LIQUIDITY PREMIUMS AND THE EXPECTATIONS HYPOTHESIS*

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Within the term structure of interest rate literature, three different quantifications of the expectations hypothesis are commonly employed. This paper demonstrates under very general conditions that the three quantifications are inconsistent. Each quantification implies a different price for the same bond. The paper concludes with a brief discussion of both the theoretical and empirical implications of these results.

1. Introduction

Three distinct theories of the term structure of interest rates have received repeated empirical testing, i.e., the expectations hypothesis, the liquidity preference hypothesis, and the market segmentation hypothesis.¹ These hypotheses involve the substitutability of default free, pure discount bonds of various maturities. The expectations hypothesis, credited to Fisher (1930) and Lutz (1940–1941), has as its basic premise that default free bonds of all maturities are perfect substitutes. Hick's (1946) liquidity preference hypothesis states that bonds of differing maturities are imperfect substitutes. The argument is that investors perceive long maturity bonds as riskier than short maturity bonds, due to larger price uncertainty over intermediate holding periods. Finally, the market segmentation hypothesis, normally associated with Culbertson (1957), argues that due to institutional factors, different maturity bonds are not substitutes. Bond prices for a particular maturity are determined in isolation from other maturity bonds.²

For convenience in empirical testing, the expectations hypothesis is normally the null hypothesis and the liquidity preference hypothesis is the

*This paper is an extension of a section contained in my Ph.D. dissertation at the Sloan School of Management, M.I.T.
¹For a good review of the theory and evidence see Van Horne (1970).
²Modigliani and Sutch's (1966) preferred habitat theory needs to be mentioned. It was formulated as a synthesis of the above three hypotheses. Their view is that investors have particular time intervals (their habitat) over which they want to hedge away wealth uncertainty. To induce an investor to hold a bond with a maturity different from his preferred habitat, he must be paid a premium in expected return. The actual size and magnitude of the premium is determined in equilibrium.

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alternative. At least three different quantifications of the expectations hypothesis can be found in the literature. The first, and perhaps most used approach equates the expectations hypothesis with the statement that the forward rate equals the expected future spot rate [see Meiselman (1962), McCulloch (1975), or Friedman (1979)]. Another approach tests whether a bond's yield is equal to the average of the expected spot rates over the life of the bond. Equality corresponds to the expectations hypothesis [e.g., see Modigliani and Shiller (1973), or Dobson, Sutch and Vanderford (1976)]. The third approach [see Cagan (1969), Santomero (1975), Fama (1976)] states that under the expectations hypothesis, bonds of different maturities have the same expected return over the same holding period. The relevant holding period is determined by the shortest maturity bond.

It has been shown by Cox, Ingersoll, and Ross (1978) in the context of a continuous time equilibrium model that the first two forms of the expectations hypothesis are inconsistent with the third. This model assumes that the state variables follow a Markov diffusion process. Richard (1978) also demonstrated this same inconsistency in an arbitrage model employing Markov-diffusion processes. This paper generalizes the above conclusions by showing the observed inconsistency is a general rule, independent of the structure of the economic model and independent of the type of uncertainty.

The proof of this statement is performed in the next section of the paper. The basic argument involves expressing a bond's price in terms of each quantification of the expectations hypothesis. Jensen's inequality is then invoked to show the resulting prices are inconsistent. The conclusion contains a discussion of the empirical implications of this result.

2. Quantifying the expectations hypothesis

This section presents the basic proposition of the paper, i.e., that the three quantifications of the expectations hypothesis discussed in the introduction are inconsistent. This proposition follows under very general conditions, and consequently generalizes statements made by Cox, Ingersoll and Ross (1978) and Richard (1978) in the context of specific economic models.

For simplicity, this paper assumes that default free, pure discount bonds of all maturities trade in frictionless markets. In particular, transaction costs in buying and selling bonds are excluded. A discrete time model is employed, however, under suitable restrictions these results can be extended to the continuous time case.³ Let \( P(t, T) \) equal the price at time \( t \) of a default free, pure discount bond that pays $1 in \( T \) time periods. By definition, \( P(t, 0) \) equals one.

The bond's yield, \( r(t, T) \), is defined by

\[
P(t, T) = \left[1 + r(t, T)\right]^{-T}, \quad r(t, T) \geq 0. \tag{1}
\]

³These restrictions will be made explicit in footnote 6.
The restriction that \( r(t, T) \geq 0 \) is consistent with the absence of arbitrage and the simultaneous trading of dollars and default free pure discount bonds. The spot rate is the yield on the shortest maturity bond, \( r(t, 1) \). It also corresponds to the known return on the shortest maturity bond over the next time interval, where a \( T \) maturity bond’s return is denoted by

\[
\frac{\Delta P(t, T)}{P(t, T)} = \frac{P(t + 1, T - 1) - P(t, T)}{P(t, T)}.
\]

(2)

Consequently, the spot rate is often called the ‘riskless rate’ [see Sharpe (1964) or Merton (1973)]. All other bond returns are random.

The forward rate over the period \([t + T, t + T + 1]\) is defined by

\[
f(t, T) = \frac{P(t, T)}{P(t, T + 1)} - 1.
\]

(3)

It corresponds to the interest rate one can contract for today (time \( t \)) on a nominal riskless loan over the period \([t + T, t + T + 1]\).\(^4\)

Given these rates, three liquidity premiums can be defined which correspond to the three quantifications of the expectations hypothesis. The first premium equals the difference between the forward rate and the expected spot rate,

\[
A(t, T) = f(t, T) - E_t(r(t + T, 1)).
\]

(4)

The operator, \( E_t(\cdot) \), is the conditional expectation operator given the information available at time \( t \). The second premium is the difference between the continuously compounded yield on a \( T \) maturity bond, \( \log(1 + r(t, T)) \), and the average of the expected future continuously compounded spot rates over \([0, T]\).\(^5\)

\(^4\)To see this correspondence, at time \( t \) construct the following portfolio: (i) buy \( P(t, T)/P(t, T + 1) \) units of the bond that matures at time \( T + 1 \), (ii) sell one unit of the bond that matures at time \( T \). The following cash flows occur:

<table>
<thead>
<tr>
<th>Time</th>
<th>Cash flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>0</td>
</tr>
<tr>
<td>( t + T )</td>
<td>-1</td>
</tr>
<tr>
<td>( t + T + 1 )</td>
<td>( \begin{bmatrix} P(t, T) \ P(t, T + 1) \end{bmatrix} )</td>
</tr>
</tbody>
</table>

This portfolio is identical to lending out $1 at time \( t + T \) for a certain nominal return of \( P(t, T)/P(t, T + 1) - 1 \).

\(^5\)Some intuition as to why \( \delta \) might be a suitable premium is given by noting that

\[
B^*(t, T) = \log(1 + f(t, T)) - E_t[\log(1 + r(t, 1))]
\]
\[ B(t, T) = \log (1 + r(t, T)) - \sum_{j=0}^{T-1} E_t[\log (1 + r(t + j, 1))] / T. \]  

Finally, the excess expected return on a \( T \) maturity bond over the riskless rate is

\[ C(t, T) = E_t\left( \frac{\Delta P(t, T)}{P(t, T)} \right) - r(t, 1). \]  

The various forms of the expectations hypotheses are given by setting each premium equal to zero,

expectations hypothesis \( L(t, T) = 0 \) for all \( T, t \), where

\[ L(t, T) \in \{ A(t, T), B(t, T), C(t, T) \}. \]

A non-zero premium that increases with maturity is usually associated with the liquidity preference hypothesis.

These three quantifications are inconsistent. The argument involves writing the pure discount, default free bond's price as a function of each premium. If each quantification implies a different bond price, then they are inconsistent.

For comparison, the bond's price as a function of each premium is presented in expression (8) prior to the algebraic derivation.\(^6\)

\[^6\]The continuous time case can be obtained by letting 'h' denote the discrete time period, and taking the limit as \( h \to 0 \). The corresponding premium definitions are:

\[ A(t, T) = f(t, T) - E_t[\log (1 + r(t + T, 0))], \]

\[ B(t, T) = \log (1 + r(t, T)) - \frac{\sum_{j=0}^{T-1} E_t[\log (1 + r(t + j, 1))]}{T}, \]

\[ C(t, T) = \lim_{h \to 0} \frac{\Delta P(t, h)}{P(t, h)} - \log (1 + r(t, 0)), \]  

where

\[ \lim_{h \to 0} \left[ \frac{\Delta P(t, h)}{h} / \frac{P(t, h)}{P(t + h)} \right] = \log (1 + r(t, 0)), \]  

and

\[ f(t, T) = \lim_{h \to 0} \left[ \frac{P(t, T)}{h} / P(t, T + h) \right] - 1. \]
\[ P(t, T) = \exp \left( - \sum_{j=0}^{T-1} \log (E_i(1 + r(t + j, 1)) + A(t, j)) \right) \]

\[ = \exp \left( - \sum_{j=0}^{T-1} E_i(\log (1 + r(t + j, 1)) - B(t, T)T) \right) \]

\[ = E_i \left( \exp \left( - \sum_{j=0}^{T-1} \log (1 + r(t + j, 1) + C(t + j, j)) \right) \right). \]

Expression (8) gives the bond's price as a function of the three premiums \( A(t, T) \), \( B(t, T) \), and \( C(t, T) \) respectively. Similar in appearance, these formulae differ only where the expectation operator and the premium appears. These equations correspond to the valuation formulas contained in Cox, Ingersoll and Ross (1978). The derivation of (8) is as follows.

To obtain the bond's price as a function of \( A(t, T) \), note that expression (4) can be rewritten as the following difference equation for \( P(t, T) \):

\[ E_i(1 + r(t + T, 1) + A(t, T))P(t, T + 1) = P(t, T), \quad P(t, 0) = 1. \]  

(9)

Holding \( t \) fixed, the solution is

\[ P(t, T) = \prod_{j=0}^{T-1} \frac{1}{E_i(1 + r(t + j, 1) + A(t, j))}, \]

(10)

which gives (8) after applying the exponential transformation \( e^{\log x} = x \). The bond's price as a function of \( B(t, T) \) follows by substituting (1) into (5), and rearranging terms. Finally, the bond's price as a function of \( C(t, T) \) is obtained by recognizing that (6) is a difference equation for \( P(t, T) \),

\[ P(t, T)(1 + r(t, 1) + C(t, T)) = E_i[P(t + 1, T - 1)], \]

\[ P(T, 0) = 1. \]  

(11)

If \( P(t, T) \) is twice continuously differentiable and follows a continuous time diffusion process, similar arguments produce

\[ P(t, T) = \exp \left( - \int_{0}^{T} E_i(\log (1 + r(t + s, 0)) + A(t, s)ds \right) \]

\[ = \exp \left( - \int_{0}^{T} E_i[\log (1 + r(t + s, 0))]ds - TB(t, T) \right) \]

\[ = E_i \left( \exp \left( - \int_{0}^{T} \log (1 + r(t + s, 0)) + C(t + s, s)ds \right) \right). \]  

(8')
Letting both \( t \) and \( T \) vary, the solution uses the fact that \( E_t(E_{t+1}(\cdot)) = E_t(\cdot) \) and the exponential transformation. This completes the derivation.

Theories for the term structure of interest rates are characterized by specifying the functional forms of either \( A(t, T) \), \( B(t, T) \), or \( C(t, T) \). The expectation hypothesis as contained in expression (7) is one such characterization. Given there is a non-trivial uncertainty about future spot rates (i.e., all the probability mass is not at a single point), and since both \( (e^x) \) and \( (-\log x) \) are strictly convex functions, Jensen’s inequality implies

\[
P(t, T) \bigg|_{A(t, T) = 0} = E_t \left[ \exp \left( - \sum_{j=0}^{T-1} \log (1 + r(t+j, 1)) \right) \right] > 0
\]

\[
P(t, T) \bigg|_{B(t, T) = 0} = \exp \left( - \sum_{j=0}^{T-1} E_t\{ \log (1 + r(t+j, 1)) \} \right) > 0
\]

\[
P(t, T) \bigg|_{C(t, T) = 0} = \exp \left( - \sum_{j=0}^{T-1} \log E_t\{ 1 + r(t+j, 1) \} \right) .
\] (12)

The three forms of the expectations hypothesis imply different bond prices. They are not consistent models of the term structure. This inconsistency is robust with respect to any economic model and any stochastic process.

Insight into this result can be obtained by examining the singular case where there is no uncertainty. In this model Jensen’s inequality reduces to an equality, and all three quantifications are equivalent.

\[
P(t, T) \bigg|_{L(t, T) = 0} = \prod_{j=0}^{T-1} \frac{1}{[1 + r(t+j, 1)]} .
\] (13)

The expectation operator is now unnecessary. The bond’s price is the standard discounted value of \$1 received at time \( T \). This result is not surprising. In this world, to avoid arbitrage, it must be the case that \( L(t, T) = 0 \) for all \( L(t, T) \in \{ A(t, T), B(t, T), C(t, T) \} \) [see Richard (1978)]. Otherwise, arbitrage profits can be obtained by either buying and selling bonds of different maturities [if \( C(t, T) \neq 0 \)] or contracting today to borrow and lend at future dates [if \( A(t, T) \neq 0 \) or \( B(t, T) \neq 0 \)]. If future spot rates are uncertain, however, these same arbitrage arguments no longer follow.

Returning to the general model, this inconsistency can further be elucidated through the following proposition. The proof of the proposition follows easily from expression (8) and Jensen’s inequality.
Proposition 1

\[ A(t, T) \equiv 0 \quad \text{implies} \quad B(t, T) > 0 \quad \text{for all} \ t, T, \]
\[ A(t, T) \geq 0 \quad \text{for all} \ t, \ T \quad \text{implies} \quad B(t, T) > 0 \quad \text{for all} \ t, T, \quad \text{and} \]
\[ L(t, T) \equiv 0 \quad \text{implies} \quad C(t, T) \neq 0 \quad \text{for some} \ t, T, \]
\[ L(t, T) \geq 0 \quad \text{for all} \ t, \ T \quad \text{implies} \quad C(t, T) \neq 0 \quad \text{for some} \ t, T \]
\[ \text{for} \ L(t, T) \in \{ A(t, T), B(t, T) \}. \]

The acceptance of the expectations hypothesis in terms of the first premium, \( A(t, T) \), rejects the expectations hypothesis in terms of \( B(t, T) \) and \( C(t, T) \). Furthermore, the acceptance of the expectations hypothesis in terms of \( B(t, T) \) [or the acceptance of the hypothesis that \( B(t, T) \) is always non-negative], rejects the expectations hypothesis in terms of \( C(t, T) \).

3. Conclusion

This paper has shown that the three standard quantifications of the expectations hypothesis contained in expression (7) are inconsistent. This implies that empirical tests of \( A(t, T) \equiv 0 \) [e.g., Meiselman (1962), Van Horne (1965), McCulloch (1975), Fama (1976), and Friedman (1979)], of \( B(t, T) \equiv 0 \) [e.g., Modigliani and Sutch (1966), Modigliani and Shiller (1973), Dobson, Sutch and Vanderford (1976)], and of \( C(t, T) \equiv 0 \) [e.g., Santomero (1975) and Fama (1976)] are testing different characterizations of the term structure. Whether any of these studies in fact tests the expectations hypothesis depends on the meaning of the phrase 'perfect substitutes'.

Setting either \( A(t, T), B(t, T) \) or \( C(t, T) \) identically equal to zero will in general not be an implication of the expectations hypothesis. To demonstrate this fact, we need only examine the simplest equilibrium asset model developed by Sharpe (1964), Lintner (1965), and Mossin (1966) and generalized to hold sequentially in time by Merton (1973). In this model, two bonds are perfect substitutes if they have identical market betas. Furthermore, identical market betas imply identical expected real returns. Unless the inflation rate has a zero variance, this implication does not imply \( C(t, T) \equiv 0 \), i.e., identical expected nominal returns. Identical market betas also does not imply \( A(t, T) \equiv 0 \) or \( B(t, T) \equiv 0 \).

Although the empirical studies testing \( A(t, T) \equiv 0, B(t, T) \equiv 0 \), and \( C(t, T) \equiv 0 \) do not necessarily test the expectations hypothesis, they do test the

\[ ^{3} \text{Briefly, the evidence suggests that} \ A(t, T) > 0, B(t, T) > 0, \text{and} \ C(t, T) > 0. \text{These results are consistent with the proposition.} \]
simplest possible characterization of the term structure. If any of these characterizations hold, then knowledge of the stochastic movement of spot rates alone would be sufficient to price all default free pure discount bonds. This is a vast simplification. Since the evidence cited is consistent with a rejection of these characterizations, we know that the valuation of default free bonds is much more complicated. The techniques of Cox, Ingersoll and Ross (1978), Richard (1978), and Langetieg (1980) are of more than just theoretical interest.

References


