THE RELATIONSHIP BETWEEN YIELD, RISK, AND RETURN OF CORPORATE BONDS

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A common statistic for a bond is its market yield or internal rate of return. Numerous articles have been written utilizing the internal rate of return to examine the price volatility of bonds [9], [14], [21]. However, all these studies have two factors in common. One, they are nonstochastic and two, they involve Macaulay's duration. This paper, by introducing uncertainty, derives the general relationship between the expected return on a bond and its market yield. The derivation is for risky corporate as well as for default free government bonds. The relationship is nonlinear and involves an expression which is an approximation to the duration of a bond.

As a corollary to the above methodology, an alternate expression for the systematic risk of a bond is discovered. The expression involves only the market yield and the approximation to a bond's duration. This corollary generalizes the previous work of Boquist, Racette, and Schalarbaume [6] in two respects. First, this paper's formulation is for any type of a bond, while Boquist, et al. examined only default free securities. Second, the single period capital asset pricing model (CAPM) is replaced by its intertemporal counterpart as formulated by Merton [15].

An outline for the paper is as follows. Section I derives the relationship between a bond's market yield and expected return. Section II examines the systematic risk of a bond, while Section III concludes the paper.

I. BOND RETURN DYNAMICS

The market yield or internal rate of return\(^1\) is not a good measure for a bond's expected return. It has serious drawbacks which have been discussed in detail elsewhere.\(^2\) This section derives the exact relationship between these two concepts by utilizing the one-to-one correspondence between a bond's price and its internal rate of return. The internal rate of return, \(r\), is defined for an arbitrary bond by expression (1).

\[
B(t) = \sum_{n=T-t-k}^{T-t} C e^{-n} + Ae^{-r(T-t)} \quad \text{where} \quad (1)
\]

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1. Market yield and the internal rate of return are synonyms.
2. Hirshleifer [8; pp. 340-357]
i) \( t = \) present date; \( T-k-1 \leq t < T-k \)
ii) \( B(t) = \) market price of bond at time \( t \)
iii) \( T = \) maturity date of the bond
iv) \( C = \) coupon payment of the bond in dollars
v) \( k+1 = \) number of coupon payments remaining
vi) \( A = \) face value of the bond
vii) \( r = \) internal rate of return on the bond

Note that in this definition the present date \( t \) does not necessarily occur at the same instant a coupon payment is received. The internal rate of return on a bond is stochastic. This is true for the following reason. Due to uncertain micro and macro economic conditions, both the term structure of interest rates and the bond’s probability of default fluctuate over time, implying the internal rate of return will vary stochastically as well. Assume the internal rate of return follows the stochastic process described in (2).

\[
\frac{dr}{r} = \alpha dt + \sigma dz \quad \text{where} \quad (2)
\]

\( dz \) is a standard Wiener process, \( \alpha \) is the instantaneous expected percent change in \( r \) per unit time, and \( \sigma^2 \) is the instantaneous variance of the percent change in \( r \) per unit time.

Since the bond price \( B(t) \) is a function of \( r \) and \( T-t \), to express the instantaneous change in the price of a bond in terms of \( r \), Ito’s Lemma\(^3\) is applied to (1).

\[
\begin{align*}
\frac{dB}{\partial r} &= \frac{\partial B}{\partial t} dt + \frac{1}{2} \frac{\partial^2 B}{\partial r^2} \sigma^2 dt \\
\end{align*}
\]

From expression (1) the following partial derivatives can be derived.

\[
\frac{\partial B}{\partial t} = rB \quad (4)
\]

\[
\frac{\partial B}{\partial r} = -D(t)B \quad (5)
\]

\[
D(t) = \sum_{n=T-t-k}^{T-t} \frac{nC e^{-r}}{B(t)} \frac{(T-t)A e^{-r(T-t)}}{B(t)} 
\]

\[
\frac{\partial^2 B}{\partial r^2} = F(t)B \quad (7)
\]

\[
F(t) = \sum_{n=T-t-k}^{T-t} \frac{n^2C e^{-r}}{B(t)} \frac{(T-t)^2A e^{-r(T-t)}}{B(t)} 
\]

Substituting (4) and (7) into (3) gives (9).

3. Astrom [1; p. 74]
4. Equation (5) is well known in the duration literature as the interest elasticity, see Weil [21].
\[
\frac{dB}{B} = \left( r + \frac{1}{2} \sigma^2 r^2 F(t) \right) dt - D(t) dr
\]  
Equation (9)

The instantaneous expectation and variance of \( B \) are (10) and (11).

\[
E\left( \frac{dB}{B} \right) = \left( r + \frac{1}{2} \sigma^2 r^2 F(t) - \alpha D(t) r \right) dt
\]  
Equation (10)

\[
\text{Var}\left( \frac{dB}{B} \right) = \sigma^2 t^2 D(t)^2 dt
\]  
Equation (11)

The instantaneous expected return on the bond is a \textbf{nonlinear} function of \( r, \alpha, \sigma^2, F(t), \) and \( D(t) \). Both \( F(t) \) and \( D(t) \) are parameters concerning the “average life” of the bond since both are weighted averages of the times coupon payments are received. In particular, \( D(t) \) as defined in (6) is an approximation to Macaulay’s duration.\(^5\) Comparative statics yield the following results. As the internal rate of return increases, ceteris paribus, \( E(dB/B) \) increases. The relationship is monotonic. As the instantaneous variance of the percent change in \( r \) increases, ceteris paribus, both \( E(dB/B) \) and \( \text{Var}(dB/B) \) increase.

The effect of changing the “average life” of the bond, as measured by \( D(t) \), on the expected return of the bond is meaningless due to the presence of \( F(t) \) in expression (10). A precise relationship between \( E(dB/B) \) and \( D(t) \) is discovered in the next section. Before proceeding to evaluate the systematic risk of a bond as a function of the internal rate of return, the above process is applied to a specific default-free government bond.

Examine a government bond that matures in the next “smallest” interval of time, \( \Delta t \). The market price of the bond is denoted \( B_G(T - \Delta t) \). After \( \Delta t \) units of time pass, the bond matures, and the cash payment \( A + C \) is received with certainty. From (1), the bond’s internal rate of return, \( i \), is defined by (12).

\[
B_G(T - \Delta t) = (C + A) e^{-i\Delta t}
\]  
Equation (12)

This is a pure discount bond. Under these conditions \( D(T - \Delta t) = \Delta t \) and \( F(T - \Delta t) = \Delta t^2 \). Substituting these facts into (9) and letting \( \Delta t \to 0 \), the stochastic process for this bond results.

\[
\frac{dB_G}{B_G} = idt
\]  
Equation (13)

There is no stochastic term in (13), thus, this specific government bond is the risk free asset. The risk free asset is the \textbf{only} case where the internal rate of return

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5. Macaulay’s duration is defined as:

\[
\text{Duration}(t) = \sum_{n=1}^{T-t} n CP(n) / B(t) + (T-t)(AP(T-t) / B(t)
\]

where \( P(n) = \) present value of $1 received at time \( n \) where the $1 has the same risk characteristics as the cash flow from owning the bond at time \( t \).

The approximation is from using \exp(-rn) to estimate \( P(n) \). (see Macaulay [13; pp. 44–53])
equals the instantaneous expected return of the bond. Government bonds of longer
maturity are not riskless due to the changing term structure of interest rates.6

II. SYSTEMATIC RISK

The preceding section examined the direct relationship between the instantaneous
expected return on a bond and its internal rate of return. This section derives an
alternate expression for the systematic risk of a bond as a function of the internal
rate of return using Merton’s intertemporal CAPM [15]. In Merton’s formulation it
is assumed all assets follow the stochastic process defined in (14).7

\[ \frac{dS_j}{S_j} = x_j dt + \sigma_j dz_j \]  

(14)

where \( x_j \) and \( \sigma_j \) are the \( j \)th asset’s instantaneous expected return and variance
respectively, and \( dz_j \) is a standard Wiener process. In particular, the market
portfolio, denoted by subscript \( M \), will follow this same process. By (9), the
instantaneous covariance between \( dB/B \) and \( dS_j/S_j \) is easily written as a function
of both \( r \) and \( D(t) \).

\[ \text{cov}\left( \frac{dB}{B}, \frac{dS_j}{S_j} \right) = -D(t) \text{cov}\left( dr, \frac{dS_j}{S_j} \right) \]  

(15)

The beta for the bond is therefore (16).8

\[ \beta_B = \frac{-D(t) \text{cov}\left( dr, \frac{dS_M}{S_M} \right)}{\text{var}\left( \frac{dS_M}{S_M} \right)} \]  

(16)

This expression is similar to that derived by Boquist, Racette, and Schlarbaum
[6]. However, (16) applies to any bond of any maturity. It is immediately obvious
that the approximation to duration (not duration itself) is a key component in the
systematic risk of a bond. Given \( \beta_B > 0 \), since \( D(t) > 0 \), \( \text{cov}(dr, dS_M/S_M) < 0 \). If the

6. In financial literature it is often assumed that the term structure of interest rates is flat (e.g. see [3]).
In the above analysis this would imply that default free government bonds of any maturity would have
the following process.

\[ \frac{dB_G}{B_G} = idt, \quad \text{i.e.} \quad di = 0. \]

7. Merton [15; p. 879]

8. Merton [15; p. 877] demonstrates that (15) will be the appropriate systematic risk measure of a
bond if the investment opportunity set is constant. If one desires a multi-parameter CAPM then
expression (15) can be used to derive the other risk parameters of the bond as a function of \( r \) and \( D(t) \).
To see this conceptually see Merton [17; p. 671].
bond beta is positive, as $D(t)$ decreases (the average life of the bond), ceteris paribus, the bond beta decreases as well. As time passes, $D(t)$ changes. This implies $\beta_B$ is unstable due to just the passing of time itself.9

III. Conclusion

This paper has examined the relationship between the bond’s yield and the bond’s expected return. The relationship was nonlinear but monotonic. The derivation was a generalization of previous work in the area due to the introduction of uncertainty. The expression applied to both risky corporate and government bonds. As a corollary to the bond return dynamics, an alternative formula for the systematic risk of a bond was discovered. The formula was a linear function of a parameter which is an approximation to the duration of a bond. From this formula, it is seen why the beta of a bond is unstable over time. The methodology of this paper could be applied to a portfolio of bonds as well.

REFERENCES


9. This same observation was stressed by Boquist, et al. [6].