cost—the bid ask spread. The condition for positive autocorrelation is that the direct risk premium effect of Proposition 2 is stronger than the lagged information effect of Proposition 3. When this is the case, the spread is larger than the amount an uninformed investor would gain from this round trip strategy and therefore the positive autocorrelation will remain.

As for the informed investors, the conditions in (21) and (22) guarantee that they in fact want to pool with the uninformed investors. The market makers’ price schedule in (20) makes any other strategy inferior. Again, there is the question of what would happen if we allow the informed investors to return to the market. This cannot be properly resolved without letting the market makers learn from the order flow about the possibility of the informed investors’ return. Such an extension is beyond the scope of this paper (and is also not addressed by the traditional sequential trade models). However, being restricted to the uninformed order sizes, and knowing that the spread is larger than the lagged information adjustment, it is unlikely that informed investors will find it profitable to arbitrage this effect away even if they were allowed to return.

Market makers should be in the best position to eliminate this effect since they are both setting the prices and trading with all investors. What makes the effect stay is that market makers are trying to equate the flow of shares they buy and sell. Market makers care about the short-term with respect to both making money and balancing their inventories. In the model, I use a simple myopic structure: market makers care about the instantaneous expected profit and the instantaneous expected flow of shares. Conceptually, every formulation in which market makers care about their profit and their ability to deliver the shares during the trading period should generate the positive autocorrelation. If we were to lengthen
the horizon of market makers to be the end of the economy (when the liquidation payoff is realized) without imposing on them any capital constraints, we would be back in the realm of the traditional sequential trade models and the transaction price process would be a martingale. The view echoed by Mayer (1988) that market makers in the real world are making money from buying and selling rather than from investing in the long-term prospects of firms makes the setup here rather attractive. It is also likely that stocks traded by market makers with different horizons will exhibit different degrees of positive autocorrelation.

Whether the transaction price process in the presence of information asymmetry is a martingale with respect to the public information set or whether short horizon returns are positively autocorrelated is ultimately an empirical question. Testing the influence of information asymmetry would necessitate employing appropriate controls for interacting effects since different components of the spread give rise to potentially contrasting return dynamics. For example, a simple model where spreads are due to order processing costs gives rise to a bid-ask bounce effect that introduces negative autocorrelation to transaction price differences. Hasbrouck and Ho (1987) control for the bid-ask bounce by using both transaction prices and quote midpoints and find evidence of positive autocorrelation in short horizon returns. Positive autocorrelation, however, can be attributed to more than one explanation. For example, specialists on the NYSE are expected to “smooth” the price process and hence may not revise their quotes immediately to reflect public information. Stale limit orders may execute at prices that no longer reflect the public information set and therefore will also create lagged information effects.²¹ Exploring the implications of information asymmetry for the transaction price process is therefore a challenge that awaits future empirical work.

The second issue discussed in the paper is the interaction of information asymmetry about future cash flows and demand uncertainty. While the issue of demand uncertainty, or uncertainty about the composition of the investor population, was investigated in Saar (2000), this paper offers an interesting result on the interaction between demand uncertainty

²¹See Hasbrouck (1991) for a discussion of these points (pp. 190–191).
and the traditional information asymmetry. The new implication is that the introduction of information asymmetry can lead to smaller, rather than larger, spreads. The intuition behind this result is that market makers use the order flow to learn about the composition of the investor population. The introduction of information asymmetry changes the trading strategies of some investors (those endowed with private information) and depending on the signal they observe, they imitate the orders of investors who belong to a different type (i.e., who have different endowments). These trading strategies impede the ability of market makers to learn about the composition of the investor population, and the spread, which is a manifestation of this learning process, can decrease.

The result that the introduction of information asymmetry may decrease the spread is important since it demonstrates that the link between spreads and information asymmetry about future cash flows may be strenuous. There is an extensive literature that attributes the spread to order processing costs, market power, inventory control, information asymmetry, and demand uncertainty. However, the implication of traditional sequential trade models that information asymmetry and the spread are positively related was used in empirical studies to justify using the spread as a direct proxy for information asymmetry. While in this paper information asymmetry and the spread are still positively related when information asymmetry is the “only game in town,” this changes in the presence of demand uncertainty where (under some conditions) there is a non-monotone relation between the spread and information asymmetry.

The problem is not resolved by using spread decomposition procedures to identify the “adverse selection” (or permanent) component of the spread and using it as a proxy for information asymmetry. Saar (2000) shows that these econometric procedures pick up the information effects of demand uncertainty and incorporate them into the “adverse selection” or information component of the spread. Hence, this component would contain the spread generated by both demand uncertainty and information asymmetry about future cash flows.

\(^{22}\)See, for example, Huang and Stoll (1997) and Madhavan, Richardson, and Roomans (1997).
In fact, since the model in this paper does not consider order processing costs, the spread result in Proposition 6 can be interpreted as describing only the information component of the spread. Therefore, the implication of Section 2.3 is that the introduction of information asymmetry can decrease rather than increase the adverse selection component identified by the spread decomposition procedures. This implication suggests that researchers may want to exercise caution when using the spread or the adverse selection component identified by these econometric procedures as measures of information asymmetry, especially since the extent of demand uncertainty changes over time and may differ across stocks depending on the maturity of the firms, the extent of coverage in the media, the exchange on which they are traded and so on. For example, stocks that are predominantly held by individuals may exhibit different patterns of demand uncertainty than stocks with extensive institutional ownership. To carry out an investigation of information asymmetry effects, researchers may want to control for uncertainty about the population of investors by using such proxies as ownership by institutions and average trade size.

One feature of the model that proved important for the investigation of information imperfections is that market makers focus on making money from facilitating trading rather than from holding the stock. This feature of the model resulted in implications that differ from those of traditional sequential trade models where market makers hold the stock until its liquidation payoff is realized. These results demonstrate that the assumptions we make about the objectives of market makers matter for the predictions that come out of our models. When market makers have different objectives from those of investors, prices and spreads have certain properties that otherwise need not be there. This observation is especially important when we attempt to predict the outcomes of current developments in the structure of markets. For example, there has been great interest recently in the electronic limit order book as an alternative to current market structures in the United States. Prices in a pure limit order book are determined, to a large extent, by all investors in the market. At least conceptually, market makers can operate in a pure limit order book environment by
submitting limit orders on both sides of the book. Still, such market makers will have competition in their liquidity provision capacity from all investors in the market. This means that prices in pure limit order markets may behave differently from prices set by market makers in a dealers’ market, even if these market makers are competitive. If we are interested in gaining insights into the question of what would happen if we replace existing market structures with a pure limit order book, further work is needed to clarify the role played by market makers in the current environment.
Appendix

Proof of Proposition 1:
We are interested in deriving the conditions under which informed investors who face the price schedule in (20) choose to carry out the trading strategy in (14). This price schedule is de-facto restricting the investors to two order sizes. Submitting any buy order different than the order size of an uninformed type 2 investor cannot be optimal due to the infinite price. Sell orders that are not equal to an uninformed type 1 order are discouraged by the zero price (with the addition of a small technical condition to guarantee that prices in this economy are non-negative, $\theta_L - \alpha \sigma^2 \tilde{X}_1 \geq 0$). The participation constraints and incentive compatibility constraints of the informed investors are as follow:

\[
\begin{align*}
\text{P.1.H.} & \quad U_1 \left( X_{2,t}, P_{2,t}^e \mid \theta_H \right) \geq U_1 \left( 0, P \mid \theta_H \right) \\
\text{IC.1.H.} & \quad U_1 \left( X_{2,t}, P_{2,t}^e \mid \theta_H \right) \geq U_1 \left( X_{1,t}, P_{1,t}^e \mid \theta_H \right) \\
\text{P.2.H.} & \quad U_2 \left( X_{2,t}, P_{2,t}^e \mid \theta_H \right) \geq U_2 \left( 0, P \mid \theta_H \right) \\
\text{IC.2.H.} & \quad U_2 \left( X_{2,t}, P_{2,t}^e \mid \theta_H \right) \geq U_2 \left( X_{1,t}, P_{1,t}^e \mid \theta_H \right) \\
\text{P.1.L.} & \quad U_1 \left( X_{1,t}, P_{1,t}^e \mid \theta_L \right) \geq U_1 \left( 0, P \mid \theta_L \right) \\
\text{IC.1.L.} & \quad U_1 \left( X_{1,t}, P_{1,t}^e \mid \theta_L \right) \geq U_1 \left( X_{2,t}, P_{2,t}^e \mid \theta_L \right) \\
\text{P.2.L.} & \quad U_2 \left( X_{1,t}, P_{1,t}^e \mid \theta_L \right) \geq U_2 \left( 0, P \mid \theta_L \right) \\
\text{IC.2.L.} & \quad U_2 \left( X_{1,t}, P_{1,t}^e \mid \theta_L \right) \geq U_2 \left( X_{2,t}, P_{2,t}^e \mid \theta_L \right)
\end{align*}
\]

(45)

where $P$ in the participation constraints denotes any arbitrary price. It can be shown that P.2.H., IC.2.H., P.1.L., and IC.1.L. always hold. Constraints P.1.H. and IC.1.H. are satisfied if:

\[
2(1 - \rho_t)(\theta_H - \theta_L) + \alpha \sigma^2 (X_{2,t}^\nu + \tilde{X}_2 - 2\tilde{X}_1) \geq \max \left\{ C \left[ -2(1 - \rho_t)(\theta_H - \theta_L) - \alpha \sigma^2 (X_{1,t}^\nu - \tilde{X}_1) \right], 0 \right\}
\]

(46)

where $C = \frac{\nu \rho_{1,t} + (1 - \nu)(1 - q)}{\nu (1 - \rho_{2,t}) + (1 - \nu)q}$. Similarly, P.2.L. and IC.2.L. are satisfied if:

\[
2\rho_t(\theta_H - \theta_L) - \alpha \sigma^2 (X_{1,t}^\nu + \tilde{X}_1 - 2\tilde{X}_2) \geq \max \left\{ \frac{1}{C} \left[ -2\rho_t(\theta_H - \theta_L) + \alpha \sigma^2 (X_{2,t}^\nu - \tilde{X}_2) \right], 0 \right\}
\]

(47)

Q.E.D

Proof of Proposition 2:
Using (18) and (19),

\[
S_{1,t}^{IA} = P_{2,t}^e - P_{1,t}^e = \frac{\theta_t - \alpha \sigma^2 X_{2,t}^\nu}{R} - \frac{\theta_t - \alpha \sigma^2 X_{1,t}^\nu}{R} = \frac{\alpha \sigma^2 (X_{1,t}^\nu - X_{2,t}^\nu)}{R} = \frac{\alpha \sigma^2 (\tilde{X}_1 - \tilde{X}_2) \rho_{2,t} \rho_{1,t}}{R}
\]

(48)
Let $m_t(\theta)$ be the market makers’ prior beliefs about $\theta$ at time $t$:

$$m_t(\theta) = \begin{cases} \rho_t & \theta = \theta_H \\ 1 - \rho_t & \theta = \theta_L \end{cases}$$  \hspace{1cm} (49)

and let $g(X_{1,t}|\theta)$ be the likelihood of observing the order $X_{1,t}$ given $\theta$:

$$g(X_{1,t}|\theta) = \begin{cases} q(1 - \nu) & \text{if } \theta = \theta_H \\ q(1 - \nu) + \nu & \text{if } \theta = \theta_L \end{cases}$$  \hspace{1cm} (50)

Using Bayes rule,

$$\rho_{1,t} = P(\theta = \theta_H | \mathcal{M}_{t,X_1}) = \frac{\rho_t q(1 - \nu)}{\rho_t q(1 - \nu) + (1 - \rho_t) [q(1 - \nu) + \nu]}$$  
$$= \rho_t \left[ \frac{(1 - \nu)q}{a_t} \right]$$  \hspace{1cm} (51)

where $a_t = \nu(1 - \rho_t) + (1 - \nu)q$. Similarly, let $g(X_{2,t}|\theta)$ be the likelihood of observing the order $X_{2,t}$ given $\theta$:

$$g(X_{2,t}|\theta) = \begin{cases} (1 - q)(1 - \nu) + \nu & \text{if } \theta = \theta_H \\ (1 - q)(1 - \nu) & \text{if } \theta = \theta_L \end{cases}$$  \hspace{1cm} (52)

Using Bayes rule,

$$\rho_{2,t} = P(\theta = \theta_H | \mathcal{M}_{t,X_2}) = \frac{\rho_t [(1 - q)q(1 - \nu) + \nu]}{\rho_t [(1 - q)(1 - \nu) + \nu] + (1 - \rho_t)(1 - q)(1 - \nu)}$$  
$$= \rho_t \left[ \frac{\nu + (1 - \nu)(1 - q)}{1 - a_t} \right]$$  \hspace{1cm} (53)

Plugging (51) and (53) into (48) and manipulating the expression yields:

$$S_{IA}^t = \frac{\alpha \sigma^2 (\bar{X}_1 - \bar{X}_2)^2 \rho_t (1 - \rho_t)}{Ra_t (1 - a_t)} > 0$$  \hspace{1cm} (54)

While the evolution of beliefs about $\theta$ depends on $\nu$, we can take current beliefs (i.e., $\rho_t$) as given and differentiate the spread expression with respect to $\nu$ to examine the effect of an exogenous change in the relative population weight of the informed investors.

$$\frac{\partial S_{IA}^t}{\partial \nu} = \frac{\alpha \sigma^2 (\bar{X}_1 - \bar{X}_2) \rho_t (1 - \rho_t) \nu}{Ra_t^2 (1 - a_t)^2} [2a_t(1 - a_t) + (2a_t - 1)\nu(1 - \rho_t - q)]$$  
$$= \frac{\alpha \sigma^2 (\bar{X}_1 - \bar{X}_2) \rho_t (1 - \rho_t) \nu}{Ra_t^2 (1 - a_t)^2} [\nu \rho_t q + \nu(1 - \rho_t)(1 - q) + 2(1 - \nu)q(1 - q)]$$  
$$> 0$$

Q.E.D
Proof of Proposition 3:

\[ P^\text{tr}(t_n) = \frac{\theta_{t_n} - \alpha \sigma^2 X_{t_n}'}{R} \]
\[ P^\text{tr}(t_{n+1}) = \begin{cases} \frac{\theta_{t_n}}{R} - \frac{\alpha \sigma^2 X_{t_n}'}{R} & \text{with probability } a_{t_n} \\ \frac{\theta_{t_n}}{R} - \frac{\alpha \sigma^2 X_{t_{n+1}}'}{R} & \text{with probability } 1 - a_{t_n} \end{cases} \]

where \( a_{t_n} = \nu(1 - \rho_{t_n}) + (1 - \nu)q \). Using (51) and (53) we can write:

\[ a_{1,t_n} = a_{t_n} + \frac{\nu^2 \rho_{t_n}(1 - \rho_{t_n})}{a_{t_n}} \]  
\[ a_{2,t_n} = a_{t_n} - \frac{\nu^2 \rho_{t_n}(1 - \rho_{t_n})}{1 - a_{t_n}} \]

Then,

\[
E \left[ P^\text{tr}(t_{n+1}) - P^\text{tr}(t_n) \mid \mathcal{M}_{t_n} \right] = \\
\begin{aligned}
& \frac{a_{t_n}}{R} \left[ \frac{\theta_{t_n} - \alpha \sigma^2 X_{t_n}'}{R} \right] + (1 - a_{t_n}) \left[ \frac{\theta_{t_n} - \alpha \sigma^2 X_{t_{n+1}}'}{R} \right] - \left[ \frac{\theta_{t_{n-1}} - \alpha \sigma^2 X_{t_{n-1}}'}{R} \right] \\
& = \frac{\theta_{t_n} - \theta_{t_{n-1}}}{R} - \frac{\alpha \sigma^2(X_1 - X_2)}{R} \left[ a_{t_n}(a_{1,t_n} - a_{t_n}) + (1 - a_{t_n})(a_{2,t_n} - a_{t_n}) \right] \\
& = \frac{\theta_{t_n} - \theta_{t_{n-1}}}{R} - \frac{\alpha \sigma^2(X_1 - X_2)}{R} \left[ a_{t_n} \nu^2 \rho_{t_n}(1 - \rho_{t_n}) - (1 - a_{t_n}) \nu^2 \rho_{t_n}(1 - \rho_{t_n}) \right] \\
& = \frac{(\rho_{t_n} - \rho_{t_{n-1}})(\theta_H - \theta_L)}{R} \\
& = \begin{cases} \\
-\frac{\nu \rho_{t_{n-1}}(1 - \rho_{t_{n-1}})(\theta_H - \theta_L)}{R[\nu(1 - \rho_{t_{n-1}}) + (1 - \nu)]} & \text{if the nth transaction was a sell} \\
\frac{\nu \rho_{t_{n-1}}(1 - \rho_{t_{n-1}})(\theta_H - \theta_L)}{R[\nu(1 - \rho_{t_{n-1}}) + (1 - \nu)]} & \text{if the nth transaction was a buy} \\
\end{cases} 
\end{aligned} \]

where the last step follows from (51) and (53).

Q.E.D

Proof of Proposition 4:

See the proofs of Proposition 2 and Proposition 4 in Saar (2000).

Q.E.D

Proof of Proposition 5:

Let \( f^t(q) \) be the market makers’ prior beliefs about \( q \) at time \( t \). This prior distribution can be rather general but its support should be in \([0,1]\). Let \( \hat{m}_t(\theta) \) be the market makers’ prior beliefs about \( \theta \) at time \( t \):

\[
\hat{m}_t(\theta) = \begin{cases} \\
\hat{\rho}_t & \theta = \theta_H \\
1 - \hat{\rho}_t & \theta = \theta_L \\
\end{cases} 
\]
and let $g(X_{1,t}|\theta)$ be the likelihood of observing the order $X_{1,t}$ given $\theta$ and $q$:

$$
g(X_{1,t}|\theta, q) = \begin{cases} 
q(1-\nu) & \text{if } \theta = \theta_H \\
q(1-\nu) + \nu & \text{if } \theta = \theta_L 
\end{cases} \quad (61)
$$

The joint distribution of $q$ and $\theta$ is just the multiplication of their marginal distributions due to the independence assumption. Using Bayes rule,

$$
\hat{q}_{1,t} = E[q \mid \mathcal{M}_{t,X_1}] = \int_0^1 \frac{q f_t(q) [\hat{\rho}_t q(1-\nu) + (1-\hat{\rho}_t) [q(1-\nu) + \nu]]}{\nu(1-\hat{\rho}_t) E[q \mid \mathcal{M}_t] + (1-\nu) E[q^2 \mid \mathcal{M}_t]} dq 
= \frac{\nu(1-\hat{\rho}_t) E[q \mid \mathcal{M}_t] + (1-\nu) E[q^2 \mid \mathcal{M}_t]}{\nu(1-\hat{\rho}_t) + (1-\nu) \hat{q}_t} 
= \hat{q}_t + \frac{(1-\nu) V_t[\hat{q}]}{\nu(1-\hat{\rho}_t) + (1-\nu) \hat{q}_t} \quad (62)
$$

where $\hat{q}_t = E[q \mid \mathcal{M}_t]$, $V_t[\hat{q}] = E[q^2 \mid \mathcal{M}_t] - \hat{q}_t^2$, and $\hat{\rho}_t = P(\theta = \theta_H \mid \mathcal{M}_t)$. Using Bayes rule,

$$
\hat{\rho}_{1,t} = P(\theta = \theta_H \mid \mathcal{M}_{t,X_1}) = \int_0^1 \frac{f_t(q) \hat{\rho}_t q(1-\nu)}{\nu(1-\hat{\rho}_t) \hat{q}_t + (1-\nu) [q(1-\nu) + \nu]} dq 
= \hat{\rho}_t \left[ \frac{(1-\nu) \hat{q}_t}{\nu(1-\hat{\rho}_t) + (1-\nu) \hat{q}_t} \right] \quad (63)
$$

Similarly, $\hat{q}_{2,t} = E[q \mid \mathcal{M}_{t,X_2}]$ and $\hat{\rho}_{2,t} = P(\theta = \theta_H \mid \mathcal{M}_{t,X_2})$ can be derived using the likelihood:

$$
g(X_{2,t}|\theta, q) = \begin{cases} 
(1-q)(1-\nu) + \nu & \text{if } \theta = \theta_H \\
(1-q)(1-\nu) & \text{if } \theta = \theta_L 
\end{cases} \quad (64)
$$

Q.E.D

**Proof of Proposition 6:**

The spread expression in (41) follows directly from (34) and (35). Plugging $\hat{q}_{1,t}$, $\hat{q}_{2,t}$, $\hat{\rho}_{1,t}$, and $\hat{\rho}_{2,t}$ from Proposition 4 into (41) and manipulating the expression yields (42).

Q.E.D
References


