A COMPARISON OF THE APT AND CAPM

A Note

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Received September 1982

The single factor version of Ross' arbitrage pricing theory and the Sharpe-Lintner-Mossin capital asset pricing model offer deceptively similar pricing relationships. This paper derives conditions for the equivalence of the two paradigms.

1. Introduction

Two asset pricing models have received repeated attention in the academic literature: the capital asset pricing model (CAPM) of Sharpe, Lintner and Mossin and the arbitrage pricing model (APT) developed by Ross (1976). This paper examines the conditions under which the pricing models are equivalent, and shows that under plausible conditions they can differ.

The APT relates asset expected returns to the expected return on a portfolio which is exposed to the single factor, whereas the CAPM relates asset expected returns to the expected excess return on the market portfolio.\textsuperscript{1} The reason for the difference between the models is based on the fact, stated by Ross (1978), that the market portfolio plays no fundamental role in the APT. Leaving aside some pathological cases; if the market portfolio is well-diversified so that it is exposed only to factor risk then the models are asymptotically equivalent. Conversely, if the market portfolio is exposed to the specific risk of its constituent assets in addition to factor risk, then the models will differ.

An outline of this paper is as follows. Section 2 presents the assumptions and results of the APT and the CAPM. Section 3 defines our measure of equivalence. Section 4 states the necessary and sufficient conditions for the equivalence of the two models. Section 5 presents an example where the models differ, while section 6 provides a brief summary.

\textsuperscript{1}We restrict attention in this paper to the single factor form of the APT and the 'simple' CAPM. We believe that direct extensions of this analysis are applicable to more realistic forms of the CAPM [e.g., Breeden (1979)], and the corresponding multiple factor version of the APT.

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2. The two pricing paradigms\(^2\)

The Sharpe–Lintner–Mossin CAPM is derived under the following assumptions:

(A.1) Frictionless markets.
(A.2) Investors have a von Neumann–Morgenstern preference function for portfolio returns which is continuous and monotone increasing.
(A.3) There exists a riskless asset with rate of return \(r\).
(A.4) Investors are risk averse, and satisfy the mean-variance criterion.
(A.5) Investors have homogeneous expectations.

The proof is contained in Haley and Schall (1979). Let asset returns be denoted by \(x_i\) \((i = 1, \ldots, R)\). The market portfolio return \(x_M\) is given by \(x_M = \sum_{i=1}^{R} w_i x_i\), where \(\sum_{i=1}^{R} w_i = 1\) and \(w_i > 0\) represents the proportion of the \(i\)th asset in the market portfolio. The CAPM relationship is

\[
E[x_i] - r = \beta_i (E[x_M] - r),
\]

where

\[
\beta_i = \frac{\text{cov}[x_i, x_M]}{\text{var}[x_M]}.
\]

Ross’ APT is derived by considering a sequence of markets having increasing numbers of assets. The pricing relationship is obtained in the limit as the number of assets tends to infinity. The following assumptions are required:\(^3\)

(B.1) There exists at least one asset with limited liability.
(B.2) There exists a riskless asset with rate of return \(r\).
(B.3) All investors are risk averse.
(B.4) There exists at least one investor who (i) has a coefficient of relative risk aversion which is uniformly bounded, (ii) is not asymptotically negligible, and (iii) believes that security returns are generated by the single factor model (2),

\[
x_i = E[x_i] + b_{f} f + u_i, \quad \text{where}
\]

\[
E[u_i] = E[f] = 0,
\]

\[
E[u_i u_j] = 0 \quad \text{if} \quad i \neq j
\]

\[
= \sigma_i^2 \leq \sigma_{\text{max}}^2 < \infty \quad \text{if} \quad i = j,
\]

\(^2\)The assumptions listed here are those most commonly used to derive the respective models. These are not necessarily the least restrictive or those used in the original derivation.

\(^3\)See Ross (1976, pp. 347–351).
and $f$ represents the random return on the factor, $b_i(<\infty)$ is the loading of the $i$th asset onto the factor, $u_i$ is the random specific return on the $i$th asset. Further, all investors have homogeneous beliefs for $E[x_i]$ ($i = 1, \ldots, R$), which are uniformly bounded (i.e., sup$_i|E[x_i]| < \infty$).

The APT pricing relation is

$$\sum_{i=1}^{\infty} (E[x_i] - r - b_i\psi)^2/\sigma_i^2 < \infty,$$

where $\psi$ is a constant. This parameter is further identified when (3) is written in the alternative form:

$$E[x_i] - r \approx b_i\psi \quad \text{for all assets } i \text{ and all } R,$$

from which it follows that $\psi$ is the approximate expected excess return on the factor.$^5$

Define $b_M = \sum_{i=1}^{R} w_i b_i$. Provided $b_M \neq 0$, we can always redefine $b_M = 1$ together with the corresponding change of scale to the factor (this step is equivalent to defining an origin for measurement).$^6$ For the remainder of the paper, we assume that this step has been taken so that $b_M = 1$.

The assumptions of both models are consistent, and both can hold simultaneously. To make the limiting form of the CAPM meaningful, however, we add

$$(C.1) \quad \lim_{R \to \infty} |E(x_M) - r| \neq 0, \quad \text{and}$$

$$\text{(C.2) there exists a constant } \beta^* < +\infty \text{ such that } |\beta_i| \leq \beta^* \text{ for all } i \text{ and } R.$$

Assumption (C.1) insures that the market portfolio requires a risk premium, even in the limiting economy. Although $b_M \neq 0$ implies $E(x_M) - r \neq 0$ for all $R$, this does not necessarily imply (C.1). Assumption (C.2) uniformly bounds the size of an asset’s beta.

Further, to be consistent with assumptions invoked in the tests of the APT [see, e.g., Roll and Ross (1980)], we make the following common assumption:

$$(C.3) \quad \text{The factor return is uncorrelated with the specific returns of every asset, i.e., } \text{cov}(f, u_i) = 0 \text{ for all } i.$$

$^4$If $\sigma_i^2 = 0$ for some $i$ then the asset can be eliminated without loss of generality. See Ross (1976, p. 348, footnote 4).

$^5$For clarity, we note that $b_i$, $\sigma_i$, and $r$ are constants (independent of $R$). We also suppress the subscript $R$ from $E(x_i) = E(x_i^R)$, $\beta_i = \beta_i^R$, and $E(x_i^M) = E(x_i^M)$ which are dependent on $R$.

$^6$Ross (1977) shows that this normalization is always possible provided investors are risk averse.

$^7$This follows from Assumption (A.4).

$^8$This could be weakened to requiring only a finite number of assets to satisfy $\text{cov}(f, u_i) = 0$. 
3. Meaning of equivalence

**Definition.** Let Assumptions (A.1)–(A.5), (B.1)–(B.4), and (C.1)–(C.3) hold simultaneously. The two models are said to be equivalent if

\[ \lim_{R \to \infty} |\beta_i - b_i| = 0 \quad \text{for all } i, \quad \text{and} \]

\[ \lim_{R \to \infty} |E(x_M) - r - \psi| = 0. \]  

(5)

This definition is the natural one for comparison since it states that the models are equivalent if, in the limit, every asset's beta equals \(b_i\) and the excess return on the market portfolio is \(\psi\).

**Theorem 1. Implication of equivalence.** If

\[ \lim_{R \to \infty} |\beta_i - b_i| = 0 \quad \text{for all } i \text{ and} \]

\[ \lim_{R \to \infty} |E(x_M) - r - \psi| = 0, \quad \text{then} \]

\[ \lim_{R \to \infty} |\beta_i (E(x_M) - r) - b_i \psi| = 0 \quad \text{for all } i. \]

**Proof.** From

\[ 0 \leq |\beta_i (E(x_M) - r) - b_i \psi| = |\beta_i (E(x_M) - r - \psi) + (\beta_i - b_i) \psi| \]

\[ \leq |\beta_i| |E(x_M) - r - \psi| + |\psi| |\beta_i - b_i| \]

we get the required result using (C.2). Q.E.D.

Theorem 1 shows that this definition of equivalence is stronger than just assuming that in the limit, for all \(i\), both models give the same value for the excess expected return. The next theorem examines the converse of Theorem 1.

**Theorem 2**

\[ \lim_{R \to \infty} |\beta_i - b_i| = 0 \quad \text{uniformly on } i, \quad \text{and} \]

\[ \lim_{R \to \infty} |E(x_M) - r - \psi| = 0 \quad \text{if and only if} \]

\[ \lim_{R \to \infty} |\beta_i (E(x_M) - r) - b_i \psi| = 0 \quad \text{uniformly on } i. \]
Proof. (Step 1.) Assume \( \lim_{R \to \infty} |\beta_i(E(x_M) - r) - b_i\psi| = 0 \) uniformly on \( i \), i.e.,

\[
\lim_{R \to \infty} \sup_i |\beta_i(E(x_M) - r) - b_i\psi| = 0.
\]

Using \( \sum_{i=1}^{R} w_i\beta_i = 1, \sum_{i=1}^{R} w_i = 1 \), and \( w_i > 0 \) it follows that

\[
\sup_i |\beta_i(E(x_M) - r) - b_i\psi| \geq \sum_{i=1}^{R} w_i |\beta_i(E(x_M) - r) - b_i\psi| \geq \left| \sum_{i=1}^{R} w_i\beta_i(E(x_M) - r) - \psi \sum_{i=1}^{R} w_i b_i \right| = |E(x_M) - r - \psi| \geq 0
\]

so \( \lim_{R \to \infty} |E(x_M) - r - \psi| = 0 \). Further,

\[
0 = \lim_{R \to \infty} (\beta_i(E[x_M] - r) - b_i\psi)
= \lim_{R \to \infty} \left[ (\beta_i - b_i)(E[x_M] - r) \right] + b_i \lim_{R \to \infty} (E[x_M] - r - \psi).
\]

So

\[
\lim_{R \to \infty} |\beta_i - b_i| = 0 \quad \text{since} \quad \lim_{R \to \infty} |E[x_M] - r - \psi| = 0
\]

and (C.1). Q.E.D.

(Step 2.) Assume \( \lim_{R \to \infty} |\beta_i - b_i| = 0 \) uniformly on \( i \) and \( \lim_{R \to \infty} |E(x_M) - r - \psi| = 0 \). From

\[
0 \leq |\beta_i (E(x_M) - r) - b_i\psi|
= |\beta_i (E(x_M) - r - \psi) + (\beta_i - b_i)\psi| \\
\leq |\beta_i| |E(x_M) - r - \psi| + |\psi| |\beta_i - b_i|
\leq \beta^* |E(x_M) - r - \psi| + |\psi| \sup_i |\beta_i - b_i|
\]

we get the required result. Q.E.D.

4. Necessary and sufficient conditions for equivalence

Theorem 3. Necessary and sufficient conditions for equivalence. Given Assumptions (A.1)–(A.5), (B.1)–(B.4), and (C.1)–(C.3) the models are equivalent if and only if

(i) \( \lim_{R \to \infty} \sum_{i=1}^{R} w_i^2 \sigma_i^2 = 0 \), or

(ii) \( \lim_{R \to \infty} \sum_{i=1}^{R} w_i \gamma_i = 0 \), and so forth.
\[
\begin{align*}
(ii) \quad \lim_{R \to \infty} w_i = b_i c / \sigma_i^2 & \quad \text{for all assets } i, \text{ where} \\
0 < c = \lim_{R \to \infty} \sum_{i=1}^{R} w_i^2 \sigma_i^2.
\end{align*}
\]

**Proof.** (a) Sufficient condition (i): Given condition (i), using the definition of the market portfolio

\[
\left| E[x_M] - r - \psi \right| = \left| \sum_{i=1}^{R} w_i (E[x_i] - r - \psi) \right|
\]

\[
= \left| \sum_{i=1}^{R} w_i (E[x_i] - r - b_i \psi) \right|
\]

since by construction \( \sum_{i=1}^{R} w_i b_i = 1 \) for all \( R \). Now applying Cauchy's inequality

\[
0 \leq \left| \sum_{i=1}^{R} w_i (E[x_i] - r - b_i \psi) \right|^2 \leq \left( \sum_{i=1}^{R} w_i^2 \sigma_i^2 \right) \left( \sum_{i=1}^{R} (E[x_i] - r - b_i \psi)^2 / \sigma_i^2 \right).
\]

In the limit, as \( R \to \infty \), the right-hand side is zero from (i) and the APT relationship (3). Hence,

\[
\lim_{R \to \infty} \left| \sum_{i=1}^{R} w_i (E[x_i] - r - b_i \psi) \right| = 0,
\]

which is one part of the equivalence definition. The second part follows from the definition of \( \beta_i = \text{cov}[x_i, x_M] / \text{var}[x_M] \). A little algebra yields

\[
\left| \beta_i - b_i \right| = \left| \frac{b_i \text{var}[f] + w_i \sigma_i^2}{\text{var}[f] + \sum_{j=1}^{R} w_j^2 \sigma_j^2} - b_i \right|
\]

But \( \lim_{R \to \infty} \sum_{i=1}^{R} w_i^2 \sigma_i^2 = 0 \) implies \( \lim_{R \to \infty} w_i \sigma_i^2 = 0 \) for all \( i \), hence, in the limit the right-hand side tends to zero.

(b) Sufficient condition (ii): Suppose case (ii) holds, then \( \lim_{R \to \infty} \left| w_i \sigma_i^2 - b_i \sum_{i=1}^{R} w_i^2 \sigma_i^2 \right| = 0 \) or equivalently \( \lim_{R \to \infty} \left| \beta_i - b_i \right| = 0 \). We next show that \( \psi = \lim_{R \to \infty} E[x_M] - r \) is consistent with the APT.

Under (ii), only a finite number of \( b_i \neq 0 \). Suppose this were false, i.e., \( b_i \neq 0 \) for an infinite number of assets indexed by a set \( J \). Since \( \{b_i, \sigma_i\} \) are independent of \( R \), \( \lim_{R \to \infty} w_i = h_i c / \sigma_i^2 \), and \( c > 0 \) implies \( \lim_{R \to \infty} w_j > 0 \) for all \( j \in J \). So \( \sum_{j \in J} w_j = + \infty \), which contradicts \( \sum_{j \in J} w_j \leq 1 \).
Using the APT relationship (3), for $J$ finite,

$$\sum_{i=1}^{J} (E[x_i] - r - b_i \psi)^2/\sigma_i^2 + \sum_{i=J+1}^{\infty} (E[x_i] - r)^2/\sigma_i^2 < + \infty$$

which implies

$$\sum_{i=1}^{J} (E[x_i] - r - b_i (\lim_{R \to \infty} E[x_M] - r))^2/\sigma_i^2 + \sum_{i=J+1}^{\infty} (E[x_i] - r)^2/\sigma_i^2 < + \infty.$$ 

Hence, $\lim_{R \to \infty} E[x_M] - r = \psi$ is consistent with the APT (in fact any $\psi$ works which is invariant with respect to $R$).

(c) Necessary conditions: By hypothesis,

$$\lim_{R \to \infty} |b_i - b| = 0 \quad \text{and} \quad \lim_{R \to \infty} |E[x_M] - r - \psi| = 0.$$ 

From the first these limits, substituting for $b_i$ gives

$$\lim_{R \to \infty} w_i \sigma_i^2 - b_i \sum_{j=1}^{R} w_j^2 \sigma_j^2 = 0 \quad \text{for all } i,$$

since $\text{var}(x_M)$ is always non-zero and finite. There are two cases to be considered:

(i) $\sum_{j=1}^{\infty} w_j^2 \sigma_j^2 = 0$, and

(ii) $\sum_{j=1}^{\infty} w_j^2 \sigma_j^2 = c > 0$, a constant. This implies that

$$\lim_{R \to \infty} w_i = b_i c / \sigma_i^2 \quad \text{for all } i.$$

The intuition behind these conditions is that in the limit, the market portfolio must be exposed only to factor risk if the APT and CAPM are to be identical. Condition (i) causes the specific risk of the market portfolio to be diversified away perfectly, while the condition that the factor and specific returns be uncorrelated [Assumption (C.3)] permits the separation of factor risk and specific risk. If (C.3) did not hold, then even given (i), every asset's factor loading need not equal its beta.

Let us consider case (ii), the situation where the market portfolio weights are restricted to being equal to a particular constant. If the specific risk component of the market portfolio is not diversified away (i.e., $c = \lim_{R \to \infty} \sum_{j=1}^{\infty} w_j^2 \sigma_j^2 > 0$) and $\lim_{R \to \infty} w_i = b_i c / \sigma_i^2$ for all $i$ together with the
other conditions holding, then it is true that the APT and CAPM are equivalent. However, this case is very unrealistic. As shown in the proof, this situation implies that only a finite number of factor loadings are non-zero (and these must be positive). This imposes a substantial restriction on the exogenous parameters of the economy \( \{b_i, \sigma_i\} \). In addition, if these parameters were slightly perturbed, this case would no longer apply. Across all possible parameter pairs \( \{b_i, \sigma_i\} \) for the economy, this situation has zero probability of ever being observed (in a randomly selected economy).

Under an additional assumption, condition (i) of theorem 3 can be shown to be equivalent to a condition involving only the portfolio weights, i.e.,

**Proposition.** If \( \sigma_{\text{max}}^2 \geq \sigma_i^2 \geq \sigma_{\text{min}}^2 > 0 \) for all \( i \), then

\[
\lim_{R \to \infty} \sum_{j=1}^{R} w_j^2 \sigma_j^2 = 0 \quad \text{if and only if} \quad \lim_{R \to \infty} \sum_{j=1}^{R} w_j^2 = 0.
\]

**Proof.** \( 0 \leq \sigma_{\text{min}}^2 \sum_{j=1}^{R} w_j^2 \leq \sum_{j=1}^{R} w_j^2 \sigma_j^2 \leq \sigma_{\text{max}}^2 \sum_{j=1}^{R} w_j^2 \). Q.E.D.

This condition will be employed in the following example.

**5. A counter example**

As the number of assets, \( R \), in the market increases, the market portfolio weights need not all converge to zero in the manner required for the equivalence of the APT and CAPM, even if the \( \text{cov} [f, u_i] = 0 \) for all assets \( i \). A plausible example occurs where one (or a finite number of) assets always constitute a positive fraction of the market portfolio.

For example, let \( w_i = w \), where \( 0 < w < 1 \), \( \sigma_i^2 \neq 0 \) and \( \sigma_i^2 = \alpha > 0 \) for \( 1 < i < R \). Notice that \( \sum_{i=1}^{R} w_i = 1 \) but the portfolio holdings do not satisfy sufficient condition (i) since

\[
\sum_{i=1}^{R} w_i^2 = w^2 + (1-w)^2 \frac{1}{(R-1)^2} \to w^2 > 0 \quad \text{as} \quad R \to \infty.
\]

For this sequence of markets, given that (C.3) holds:

\[
|\beta_i - b_i| = \left| \frac{b_i \text{ var } [f] + w \sigma_i^2}{\text{ var } [f] + w^2 \sigma_i^2 + (1-w)^2 \sum_{j=2}^{R} \sigma_j^2/(R-1)^2} \right| b_i
\]

\[
\to \left| \frac{w \sigma_i^2 (1-b_i w)}{\text{ var } [f] + w^2 \sigma_i^2} \right| \quad \text{as} \quad R \to \infty.
\]
That is, the limit is a function of $b_1$ and not necessarily zero. Similarly, $|\beta_i - b_i|$ is a function of $b_i$ for $i > 1$, so the APT and CAPM are not equivalent.

6. Conclusion

There are two conditions for the single factor APT and ‘simple’ CAPM to be asymptotically equivalent; namely, the factor must be uncorrelated with the residuals and the market must be well-diversified. It is of interest to question whether these conditions hold in the capital markets so that the APT and CAPM would yield identical predictions. Unfortunately, it is difficult (if not impossible) to find information in our strictly finite capital markets which would provide evidence as to whether the APT and CAPM are equivalent in infinite capital markets.

References