A CHARACTERIZATION THEOREM FOR UNIQUE RISK NEUTRAL PROBABILITY MEASURES

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Given the existence of a unique risk neutral probability measure, this paper provides a characterization theorem which utilizes prices on traded call options to infer the implied probability law over asset payoffs.

1. Introduction

Arbitrage pricing has become a very popular methodology in financial economics. It is used to price contingent claims [see Merton (1973), and Harrison and Kreps (1979)], securities [see Ross (1976), and Chamberlain and Rothschild (1983)], and establish norms for corporate dividend and liability policy [see Merton (1977) and Ross (1978)]. Central to much of this analysis is the existence of a unique risk neutral probability measure which prices assets according to their expected value (with respect to this probability measure). It prices the assets as if they traded in an economy populated with identical risk neutral investors whose probability beliefs are given by the risk neutral probability measure.

Given the existence of a unique equivalent probability measure in an economy, this paper provides a characterization theorem. The characterization theorem utilizes prices on traded assets to infer the probability laws implied by the risk neutral probability measure over asset payoffs. The traded assets employed are call options. In this sense, the paper generalizes the earlier work of Breeden and Litzenberger (1978) and Bick (1982).

The characterization theorem is formulated for a heterogeneous beliefs economy. This theorem, therefore, provides an alternative approach to pricing call options using arbitrage pricing theory in a heterogeneous beliefs economy. The alternative approach would be to use a subset of traded calls and the characterization theorem to estimate the risk neutral probability law for the underlying stock, and then to price the remaining calls (or contingent claims) on the stock using the estimated probability law. This also provides an alternative procedure for testing the existing option pricing formulas. The empirical estimated probability laws obtained from the characterization theorem can be compared to the theoretical probability laws implied by particular models, like the Black–Scholes option model. This would overcome a bias in the majority of the existing literature. Most studies use implied volatilities to test the model, a procedure which introduces a bias in favor of accepting the model, see Cox and Rubinstein (1985) for a review. This is an area for future research.

An outline of the paper is as follows. Section 2 presents a single period economy. Section 3 presents the characterization theorem, while section 4 summarizes the paper.

2. Single period model

This section presents the single period version of the model. Following the procedure of Harrison and Kreps (1979), however, this model can easily be extended to a multi-period economy. The extension procedure transforms the multi-period economy into an 'equivalent' single period economy. Hence, there is really no loss in generality from proving the characterization theorem in this simpler case.

The economy consists of two dates, time 0 and time 1. The uncertainty in the economy at date 1 is characterized by a state space \( \Omega \). At each date, a set of assets denoted by \( M \) trades. An asset \( x \in M \) is a mapping from the probability \( \Omega \) into \( R \).

There are a finite number of investors in the economy indexed \( i = 1, \ldots, I \). Each investor is endowed with a probability space \(( \Omega, F_i, \tilde{P}_i)\), where \( F_i \) is a \( \sigma \)-algebra over \( \Omega \) and \( \tilde{P}_i \) is a complete probability measure. These information sets and beliefs are restricted as follows:

A.1. Homogeneity of information. All investors agree upon the functional form of \( x(w) \) for all \( w \in \Omega \) and all \( x \in M \).

This implies that the information set generated by \( M, \sigma(M) \), is contained in \( F_i \) for all \( i \). Let \( \tilde{P}_i \) be the restriction of \( \tilde{P}_i \) to \( \sigma(M) \).

A.2. Homogeneity of beliefs. For any \( H \in \sigma(M) \), for \( i \neq j \), \( P_i(H) = 0 \) if and only if \( P_j(H) = 0 \).

This assumption requires investors to agree upon zero probability events from the set \( \sigma(M) \). Formally, their beliefs are absolutely continuous with respect to each other on \( \sigma(M) \). In the remainder of the text, define \( P : \sigma(M) \to [0, 1] \) by \( P = \sum_{i=1}^{I} P_i / I \). \( P \) is absolutely continuous with respect to \( \tilde{P}_i \) and vice-versa. Furthermore, \( E |x| < + \infty \) iff \( E_i |x| < + \infty \) for all \( i \) where \( E(x) = \int_{\Omega} x \ dP \).

A.3. Finite expectations. \( E(|x|) < + \infty \) for all \( x \in M \) where \( E(|x|) = \int_{\Omega} |x| \ dP \).

A.4. Frictionless markets. Trading in the assets \( M \) entails no transaction costs, no taxes, and no restrictions on short sales.


A.6. Traded assets. \( M \) is a linear space, i.e., if \( x, y \in M \) then \( a x + \beta y \in M \) for any \( a, \beta \in R \) and \( x_{\Omega} \in M \) where \( x_{\Omega}(w) = 1 \) for all \( w \in \Omega \).

Assumption 6 requires that if any two assets trade, then portfolios containing \( \alpha \) shares of \( x \) and \( \beta \) shares of \( y \) trade as a package for all \( \alpha, \beta \in R \). It also requires the riskless asset, \( x_{\Omega} \), to trade.

Prices for the assets at time 0 are represented by the price functional \( \Pi : M \to R \) where \( \Pi(x) \) is the time 0 dollar price of asset \( x \).

A.7. No arbitrage opportunities. (i) \( \Pi \) is linear on \( M \), (ii) if \( x \in M \), \( x = 0 \) a.e. \( P \) then \( \Pi(x) = 0 \), and (iii) \( \Pi \) is strictly positive on \( M \).

The linearity of \( \Pi \) over \( M \) implies that self constructed portfolios sell for the same price as traded packages, i.e., \( \Pi(\alpha m_1 + \beta m_2) = \alpha \Pi(m_1) + \beta \Pi(m_2) \) for \( m_1, m_2 \in M \) and \( \alpha, \beta \in R \). Condition (ii)
says that zero cash flow assets have zero prices. Last, strict positivity means that if \( P(m > 0) > 0 \) and \( P(m \geq 0) = 1 \) then \( \Pi(m) > 0 \), so, limited liability assets have positive prices.

A.8. Existence of an equivalent risk neutral measure. There exists a unique probability measure \( \Pi^*: \sigma(M) \rightarrow [0, 1] \) such that

(i) \( \Pi(m)/\Pi(x_0) = \int_\Omega m \ d P^*(w) \) for all \( m \in L^1(\Omega, \sigma(M), P) \), and

(ii) \( P^*(A) = 0 \) if and only if \( P(A) = 0 \) for all \( A \in \sigma(M) \).

Sufficient conditions for the existence of such a measure are that \( M \) is complete, i.e., \( M = L^1(\Omega, \sigma(M), P) \).\(^1\) Traded call options \( C(K) = \max(0, x - K) \in M \) for all \( K \) are not sufficient to ensure that \( M = L^1(\Omega, \sigma(M), P) \). \( M \) must also be closed as a subspace of \( L^1(\Omega, \sigma(M), P) \).\(^2\) Alternatively, \( \Pi \) is continuous and \( C(K) \in M \) for all \( K \) are also sufficient for A.8.\(^3\)

3. The characterization theorem

This section generalizes Breeden and Litzenberger (1978) and Bick (1982). If \( M = L^1(\Omega, \sigma(M), P) \), then the characterization of \( \Pi^* \) is straightforward. Given \( x_A \in M \) for \( A \in \sigma(M) \) where \( x_A(w) = 1 \) if \( w \in A \), 0 otherwise, we know that \( \Pi^*(A) = \Pi(x_A)/\Pi(x_0) \). If \( M \) is not complete, an alternative procedure may be employed. For many applications it is sufficient to know only the probability law for a specific asset \( x \in M \) where \( x(w) \geq a.e. P \). This is true for most of the contingent claim valuation techniques employed in financial economics, see Merton (1973).

Define \( P^*: (R, B(R)) \rightarrow [0, 1] \) to be the probability measure on the Borel subsets of \( R \), \( B(R) \), satisfying \( \Pi^*(A) = P^*(w \in \Omega: x(w) \in A) \) for all \( A \in B(R) \). This is the probability law for \( x \in M \). Define \( G_x: R \rightarrow [0, 1] \) by \( G_x(a) = P_x^*(w \in \Omega: x(w) \leq a) \), \( G_x \) is the probability distribution function of \( x \) [see Royden (1968, p. 262)]. We characterize \( G_x \).

**Theorem 1** (Characterization of the distribution \( G_x \)). Under A.1–A.8, if \( C(K) = \max(x - K, 0) \in M \) for all \( K \geq 0 \) then

(i) \( d\Pi(C(K))/dK \) exists a.e., and

(ii) \( G_x(K) = [1/\Pi(x_0)]d\Pi(C(K))/dK \) \(-1 \) a.e. for all \( K \geq 0 \).

**Proof.** In the appendix.

This theorem asserts that given all call options on \( x \) with exercise prices \( K \geq 0 \) trade, then the distribution \( G_x(K) \) equals the derivative of the call's price at time 0 with respect to the exercise price, divided by \( \Pi(x_0) \), minus one. If more smoothness of the distribution \( G_x(K) \) can be assumed, then the characterization takes a simpler form.

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\(^1\) We prove A.8 under this hypothesis.

**Proof.** Since \( M = L^1(\Omega, \sigma(M), P) \), \( \Pi \) is linear and positive, by Schaefer (1980, p. 238) \( \Pi \) is continuous. Define \( \Pi(m) = \Pi(m)/\Pi(x_0) \) for all \( m \in M \). I inherits linearity, positivity and continuity from \( \Pi \). By Meyer (1966, p. 21) there exists \( \Pi^* \) on \( \sigma(M) \) such that \( \Pi(m) = \Pi^*(m) \) for all \( m \in M \). Suppose \( \Pi^*(A) = 0 \) for \( A \in \sigma(M) \). Then \( \Pi(x_0) \Pi^*(x_A) = \Pi(x_A)\Pi(x_0) \) where \( x_A = 1 \) if \( w \in A \), 0 otherwise. Suppose \( P(x_A > 0) > 0 \). This implies \( \Pi(x_A) > 0 \), contradiction. Suppose \( P(A) = 0 \) for \( A \in \sigma(M) \). Then \( P(x_A > 0) = 0 \), \( P(x_A \geq 0) = 1 \) so \( P(x_A = 0) = 1 \). By A.7(ii), \( \Pi(x_A) = 0 \), but \( \Pi(x_A) = \Pi(x_0) \Pi(x_A)/\Pi(x_0) \) d \( P^* \) which implies \( \Pi(x_A) = 0 \).

\(^2\) See Green and Jarrow (n.d.).

\(^3\) In footnote 1, \( M = L^1(\Omega, \sigma(M), P) \) was only used to provide continuity. If we assume continuity, closure under call options is enough to apply the same theorem from Meyer (1966, p. 21).
Corollary 1. If $G_x(K)$ is continuous for all $K \geq 0$, then

$$G_x(K) = \frac{1}{\Pi(x_u)} \frac{d\Pi(C(K))}{dK} - 1.$$ 

Corollary 2 [Breeden and Litzenberger (1978)]. If a density function $dG_x(K)/dK$ exists for all $K \geq 0$ then

$$\frac{dG_x(K)}{dK} = \frac{1}{\Pi(x_u)} \frac{d\Pi^2(C(K))}{dK^2}.$$ 

This characterization is useful for pricing contingent claims on $x$ which trade. A direct application of Royden (1968, p. 263) gives

Theorem 2 (Pricing contingent claims). If $g : R \rightarrow R$ is Borel measurable and $g(x) \in M$, then

$$\frac{\Pi(g(x))}{\Pi(x_u)} = \int_R g(x) \, dG_x(x).$$ 

This theorem prices contingent claims where a contingent claim on $x$ is represented by a Borel measurable function of $x$.

4. Summary

In a single-period economy, given the existence of a unique equivalent risk neutral probability measure, this paper provides a theorem which characterizes the probability laws for traded assets (Theorem 1). This characterization theorem can be used to price contingent claims on traded assets. This is the content of Theorem 2. This analysis is easily extended to a multiperiod economy where the preceding theorems can be employed. In this context, the theorems provide an alternative approach to empirically estimate call option prices and to test existing call option models. The new procedure would make use of the estimated probability laws from the risk neutral probability measure and thereby avoid the biases and circularity inherent in existing tests of call valuation models which use implied volatilities. This is an exciting area for continuing research.

Appendix

Proof of Theorem 1. Since $C_K = \max(x - K, 0) \in M$ and $x - K \in M$, we have $r_K = \max(K - x, 0 = C_K - (x - K) \in M$. This is a put option. Using A.8,

$$\Pi(r_K) = \Pi_0(x_u) \int_0^\infty \max(K - x(w), 0) \, dP^*(w)$$

$$= \Pi_0(x_u) \int_0^\infty \max(K - x, 0) \, dP_x^*(x)$$

$$= \Pi_0(x_u) \left[ KP_x^* [x \leq K] - \int_0^K x \, dP_x^*(x) \right]$$

$$= \Pi_0(x_u) \left[ KP_x^* [x \leq K] - \int_0^K P_x^*(y < x \leq K) \, dy \right]$$
by Fubini's theorem where \(dy\) is Lebesque measure,

\[
= \Pi_0(x_B) \int_0^K G_x(y) \, dy.
\]

By Kolmogorov and Fomin (1970, p. 316) \(G_x\) is continuous a.e. (with respect to \(dy\)). By the proof of the fundamental theorem of calculus [Lang (1983, p. 105)], at points of \(G_x\) continuity,

(i) \(d\Pi_0(r_K)/dK\) exists, and
(ii) \(d\Pi_0(r_K)/dK = \Pi_0(x_B)G_x(K)\). Q.E.D.

References

Lang, Serge, 1983, Real analysis, second ed. (Addison-Wesley, Reading, MA).