Consensus Beliefs Equilibrium and Market Efficiency

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ABSTRACT

This paper presents an analysis of the concept of consensus beliefs and its relation to market efficiency. We show that unless traders have rational expectations, the two published interpretations of consensus beliefs are not useful for considerations of market efficiency. One interpretation (see Verrecchia [6]) has no implication for market efficiency. Under the second interpretation (see Verrecchia [7, 8]) consensus beliefs equilibria are efficient, but they typically do not exist unless traders have rational expectations.

A PROMINENT APPROACH IN finance for characterizing market efficiency is Rubinstein’s notion of consensus beliefs [5]. Rubinstein defines consensus beliefs to be “those beliefs which, if held by all individuals in an otherwise similar economy, would generate the same equilibrium prices as in the actual heterogeneous economy.”1 As shown by Verrecchia [6], if arbitrarily selected beliefs are considered, then there will typically exist homogeneous beliefs which result in the same prices as those resulting from the actual heterogeneous beliefs. Such beliefs qualify as consensus beliefs, but the existence of such consensus beliefs has no relation to informational efficiency. The reason for this is that the logic of Verrecchia’s existence argument applies regardless of the information sets in the economy, and hence of market efficiency considerations.

Verrecchia [8] introduces an alternate interpretation of consensus beliefs which utilizes information and is consistent with Rubinstein’s [5, p. 818] original motivation. Let an individual’s information be represented by an observation of a random variable which is correlated with future returns, and suppose that each agent’s beliefs are generated by a revision of his prior beliefs given this information. Let p be an equilibrium price for this economy. If the concept of consensus beliefs is to be useful for deciding whether an agent perceives that his information is “fully reflected” in p, then the consensus belief must not be inconsistent with the agent’s information. Hence, an agent’s beliefs could be called a consensus belief if p would not change if all agents had everyone else’s information. This is essentially Verrecchia’s [8] definition, and we call this type of consensus beliefs efficient consensus beliefs.

With this reinterpretation, there is a link between efficient consensus beliefs and informational efficiency. If traders have efficient consensus beliefs, then the market is informationally efficient; however, the requirement of efficient consen-

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1 See Rubinstein [5], page 818.
sus beliefs is very stringent. In fact, we show that if a standard (or in Radner's [4] terminology an unsophisticated) equilibrium concept is used, then efficient consensus beliefs equilibria generally do not exist. Hence, this interpretation does not appear to be a useful concept except in very rare circumstances.

Due to the equilibrium concept used, efficient consensus beliefs assumes that individuals are naive in their understanding of the market process. If instead we assume traders are sophisticated and use the rational expectations equilibrium concept, then revealing rational expectations equilibria will generally exist for the economies we consider. In a revealing rational expectations equilibrium all agents have the same information, so if they have homogeneous priors they necessarily have efficient consensus beliefs.

In summary, as efficient consensus beliefs equilibria typically exist only when agents have rational expectations, consensus beliefs cannot be used as an alternative theory to rational expectations for considerations of market efficiency. In fact, it has almost no economic content unless rational expectations are employed.

The remainder of the paper is organized as follows. Section I introduces the model and the notation employed. Section II defines three related equilibrium concepts, and Section III discusses consensus beliefs equilibrium. Section IV presents the concept of a rational expectations equilibrium, while Section V illustrates the previous concepts with an example. Finally, Section VI presents a brief conclusion.

I. The Economy

The basic structure of the economy we examine is drawn from Radner [4]. We consider a pure exchange economy in which trader \( i (i = 1, \ldots, I) \) chooses \( (c_i, z_i) \), where \( c_i \in R_+ \) is his consumption and \( z_i \in R^K_+ \) is his portfolio of \( K \) assets. At the time of the current market, the return vector for the \( K \) assets, \( r \), is unknown. Let the set of possible states or asset return vectors, \( A \), be finite (cardinality \( n \)). Traders \( i = 1, \ldots, J \) receive a signal \( s \) from the finite set \( S \) (cardinality \( m \)). This signal is correlated with \( r \) and consequently these traders are called informed. Traders \( i = J + 1, \ldots, I \) receive no signal and are called uninformed. Each trader has a subjective joint probability distribution on \( S \times A \) which is represented by a \( m \times n \) matrix \( Q \), where a typical element \( Q_{sr} \) is the joint probability of signal \( s \) and return \( r \), \( Q \in \Phi = \{ Q \in R^{mn} : Q_{sr} > 0 \text{ for all } s \text{ and } r, \text{ and } \sum_{(s,r) \in S \times A} Q_{sr} = 1 \} \). Finally, each trader has an endowment vector \( w_i = (w_i^0, w_i^1, \ldots, w_i^K) \in R^{K+1} \). The assumptions that all informed traders receive the same signal and that all traders have the same beliefs on \( S \times A \) are made only to simplify the exposition; they do not affect the nature of the conclusions.

\(^2\) Consensus beliefs and rational expectations equilibria are explicitly posed as competing theories by Verrecchia [8].

\(^3\) The assumption that \( S \) is a finite set is important for the statements that we make about rational expectations equilibrium. This assumption can be relaxed to \( dim S = K \) without destroying the generic existence of revealing rational expectations equilibria. See Jordan and Radner [3] for a survey of the literature.

\(^4\) This economy simplifies Radner [4] in that Radner allows informed traders to receive different signals. With the exception of this simplification, however, the two economies are identical.
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If return \( r \) obtains, trader \( i \)'s von Neumann-Morgenstern utility is denoted \( U_0(c_i) + U_i(r'z_i) \) where \( r'z_i \) is the return on his asset portfolio. For an informed trader, expected utility conditional upon signal \( s \) is

\[
U_0(c_i) + E_q[U_i(r'z_i)|s] = U_0(c_i) + V_i(z_i, s)
\] (1)

Uninformed traders do not know \( s \), so expected utility for traders \( i = J + 1, \ldots, I \) is

\[
U_0(c_i) + E_q[U_i(r'z_i)] = U_0(c_i) + \sum_{s \in S} \pi_s V_i(z_i, s)
\] (2)

where

\[
\pi_s = \sum_{r \in R} Q_{sr}
\]

is the marginal probability of signal \( s \).

We normalize prices so that the price of one unit of current consumption is 1. The vector of current asset prices is denoted \( p \in R^K \). Given his endowment, trader \( i \)'s budget constraint is then

\[
c_i + p'z_i \leq w_i^0 + p'(w_i, \ldots, w^K)
\] (3)

The following assumptions are imposed on the market economy.

(A.1) \( U_{0i} \) and \( U_i \) are twice continuously differentiable, strictly concave, and \( \lim_{y \to 0} U'_i(y) = +\infty \) for all \( i \).

(A.2) (a) \( n > K \), any \( K \) vectors in \( A \) span \( R^K \), and \( r \gg 0 \).

(b) At any equilibrium, for every \( i \) there exists no \( x \in R^K \), such that

for every \( r \), \( (r'x)U'_i(r'z_i) = 1 \).

Assumptions (A.1) and (A.2a) ensure the existence of equilibrium, that demands are single valued, and that \( c_i > 0 \) for all \( i \). Assumption (A.2a) also requires incomplete markets since there are more states of the economy (\( n \)) than securities (\( K \)). In addition, the return vectors are strictly positive and satisfy a spanning condition which requires that no two assets are redundant, i.e., span the same subset of \( A \). Finally, Assumption (A.2b) excludes situations where a small change in a portfolio at equilibrium leaves marginal utility constant across states. This condition essentially ensures that new information which differentiates states will induce a change in an investor's optimal portfolio and thereby change aggregate demand and equilibrium prices.\(^5\)

II. Equilibrium

Two related concepts of market equilibrium can be distinguished. The first is called unsophisticated equilibrium and the second full information equilibrium (both terms are due to Radner [4]). This section defines both equilibria.

Let us introduce the following notation: \( \pi_{sr} = Q_{sr}/\sum_{r \in A} Q_{sr} \) is the traders conditional probability for \( r \), given \( s \), and \( \pi_r = \sum_{s \in S} Q_{sr} \) is the traders marginal probability for \( r \). Let \( \pi_s = (\pi_{sr}, \ldots, \pi_{sr_{n-1}}) \) and \( \pi = (\pi_{r_1}, \ldots, \pi_{r_{n-1}}) \) where \( A = (r_1, \ldots, r_n) \).

An unsophisticated equilibrium is defined as follows. Informed trader \( i \) maximizes \( E_q[U_0(c_i) + U_i(r'z_i)|s] \) subject to (3). Denote the informed trader \( i \)'s asset

\(^5\) See Radner [4] for a discussion of these assumptions.
demand function by
\[ z_i(p, \pi_s) \in R^K \]
where \( \pi_s \) is the conditional probability vector for the informed traders. Similarly, uninformated trader \( i \) maximizes \( E_q[U_i(c_i) + U_i(r'z_i)] \) subject to (3), generating the asset demand function \( z_i(p, \pi) \) where \( \pi \) is the unconditional probability vector for the uninformed traders. Note that both groups of traders use only their private information.

The total market excess demand vector for the \( K \) assets is
\[ Z(p_s, \pi_s, \pi) = \sum_{i=1}^J z_i(p_s, \pi_s) + \sum_{i=J+1}^I z_i(p, \pi) - \sum_{i=1}^J (w_i \ldots, w_i^K) \]
A unsophisticated market equilibrium is a collection of price vectors \( (p_s)_{s \in S} \) such that, for each \( s \in S \),
\[ Z(p_s, \pi_s, \pi) = 0. \tag{4} \]
By assumptions (A.1)–(A.2) such an equilibrium exists (see Radner [4, p. 666]).

The full information equilibrium is obtained by letting every trader maximize \( E_q[U_i(c_i) + U_i(r'z_i)|s] \) subject to (3). Here the uninformed traders also know the signal \( s \). The corresponding market excess demand function is \( Z(p, \pi_s, \pi) \) evaluated at \( \pi = \pi_s \), i.e., \( Z(q, \pi_s, \pi) \).

A full information equilibrium is a collection of price vectors \( (p_s)_{s \in S} \) such that, for each \( s \in S \),
\[ Z(p_s, \pi_s, \pi_s) = 0. \tag{5} \]
Since this is a particular case of (4), such an equilibrium always exists under (A.1)–(A.2).

### III. Consensus Beliefs

According to Verrecchia’s [6] interpretation, the definition of consensus beliefs equilibrium requires only that there exist some homogeneous belief \( \hat{\pi} \) which, if held by all traders, would result in the same equilibrium price as in the actual heterogeneous economy. The intended definition of market equilibrium is what we call an unsophisticated equilibrium. In our notation, this definition requires that for each \( s \in S \) there exist a \( \hat{\pi}(s) \) such that
\[ Z(p_s, \pi_s, \pi) = 0 \quad \text{and} \quad Z(p_s, \hat{\pi}(s), \hat{\pi}(s)) = 0 \tag{6} \]
The existence of such consensus belief, \( \hat{\pi}(s) \), is demonstrated in Verrecchia [6]. Thus any equilibrium is a consensus beliefs equilibrium and the fact that such a \( \hat{\pi}(s) \) exists says nothing about the informational efficiency of the market. It will typically exist regardless of the concept of equilibrium used to determine \( p_s \), or the sophistication of the traders. In fact, \( \hat{\pi}(s) \) may even involve traders ignoring their private information.

Verrecchia [8] in a subsequent analysis utilizes Rubinstein’s [5, p. 818] motivation for consensus beliefs and relates this concept to informational efficiency as follows: “A market is said to be efficient with respect to an information set \( A \) if the prices it generates are identical to those generated in an otherwise identical
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economy in which the set $A$ accurately describes the information available to each and every market participant; the common, or homogeneous, belief induced by knowledge of $A$ is the consensus belief.” In our model, the only information of interest is the signal received by the informed traders; so the consensus belief would be $\pi_s$. If traders have such consensus beliefs, which we call efficient consensus beliefs, equilibrium prices remain unchanged regardless of whether uninformed investors know the true signal, $s$. The market price “reflects” the true signal, even without uninformed traders using equilibrium prices to infer information about the informed traders’ signal. This insight differentiates this concept of efficiency from the concept of rational expectations equilibrium (for a definition of rational expectations equilibrium see Section IV).

An efficient consensus beliefs equilibrium is a collection of prices $(p_s)_{s \in S}$ such that, for each $s \in S$,

$$Z(p_s, \pi_s, \pi_s) = 0 \quad \text{and} \quad Z(p_s, \pi_s, \pi_s) = 0$$

(7)

This definition formalizes the notion that prices would remain unchanged if all traders had the consensus belief $\pi_s$, where $\pi_s$ is the full information belief.

The following proposition shows that if the unsophisticated equilibrium concept is used, then efficient consensus beliefs equilibria typically do not exist.7,8

**Proposition (Generic Nonexistence of Full Information Consensus Beliefs):**

For the economy presented in Section I, except for a negligible subset (a set whose closure has Lebesgue measure zero) of $\Phi$, an efficient consensus beliefs equilibrium does not exist.

This result follows from the fact that changing the beliefs of some traders will typically change equilibrium prices. The conclusion does not depend on our assumption that there are informed and uninformed traders with all informed traders receiving the same signal. In an economy where all traders receive (perhaps different) signals, the same result follows provided that the information set that the market is to be efficient with respect to, contains information which is not initially known by some traders.9

A nongeneric case in which an efficient consensus beliefs equilibrium does exist is when the actual joint distribution $Q$ on $S \times A$ is such that the informed traders know (for each $s \in S$) that a particular $r_i \in A$ will occur, i.e., for each $s \in S$, $\pi_s = (0, 0, \ldots, 1, \ldots, 0)$ where the 1 appears in the $i$th co-ordinate and all other co-ordinates are zero. In this case, $p_s = r_i$ solves (7), and hence an efficient consensus beliefs equilibrium exists.

The proposition does not apply to an economy with risk-neutral traders as our assumption of strict concavity (A.1) excludes this case. If any informed trader is

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6 Verrecchia [8], p. 874. See also the following paragraph and footnotes 2 and 3.
7 A proposition is said to be generic if the property holds except on a negligible set. A subset $B \subseteq \mathbb{R}^n$ is negligible if its closure has Lebesgue measure zero (see Radner [4], p. 658).
8 The proof of this proposition is a straightforward extension of a proposition contained in Radner [4, p. 668] and is therefore omitted. The proof is available from the authors upon request.
9 The proof of this claim is essentially the same as the proof of the proposition. If the information set contains new information for some traders, their beliefs will typically change and the equilibrium price will therefore typically change.
risk neutral, then if there is an equilibrium, the equilibrium price vector must be \( p_s = E[r | s] \) for each \( s \in S \). Since this follows regardless of the beliefs of the uninformed traders, prices \( \{ E[r | s] \}_{s \in S} \) are efficient consensus beliefs equilibrium prices. We exclude this case because if there are two or more risk-neutral traders with different beliefs, then they would want to go infinitely long in some assets and infinitely short in others, and the constraint that \( z \in R^k_+ \) would be binding.

IV. Rational Expectations Equilibrium

An alternative approach to market efficiency is implicit in the rational expectations literature (see Jordan and Radner [3] for a survey). The link between rational expectations equilibria and market efficiency (in the sense of Fama [1]) is that revealing REE prices guarantee that informed traders cannot make profits from their information because uninformed traders can infer the information from the price.

The formal definition of rational expectations follows. As in Section I, informed traders maximize

\[
E_q[U_w(c_i) + U_i(r'z_i) | s]
\]

subject to (3), generating demands \( z_i(p_s, \pi_s) \). Since the informed traders know \( s \), knowledge of the equilibrium price \( p_s \) does not change their expectations.

The same indifference does not follow for uninformed traders. Acting as price takers, they will use the price \( p_s \) (and the "forecast function" \( \phi : S \to R^k_+ \) defined by \( p_s = \phi(s) \)) to infer information about \( s \). Therefore, uninformed traders maximize

\[
E_q[U_w(c_i) + U_i(r'z_i) | p_s, \phi]
\]

subject to (3), generating demands \( z_i(p_s, \pi_s, \phi) \) where \( \pi(p_s, \phi) \) is the conditional probability of \( r \) given \( p_s \) and \( \phi \).

The total excess demand function (following Section I) is denoted \( Z(p_s, \pi_s, \pi(p_s, \phi)) \).

A rational expectations equilibrium (REE) is a mapping \( \phi : S \to R^k_+ \) such that

\[
Z(\phi(s), \pi_s, \pi(\phi(s), \phi)) = 0 \quad \text{for all} \quad s \in S. \tag{8}
\]

If \( \phi \) is a one-to-one mapping from \( S \) into \( R^k_+ \), then knowledge of \( \phi(s) = p_s \) is equivalent to knowledge of \( s \), i.e., \( \pi_s = \pi(p_s, \phi) \). When this situation occurs, along with (6.1), the rational expectations equilibrium is called revealing.

A revealing rational expectations equilibrium is a mapping \( \phi : S \to R^k_+ \) such that

\[
Z(\phi(s), \pi_s, \pi(\phi(s), \phi)) = 0 \quad \text{and} \quad \pi(\phi(s), \phi) = \pi_s \quad \text{for all} \quad s \in S. \tag{9}
\]

Revealing rational expectations are full information equilibria. Hence, unlike efficient consensus beliefs equilibria, how uninformed traders come to have the same beliefs (or at least the same information) as informed traders is explained.
in rational expectations equilibria. They use market prices to make inferences about other traders' information.

Revealing rational expectations equilibria are clearly strong form efficient (see Fama [1]), and if traders have homogeneous priors they will have consensus beliefs in equilibrium. However, unlike efficient consensus beliefs equilibria (with unsophisticated traders) revealing rational expectations equilibria generically exist for the economy of Section I. This result is demonstrated in Radner [4]. Therefore, one should expect efficient consensus beliefs equilibria to exist only if traders have "rational expectations." Since (revealing) rational expectations equilibria are automatically efficient, the concept of consensus beliefs adds no additional insights into market efficiency beyond rational expectations, but does add the assumption that traders have homogeneous priors. Thus, consensus beliefs and rational expectations cannot be thought of as competing theories of informational efficiency.

V. An Example

This section uses Radner's [4, p. 661] example to illustrate the concepts in the previous sections.

A. Economy

Consider an economy with only two assets. Let trader $i$'s ($i = 1, \ldots, I$) endowment of these two assets be denoted $(t_i, v_i)$ and his portfolio choice $(y_i, z_i)$. The normalized price vector is denoted $(p, 1 - p)$.

Suppose there are two signals of the economy that can occur, called $s'$ and $s''$. Traders $i = 1, \ldots, J$ are informed in that they know the true signal, while traders $i = J + 1, \ldots, I$ are uninformed, i.e., they do not know the signal. Each trader has a probability belief $\psi'$ for signal $s'$ and $\psi''$ for signal $s''$ where $\psi' + \psi'' = 1$. Homogeneity of beliefs is assumed.

Each trader's expected utility function $V(\cdot)$ is of the "Cobb-Douglas" form. Given signal $s'$ occurs, each trader's expected utility function of his portfolio $(y_i, z_i)$ is

$$V_i(y_i, z_i, \alpha_i') = \alpha_i' \log y_i + (1 - \alpha_i') \log z_i, \quad 0 < \alpha_i' < 1 \tag{10}$$

If signal $s''$ occurs,

$$V_i(y_i, z_i, \alpha_i'') = \alpha_i'' \log y_i + (1 - \alpha_i'') \log z_i, \quad 0 < \alpha_i'' < 1 \tag{11}$$

The informed trader's expected utility conditional upon the signal is either (10) or (11). These correspond to (1) in the text.

The uninformed trader's expected utility is

$$V_i(y_i, z_i, \tilde{\alpha}_i) = \tilde{\alpha}_i \log y_i + (1 - \tilde{\alpha}_i) \log z_i, \quad \text{where} \quad \tilde{\alpha}_i = \psi' \alpha_i' + \psi'' \alpha_i'' \tag{12}$$

He takes his expectation across signals; see (2).
Trader $i$’s portfolio constraints are
\[ y_i \geq 0, \quad z_i \geq 0, \quad py_i + (1-p)z_i \leq pt_i + (1-p)v_i \] (13)

B. Equilibrium

Consider the unsophisticated market equilibrium, given signal $s'$ has occurred. The informed traders maximize (10) subject to (13) giving demands:
\[ y_i = \alpha'_i (pt_i + (1-p)v_i)/p \]
\[ z_i = (1 - \alpha'_i)(pt_i + (1-p)v_i)/(1-p) \quad i = 1, \ldots, J \] (14)
The uninformed investors maximize (12) subject to (13) giving demands:
\[ y_i = \hat{\alpha}_i (pt_i + (1-p)v_i)/p \]
\[ z_i = (1 - \hat{\alpha}_i)(pt_i + (1-p)v_i)/(1-p) \quad i = J + 1, \ldots, I \] (15)

Equilibrium requires only that
\[ \sum_{i=1}^{I} y_i = \sum_{i=1}^{I} t_i \] (16)
since Walras Law is satisfied. Solving (16) for the unsophisticated market equilibrium price,
\[ p_u(s') = \frac{\sum_{i=1}^{I} \alpha'_i v_i + \sum_{i=j+1}^{I} \hat{\alpha}_i v_i}{\sum_{i=1}^{I} [\alpha'_i v_i + (1 - \alpha'_i)t_i] + \sum_{i=j+1}^{I} [\hat{\alpha}_i v_i + (1 - \hat{\alpha}_i)t_i]} \] (17)

An analogous price holds under signal $s''$ (just replace the prime with double primes).
The full information equilibrium price is obtained in a similar fashion. Here all investors know the true signal, say $s'$, so they all maximize (10) subject to (13). The full information equilibrium price, $p_F$, is
\[ p_F(s') = \sum_{i=1}^{I} \alpha'_i v_i/\sum_{i=1}^{I} [\alpha'_i v_i + (1 - \alpha'_i)t_i] \] (18)

An analogous price holds under signal $s''$, $p_F(s'')$.

C. Consensus Beliefs

In this economy, an efficient consensus beliefs equilibrium is a collection of price vectors $p(s)$ such that, for each $s \in (s', s'')$,
\[ p(s) = p_u(s) \quad \text{and} \quad p(s) = p_F(s) \] (19)

Using (17) and (18), this implies that
\[ p_u(s') = p_F(s') \quad \text{and} \quad p_u(s'') = p_F(s'') \] (20)

This condition imposes restrictions on the set of parameters for this economy: $[(\alpha'_i, \alpha''_i)]_{i=1}^{I}, \psi \in (0, 1)^{2I+1}$. The set of parameters $[(\alpha'_i, \alpha''_i)]_{i=1}^{I}, \psi$ such that (20) is true is a manifold of dimension $2I - 1$ in the space $(0, 1)^{2I+1}$. Hence the set of parameters such that (20) is true is a closed set of Lebesgue measure zero. Therefore, an efficient consensus beliefs equilibrium generically does not exist for the economies of this example.
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VI. Conclusion

This paper shows that unless traders have rational expectations, the concept of consensus beliefs is not useful for considerations of market efficiency. Under one interpretation, consensus beliefs equilibria exist but have no relation to market efficiency. Under a stricter interpretation, consensus beliefs equilibria are efficient, but they typically do not exist unless traders have rational expectations.

As an aide to understanding market efficiency, consensus beliefs add nothing beyond rational expectations. They typically exist only when traders have rational expectations, but revealing rational expectations equilibria are efficient even if traders do not have consensus beliefs. So consensus beliefs cannot be considered as a viable competing theory of market efficiency.

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