FORWARD CONTRACTS AND FUTURES CONTRACTS

Robert A. JARROW and George S. OLDFIELD*
Cornell University, Ithaca, NY 14853, USA

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This paper provides a detailed discussion of the similarities and differences between forward contracts and futures contracts. Under frictionless markets and continuous trading, simple arbitrage arguments are invoked to value forward contracts, to relate forward prices and spot prices, and to relate forward prices and futures prices. We also argue that forward prices need not equal futures prices unless default free interest rates are deterministic.

1. Introduction

Forward contracts and futures contracts are deceptively similar securities. Each conveys the right to purchase a specified quantity of some asset at a fixed price on a fixed future date. The contract's fixed price is called the exercise or delivery price and the contract's maturity date is called the delivery day. Both forward and futures contracts must be exercised if held until the delivery day. By convention, when either contract is initiated the exercise price is set so that each has a zero initial value. However, the two contracts' cash flows differ. A futures contract makes interim payments during its life while a forward contract does not. The form of the payments means a futures contract is not simply a forward contract rewritten each day.¹ Moreover, this difference implies that a forward contract's exercise price, called the forward price, and a futures contract's exercise price, called the futures price, may be quite different. A paper by Cox, Ingersoll and Ross (1981) presents the same result.²

This paper's purpose is to emphasize the difference between forward and futures contracts and forward and futures prices. We assume both contracts

*We thank Fischer Black for helpful discussions on this issue, especially concerning forward and futures prices.
¹This is a common misconception. See Black (1976, p. 170) and Oldfield and Messina (1977, p. 476).
²The insight that the forward price can differ from the futures price used in Cox, Ingersoll and Ross (1981) first appeared in an earlier manuscript of theirs (1977). We independently discovered the same idea.
are available for specified assets. This is a considerable simplification. Forward and futures contracts are only jointly available for some U.S. Treasury bills, GNMA pass-through securities, and some foreign currencies. Most commodity contracts are futures contracts. We also assume frictionless and continuous trading occurs so that new contracts can be written anytime during the trading day at the quoted exercise price. Margin requirements, transactions fees, taxes, and institutional trading restrictions may substantially influence the following value relationships. We do not analyze the effects of these impediments to free trading.

With our frictionless market assumptions we demonstrate an arbitrage valuation method for forward contracts that does not require specific stochastic processes for either forward prices or default-free discount rates. This is done in section 2. Section 3 shows the conditions under which forward prices and futures prices must be identical. The fourth section treats a special futures contract, the Treasury bill futures. With this contract we give a direct example of the difference between forward and futures contracts. The example also illustrates the term structure implications of contract differences. A summary completes the paper.

2. Values of forward and futures contracts

A forward contract obligates its owner to purchase one unit of a specified asset at a fixed price, called the exercise price, on a fixed future date, called the delivery date. The exercise price is fixed so that the initial value of the contract is zero. At the date a contract is written, the exercise price is called the forward price. These definitions and contract provisions are clarified using the following notation. Let \( t \) be the date the contract is written, \( s \) the current time, and \( t^* \) the delivery date of the contract, where \( t \leq s \leq t^* \). The exercise price is denoted \( k(t) \), the forward price at time \( s \) is \( p(s,t^*) \), and the forward contract's market value at time \( s \) is \( f(s) \). The forward contract's market value at time \( s \) is a function of the delivery date, the exercise price, and the forward price: \( f(s) = f(p(s,t^*); k(t), t^*) \).

The exercise price is defined by

\[
f(p(t,t^*);k(t),t^*) = 0 \quad \text{and} \quad k(t) = p(t,t^*).
\]

When written at time \( t \), the exercise price gives the forward contract a zero value. Also, by definition the forward price equals the exercise price.

\(^3\)In addition, commodity futures contracts specify the delivery month rather than day. Short contracts may be exercised any time during the month at the short investor's option. Long contracts are matched with exercised shorts by the exchange clearing house. Oldest long contracts are selected first.
After time $t$, the exercise price remains fixed for this contract. However, the exercise price for new contracts may differ from the previous exercise prices. By definition $p(s,t^*)$ is the exercise price for a new forward contract written at time $s$, but with the same delivery date $t^*$. Consequently, $p(s,t^*)$ need not equal $p(t,t^*)=k(t)$ for $s > t$. At time $t^*$, the forward price has a special value. It equals the spot price for the asset at time $t^*$. This follows directly from condition (1). A forward contract initiated at time $t = t^*$, with delivery date $t^*$ has zero value if and only if the exercise price (forward price time $t^*$) equals the spot price. For convenience, we let $p(s,s)$ denote the spot price of the asset at any time $s$.

Along with condition (1), the forward contract's provisions are completely specified by the following terminal condition:

\[
 f(p(t^*,t^*); k(t), t^*) = p(t^*,t^*) - k(t). \tag{2}
\]

A forward contract's value at the delivery date is the value of exercising at $k$ and immediately reselling the commodity at the spot price. This value can be negative because the contract must be exercised.

A general forward contract valuation formula can be derived with a simple arbitrage argument. Suppose a long position in a forward contract is taken at time $t$ with delivery price $k(t) = p(t,t^*)$. Some time later, at time $s$, a short position in a new contract is written with delivery price $k(s) = p(s,t^*)$. At the delivery date, time $t^*$, the contract which is long has a value $[p(t^*,t^*) - p(t,t^*)]$ and the contract which is short has a value $[p(s,t^*) - p(t^*,t^*)]$. The combined position's value at time $t^*$ is the sum of long and short values or $[p(s,t^*) - p(t,t^*)]$. This payoff is certain at time $s$ since it is independent of $p(t^*,t^*)$. To preclude profitable arbitrage, the value of the long and short position together must be $[p(s,t^*) - p(t,t^*)]B(s,t^*)$ at time $s$. The term $B(s,t^*)$ is the value at time $s$ of a default-free discount bond that pays one dollar at time $t^*$. Thus the difference in forward contract values is the discounted difference of the forward prices. But since $f(p(s,t^*); k(s), t^*) = 0$ at time $s$, this gives the value at time $s$ of the forward contract written at time $t$,

\[
 f(p(s,t^*); k(t), t^*) = \{p(s,t^*) - k(t)\}B(s,t^*), \tag{3}
\]

where $k(t) = p(t,t^*)$. In words, a forward contract's value is the default-free discounted value of the forward price difference.

Eq. (3) generalizes both Black's (1976) and Ross's (1978) model for valuing forward contracts. Black's approach highlights the similarities between a forward contract and a European call option. He assumes that a commodity's forward price follows a Gauss–Wiener diffusion process with
constant parameters, and that the default-free term structure is flat and invariant through time. Ross generalizes Black's model by removing the diffusion process assumption but he retains the constant default-free rate assumption (although he mentions that it is not required). Expression (3) requires neither of these assumptions and is identical to both Black's and Ross's models when default-free rates are non-stochastic.

The previous arbitrage argument involves only contracts traded in frictionless markets. The argument works even if there are storage costs associated with holding the underlying asset. Considering the underlying asset and its storage costs, we can obtain an alternate valuation formula for a forward contract. Let \( D(t,s) \) be the present value of the known storage costs associated with the asset over \([t,s]\).\(^4\) These costs can be either positive or negative depending upon the asset. For example, \( D(t,s) > 0 \) implies negative cash flows due to storage of the asset over \([t,s]\). Without loss of generality, it is assumed that the storage cost outlays occur at time \( s \).\(^5\) [We define \( D(t,t) \equiv 0 \).]

Consider the following portfolio strategy: buy the spot asset to store over \([t,t^*]\) and sell \((p(t,t) / B(t,t^*))\) units of the default-free discount bond paying one dollar at time \( t^* \). The value of this portfolio at time \( t \) is zero. The portfolio has no cash flows until time \( t^* \). At \( t^* \), the value of the portfolio equals the sum of the spot asset's value \( p(t^*,t^*) \), the storage costs with value \( D(t,t^*) B(t,t^*) \), and the short position in the bonds \( p(t,t) / B(t,t^*) \). The total is \( p(t^*,t^*) - (D(t,t^*) + p(t,t)) / B(t,t^*) \).

Suppose at time \( t \) a forward contract can be written with an exercise price of \( k(t) = p(t,t^*) > (D(t,t^*) + p(t,t)) / B(t,t^*) \). An arbitrage opportunity now exists. Simply sell a forward contract and adopt the previous portfolio strategy. The value of this position at time \( t \) is zero, but it assures a certain value of \( - (D(t,t^*) + p(t,t)) / B(t,t^*) + p(t,t^*) > 0 \) at time \( t^* \). Therefore to preclude arbitrage, at any time \( t, t \leq t^* \),

\[
D(t,t^*) + p(t,t) \geq B(t,t^*) p(t,t^*). \tag{4}
\]

The opposite inequality also holds. If \( k(t) = p(t,t^*) < (D(t,t^*) + p(t,t)) / B(t,t^*) \), then to exploit this opportunity one needs to reverse the previously defined portfolio strategy and simultaneously buy a forward contract. This implies that an investor must be able to store negative

\(^4\)The following approach was suggested by the referee, Kenneth French.

\(^5\)Given any known sequence of cash flows over \([t,s]\), under the frictionless market assumption, it is always possible to transform this sequence into a single certain flow at time \( s \). To see this, suppose an outflow of \( x \) dollars occurs at time \( s^* \) where \( t \leq s^* < s \), with present value \( D(t,s) \). To transfer this flow to time \( s \) simply buy \( x \) bonds that mature time \( s^* \), value \( x \cdot B(t,s^*) \), and sell \( x \cdot B(t,s^*) / B(t,s) \) bonds that mature time \( s \). The initial cash flow of the portfolio is zero, a positive cash flow occurs at time \( s^* \) of \( x \), and a negative cash flow occurs at time \( s \) of \( x \cdot B(t,s^*) / B(t,s) \) dollars. The present value of this cash flow time \( s \) is \( x \cdot B(t,s^*) = D(t,s) \).
amounts of an asset. For a financial security, e.g. a Treasury bill, this is possible with a short sale. For a physical commodity like wheat, it is also possible. Again, the investor need only short sell the commodity. The steps of the transaction are identical to that of short selling a financial security. We assume that someone stores a positive amount of the commodity. The investor borrows the stored commodity, assumes the storage costs, and sells it. To preclude profitable arbitrage, the present value of the known storage costs plus the spot price must be less than or equal to the discounted value of the current forward price,

\[ D(t, t^*) + p(t, t) \leq B(t, t^*)p(t, t^*). \]  

(5)

Combined, expressions (4) and (5) give an equality between forward prices, spot prices, and storage costs,

\[ D(t, t^*) + p(t, t) = B(t, t^*)p(t, t^*), \]  

(6)

for \( t \leq t^* \). The present value of the known storage costs plus the spot price must equal the discounted forward price.

The above arbitrage argument demonstrates the equivalence between buying a forward contract and storing the asset over \([s, t^*]\) while simultaneously short selling \( k(t) \) default-free discount bonds maturing at time \( t^* \). We substitute eq. (6) into eq. (3) and get the forward contract's value,

\[ f(p(s, t^*); k(t), t^*) = p(s, s) + D(s, t^*) - k(t)B(s, t^*). \]  

(7)

Expression (7) may partially explain the infrequent occurrence of forward contract markets. Given the existence of frictionless markets for both the spot commodity and default-free discount bonds of all maturities, the forward contract is a redundant security. The existence of forward markets is a matter of indifference to investors. It is shown below, however, that a similar argument cannot be made with respect to futures contracts.6

A futures contract's cash flow stream differs from the forward contract's. While the forward contract's provisions specify its value at two particular times \((t \text{ and } t^*)\), a futures contract's provisions specify its value at time \( t \) and its cash flows at the end of every trading day. A futures contract gives its owner the right to purchase one unit of an asset at the exercise price, \( K(t) \), on the delivery date \( t^* \), where \( t \leq t^* \). If held until maturity the contract must

6The above argument does not follow if storage costs are stochastic. With stochastic storage costs, it is not possible to construct a perfectly hedged portfolio with the spot commodity and default-free bonds alone.
be exercised. The exercise price of the futures contract, called the futures price \( P(t, t^*) \), is determined such that the value of the futures contract when it is written (time \( t \)) is zero. Letting \( F \) equal the value of the futures contract, the exercise price is defined by

\[
F(P(t, t^*); K(t), t^*) = 0, \tag{8}
\]

where \( K(t) = P(t, t^*) \). This condition is identical to the forward contract's initial condition. The exercise price \( K(t) \) for new futures contracts is the quoted futures price on the exchange. Clearly \( P(s, t) \) need not equal \( P(t, t^*) \) for \( t < s \leq t^* \). Also, at time \( t^* \), the futures price \( P(t^*, t^*) \) necessarily equals the spot price of the asset.

Unlike a forward contract, a futures contract's provisions specify its cash flow at the end of every trading day. After a futures contract is written, if it is held until the day's end, it is rewritten by the exchange clearing house at the closing futures price. The difference in contract futures prices is added (or subtracted) from the trader's account. If the position in the futures contract is closed during the day by entering an offsetting contract, the difference between futures prices is earned immediately. Let the end of the first day be denoted \( T \). Then the change in futures contract value during a day is

\[
F(P(t, t^*); K(t), t^*) - F(P(t, t^*); K(t), t^*) = P(s, t^*) - P(t, t^*), \tag{9}
\]

where \( t \leq s \leq T \). By definition the futures contract gives its owner the change in the value of the futures prices either when the position is closed prior to time \( T \), or at time \( T \). Since its initial value is zero, condition (9) implies

\[
F(P(s, t^*); K(t), t^*) = P(s, t^*) - P(t, t^*), \tag{10}
\]

where \( t < s < T \). This is the futures contract's value at time \( s \) after it is written and before the cash payout is distributed at time \( T \). Expression (10) makes explicit the provision that the cash flows only occur at the end of each trading day, at which time \( F(P(T, t^*); K(t), t^*) \) returns to zero value.

3. Forward prices and futures prices

Forward prices (as defined in forward contracts) and futures prices (as defined in futures contracts) are often taken to be synonymous. Because a forward contract differs from a futures contract, it is not generally true that these prices must be the same. This section examines sufficient conditions under which the absence of profitable arbitrage forces forward and futures prices to be equal.
At the delivery date $t^*$, forward prices always equal futures prices since both $p(t^*, t^*)$ and $P(t^*, t^*)$ equal the spot price. Prior to $t^*$, if the default-free discount rate is deterministic, then $p(t^*, t^*)$ equals $P(t, t^*)$ as well. A series of arbitrage positions demonstrates this relationship. Begin one trading day before $t^*$, at $t^* - 1$. A long forward contract can be written at the market exercise price $p(t^* - 1, t^*)$. At the same time, a short position in a futures contract can be written at the market futures price $P(t^* - 1, t^*)$. At time $t^*$, the payoff of long and short contracts together is the difference $(p(t^*, t^*) - p(t^* - 1, t^*)) - (P(t^*, t^*) - P(t^* - 1, t^*))$. Since $p(t^*, t^*) = P(t^*, t^*)$, the net payoff at $t^*$ is $(P(t^* - 1, t^*) - p(t^* - 1, t^*))$. This means a certain profit on zero investment can be arranged at $t^* - 1$ unless $p(t^* - 1, t^*)$ equals $P(t^* - 1, t^*)$.

Now move back another trading day. At time $t^* - 2$, the forward price is $p(t^* - 2, t^*)$ and the futures price is $P(t^* - 2, t^*)$. Let default-free rates be deterministic so next period’s default free discount bond price $B(t^* - 1, t^*)$ is known at $t^* - 2$. A long position of $(1/B(t^* - 1, t^*))$ forward contracts can be established at forward price $p(t^* - 2, t^*)$. A short position in a futures contract can be initiated simultaneously at futures price $P(t^* - 2, t^*)$. At time $t^* - 1$, eq. (3) gives the forward contract's value, which when multiplied by the number of contracts long is $(p(t^* - 1, t^*) - p(t^* - 2, t^*))$. The short position in the futures contract has a value of $(P(t^* - 2, t^*) - P(t^* - 1, t^*))$. Since $p(t^* - 1, t^*) = P(t^* - 1, t^*)$, the combined long and short position’s value at $t^* - 1$ is $(P(t^* - 2, t^*) - p(t^* - 2, t^*))$. This is certain at time $t^* - 2$ so $P(t^* - 2, t^*)$ must equal $p(t^* - 2, t^*)$ to prevent a free arbitrage profit.

If default-free rates are deterministic the previous argument can be repeated successively to demonstrate $p(s, t^*) = P(s, t^*)$ for all $s$. This is because the proper hedge ratio for forward contracts is always known. However, if default-free rates are stochastic, the hedge factor is a random variable and the above riskless hedge cannot be constructed. In this case, our arbitrage procedure cannot be used to force forward and futures prices to equality. In fact, if interest rates are stochastic, $P(t, t^*)$ need not equal $p(t, t^*)$ prior to time $t^* - 1$. For an example of a continuous time equilibrium model in which forward and futures prices can differ, see Richard and Sundaresan (1981).

Since forward and futures prices need not be equal before $t^* - 1$, forward and futures contracts need not have equal value over the initial trading day. The exact relationship between the two contract's values depends upon the unknown relationships between forward and futures prices. When default-free rates are deterministic, however, this relationship is known. Combining eqs. (3) and (10) gives a forward contract's price in terms of a futures contract,

\[ f(p(s, t^*); k(t), t^*) = F(P(s, t^*); K(t), t^*)B(s, t^*), \] (11)
where \( t \leq s < T \). The absolute value of the forward contract's value over the initial day of trading is less than the absolute value of the corresponding futures contract's value, although \( F = p \). Consequently, it is incorrect to view a futures contract as the corresponding forward contract rewritten at the end of each day even if forward and futures prices are equal and the default-free rate is deterministic.

The discrepancy between \( f \) and \( F \) arises from the different cash flow streams the securities provide. With the forward contract, no cash flows occur until the delivery date, \((t^* - s)\) periods in the future. Changes in the forward price are accumulated within the contract until it is exercised. Contrast this situation with the futures contract in which changes in the futures price are distributed each day. Since any past changes in the forward contract are settled \((t^* - s)\) days in the future, the present value of this change must reflect the interest rate [see eq. (3)].

In contrast to the forward contract, with stochastic interest rates, the futures contract cannot be shown to be equivalent to a portfolio composed of the spot commodity and default-free discount bonds alone. Given the existence of frictionless markets for these assets, a futures contract is not a redundant security.

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7The above analysis only considers the equivalence of the forward and futures contract over the initial day of trading. For any subsequent day, the relationship between corresponding contracts with identical maturity dates and initiation dates can be examined as follows. For simplicity denote \( f(s) = f(p(s, t^*); k, t^*) \). \( \Delta f(s) = f(s) - f(s - \Delta) \), and \( \Delta B(s, t^*) = B(s, t^*) - B(s - \Delta, t^*) \), where \( \Delta > 0 \) and both \( s \) and \( s - \Delta \) occur within the same day. By expression (3),

\[
\Delta f(s) = [p(s, t^*) - k]B(s, t^*) - [p(s - \Delta, t^*) - k]B(s - \Delta, t^*).
\]

Add and subtract \( B(s, t^*)[p(s - \Delta, t^*) - k] \) to get

\[
\Delta f(s) = [p(s, t^*) - p(s - \Delta, t^*)]B(s, t^*) + \{
[p(s, t^*) - k]B(s - \Delta, t^*)\} \frac{\Delta B(s, t^*)}{B(s - \Delta, t^*)}.
\]

But \( \Delta F(s) = (P(s, t^*) - P(s - \Delta, t^*)) \) so in general, the change in value of the forward contract does not equal the change in value of the futures contract. If default-free rates are deterministic, then since \( P(s, t^*) \) equals \( p(s, t^*) \)

\[
\Delta f(s) = \Delta F(s)B(s, t^*) + \{(P(s, t^*) - K)B(s - \Delta, t^*)\} \frac{\Delta B(s, t^*)}{B(s - \Delta, t^*)}.
\]

This formula demonstrates that changes in the forward contract value equal the present value of changes in the futures price (or futures contract) plus the riskless return on the accumulated changes in the futures prices. Note when \( s = t \), \( P(s, t^*) = K \), and this formula corresponds to eq. (11).

8A futures contract could be defined differently. If the day's payoff (or the payoff at a position's closeout) is defined as the discounted difference between futures prices, arbitrage can force forward prices and futures prices into equality despite default rates. Then a futures contract would become a forward contract rewritten each day and the interim payments themselves make no difference.
4. Treasury bill futures contract: An example

The previous section demonstrates that a forward contract's value differs from a futures contract's value over the initial day of trading. This section shows how this affects Treasury bill futures contracts. As before, the valuation formula for a forward contract is derived without imposing any simplifying assumptions about the stochastic price movements. The valuation technique is interesting because there exists a portfolio of spot securities that precisely duplicates the cash flow pattern of the forward contract.

Consider a forward contract on an \( n \)-day maturity Treasury bill to be delivered at date \( t^* \) with an exercise price of \( k \). The U.S. Treasury bills maturing in \((t^*-t)\) days and \((t^*-t)+n\) days can be combined in a portfolio to duplicate the forward contract's cash flows. Define \( B(t,s) \) as the spot price at time \( t \) of one dollar received with certainty at time \( s \). This corresponds to the spot price at \( t \) per dollar maturity value of a Treasury bill that matures at \( s \).

A portfolio can be formed with one share long in the Treasury bill that matures in \((t^*-t)+n\) days, and \( \{B(t,t^*+n)/B(t,t^*)\} \) shares short in the Treasury bill that matures in \((t^*-t)\) days. The value of the portfolio at time \( s \), denoted \( V(s) \), is

\[
V(s)=B(s,t^*+n)-\{B(t,t^*+n)/B(t,t^*)\}B(S,t^*).
\]  
(12)

At time \( t \), \( V(t) \) equals zero, and at time \( t^* \), \( V(t^*) \) equals \( B(t^*,t^*+n)-B(t,t^*+n)/B(t,t^*) \). This is equivalent to a forward contract initiated at time \( t \) on a Treasury bill maturing at time \( t^*+n \), with an exercise price of \( B(t,t^*+n)/B(t,t^*) \). In a frictionless market the absence of arbitrage opportunities means the forward price must relate simply to spot prices [see expression (6) and note that \( D(t,t^*)=0 \) in this case],

\[
p(t,t^*)=B(t,t^*+n)/B(t,t^*).
\]  
(13)

The current forward price is the spot price ratio of the Treasury bills. Furthermore, when eq. (13) is true, \( V(t) \) exactly duplicates the cash flow stream of a forward contract. Consequently, the value of a Treasury bill forward contract must equal the value of the portfolio,

\[
f(p(s,t^*);k(t),t^*)=B(s,t^*+n)-\{B(t,t^*+n)/B(t,t^*)\}B(s,t^*).
\]  
(14)

To express eq. (14) in a form equivalent to eq. (3), use eq. (13) and factor out \( B(s,t^*) \),

\[
f(p(s,t^*);k(t),t^*)=(p(s,t^*)-p(t,t^*))B(s,t^*).
\]  
(15)
The relationship between $f$ and $F$ depends on $B(s, t)$. If the default-free rate is deterministic the Treasury futures contract value follows directly from eq. (15),

$$f(p(s, t^*); k(t), t^*) = F(P(s, t^*); K(t), t^*) \cdot B(s, t^*),$$

(16)

for $t \leq s < T$. However, if default-free rates are stochastic, $p(s, t^*)$ and $P(s, t^*)$ are not necessarily equal and $f$ and $F$ are not simply related. This means that Treasury bill futures prices may not be identical to forward prices implied in the default-free term structure. Consequently, spot Treasury bills cannot be used to construct an exact hedge with Treasury bill futures contracts.

5. Conclusion

Several recent papers confuse forward and futures contracts by stating that a futures contract is identical to a forward contract rewritten at the end of each day. This paper demonstrates that this is not the case. We show that the value of the forward contract over the initial day of trading is not necessarily related to the value of the corresponding futures contract. If default-free rates are deterministic, a forward contract's value is a proportion of a futures contract's value. The proportionality factor is the present value of receiving one dollar at the delivery date. A Treasury bill futures contract is examined to demonstrate that futures prices may not be useful for hedging forward prices implied in the default-free term structures.

References