Spanning and Completeness in Markets with Contingent Claims*

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We characterize financial markets that are "complete" or that contain portfolios which "span" all measurable functions of a particular asset payoffs, either finite and infinite dimensional. These results are then employed to describe the extent to which options trading is sufficient to complete markets, to investigate the existence of "efficient funds," and to establish the extent of market completeness required to ensure the unanimity and irrelevance results of modern corporation finance. Journal of Economic Literature, Classification Numbers 026, 514, 521. © 1987 Academic Press, Inc.

I. INTRODUCTION

We provide a characterization of market completeness for general linear spaces of asset payoffs. A market is said to be complete when the marketed assets span all the measurable contingent claims that could be written on them. We show that a necessary condition for "market completeness" is that the space of marketed assets constitute what is known as a

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"sublattice." From an economic perspective, sublattices can be shown to include various types of call options. When the space also exhibits closure under pointwise convergence of monotone sequences, we have conditions which are both necessary and sufficient for "market completeness."\(^2\)

Call options thus serve an important role in achieving market completeness. In this regard, our results extend those obtained by Ross [16] and Arditti and John [1] for finite state-spaces. We also analyze conditions under which the variety of options needed to complete markets can be reduced. The extent to which options must be compounded to complete a market depends on the dimensionality of the portfolio space. These conditions lead to a discussion of the existence of "efficient funds," which in this more general setting may not exist. Finally, because of their similarity to options, corporate securities also generate sublattices, and we outline the implications of this for some issues in corporation finance.

An outline of the paper is as follows. Section II sets forth the assumptions of our model and the basic results. Section III discusses special properties of discrete state-spaces and efficient funds. Section IV completes the paper with a study of the trading opportunities generated when firms issue debt and equity.

II. THE MODEL

We consider a two date economy, where all trading takes place at date 0. At date 1 a state, \(\omega\), is revealed from a set of elementary outcomes, \(\Omega\). Financial assets are identified by their exogenous state-dependent payoffs (in a numeraire commodity) at date 1. Thus, a typical asset is a function \(m: \Omega \to R\). Investors are assumed to agree on the state-space representation of \(m\), that is, on the value of \(m(\omega) \in R\). The set of all possible assets is \(R^\Omega\), the vector space of real valued functions from \(\Omega\) to \(R\). It may be that some assets are not traded. We denote by \(M \subset R^\Omega\) the subset of marketed assets, and assume \(M\) is a linear subspace of \(R^\Omega\). This implies that markets are frictionless and short sales are unrestricted. \(M\) can be viewed as the space

\(^1\) In an unpublished note, Brown and Ross [5] have also observed that vector lattices span nonlinear functions of themselves. Their approach differs from ours mathematically, however. They topologize the set \(\Omega\), and appeal to theorems which show sublattices are dense in the space of continuous functions of the asset payoffs. With no assumptions on \(\Omega\), we show the sublattices are dense in the space of measurable functions, and discuss the added closure conditions required for complete spanning of these function spaces.

\(^2\) Nachman [14] investigates these same issues and others in a restricted version of our economy where the space of assets is a Banach lattice. This added structure permits a study of a particular process of market completion (expansion).
of portfolio payoffs generated by a set of basis assets, which may be either finite or infinite.

We are interested in conditions which ensure that these assets span some other exogenously given set of cash flows. Let $N$ be an arbitrary subset of $\mathbb{R}^2$, let $\sigma(N)$ be the smallest sigma-algebra with respect to which all $n \in N$ are measurable, and let $\chi_A$ be the indicator function for an event $A \in \Omega$.

**Definition 1.** A financial market $M$ is $\sigma(N)$-complete if, for any $A \in \sigma(N)$, $\chi_A \in M$, and $M$ is closed under pointwise convergence of monotone sequences.\(^3\)

This definition implies $M$ is $\sigma(N)$-complete if and only if it spans any measurable function of the payoffs in $N$ (see Breiman [4, Proposition A.20]). Closure of $M$ under weaker types of convergence will imply weaker types of spanning. These issues are pursued in footnotes.\(^4\) If $N = \mathbb{R}^2$, the market is complete in the Arrow–Debreu sense. If $N = M$, then $\sigma(M)$-completeness implies the market spans all derivative assets on itself. As discussed in Harrison and Kreps [9], in $\sigma(M)$-complete markets derivative claims can be priced through “arbitrage.” Thus these markets play an important role in the theory of finance.

**Theorem 1.** The set of marketed assets is $\sigma(M)$-complete if and only if (i) $M$ contains the riskless asset, (ii) $M$ is a sublattice on $\mathbb{R}^2$, and (iii) $M$ is closed under pointwise convergence of monotone sequences.\(^5\)

\(^3\) By closure under pointwise convergence of monotone sequences we mean that if $(m_n)_{n=1}^\infty \in M$, $\lim_{n \to \infty} m_n(\omega)$ exists $\forall \omega \in \Omega$, and $m_n(\omega) \geq m_m(\omega)$, then $\lim_{n \to \infty} m_n \in M$.

\(^4\) Given a probability space $(\Omega, \sigma(M), P)$ where $\sigma(N) = \sigma(M)$ two modifications are useful. Identity two assets $x, y \in M$ as equivalent if $x = y$ a.e. $P$. The first modification is to replace “pointwise convergence” with “$P$ a.e. convergence” in definition 1. The second modification is to consider $M \subseteq L^q(\Omega, \sigma(M), P)$, for $1 \leq q$, with the closure condition changed to “closure under $L^q(\Omega, \sigma(M), P)$ convergence.” For $L^1$ closure, $M$ will span any $L^\infty(\Omega, \sigma(M), P)$ function.

\(^5\) Theorem 1 can be restated under the two modifications of footnote 4. For $P$ a.e. convergence, replace “pointwise convergence” with “$P$ a.e. convergence.” For $L^q(\Omega, \sigma(M), P)$ convergence, condition (ii) becomes “$M$ is a sublattice of $L^q(\Omega, \sigma(M), P)$,” and (iii) “$M$ is closed under $L^q(\Omega, \sigma(M), P)$ convergence.”

**Proof.** We sketch the proof. If $M$ is $\sigma(M)$ complete, then the implications follow by the properties of $L^1$ space. Conversely, if $M$ satisfies (i)–(iii), show $L^\infty(\Omega, \sigma(M), P) \subseteq M$. Take $m \in L^\infty(\Omega, \sigma(M), P)$. Since $m$ is $\sigma(M)$-measurable, by the proof of Theorem 1, we know there exist $(\phi_n) \in M$ such that $\phi_n \uparrow \chi_{[\omega \in \Omega : m(\omega) > n]}$ a.e. $P$. By the monotone convergence theorem, $\phi_n \uparrow \chi_{[\omega \in \Omega : m(\omega) > n]}$ in $L^1$. Hence, $\chi_{[\omega \in \Omega : m(\omega) > n]} \in M$. From here, by standard arguments, it can be shown that $m \in M$.

\(^6\) Breeden and Litzenberger [3] use a construction similar to (1) in their derivation of state-prices with option pricing formulas. Using their method, one could value the indicator $\chi_{[t < n]}$ as the first derivative of the call's value with respect to the exercise price.
Proof. Necessity. Let $M$ be $\sigma(M)$-complete. By definition, $M$ is closed under pointwise convergence of monotone sequences. Since $\Omega \in \sigma(M)$, $\chi_\Omega \in M$. Finally, we show that if $f \in M$, then $f^+ = \max(f, 0) \in M$. Since $f^+$ is $\sigma(M)$-measurable, there exists a sequence of simple functions $\{\phi_n\}_{n=1}^\infty \in M$ (portfolios of $\chi_A$ for $A \in \sigma(M)$) such that $\phi_{n+1}(\omega) \geq \phi_n(\omega)$ for all $\omega \in \Omega$ and $\lim_{n \to \infty} \phi_n(\omega) = f^+(\omega)$. Since $M$ is closed under pointwise convergence of monotone sequences, $f^+ \in M$. Hence, $M$ is a sublattice of $R^\Omega$.

Sufficiency. Define $E = \{A \in \Omega: \chi_A \in M\}$. We will show that $E = \sigma(M)$ by showing that $\sigma(M) \subseteq E$. The other containment is always true.

Step 1. Show $E$ is an $\sigma$-algebra. $E$ is an algebra since: (i) $\chi_\Omega \in M$ implies $\Omega \in E$, (ii) if $A \in E$ then $A^c \in E$ since the linearity of $M$ implies $\chi_A^c = \chi_\Omega - \chi_A \in M$, and (iii) if $A_1, \ldots, A_n \in E$ then $\bigcup_{i=1}^n A_i \in E$ since $\chi_{A_1}, \ldots, \chi_{A_n} \in M$ imply $\sum_{i=1}^n \chi_{A_i} = \chi_{\bigcup_{i=1}^n A_i} \in M$. But, $E$ is also a monotone class since if $A_n \in E$ and $A_n \uparrow A$ then $A \in E$. Indeed, if $\chi_{A_n} \in M$ and $\lim_{n \to \infty} \chi_{A_n} = \chi_A$ monotonely then $\chi_A \in M$. By the monotone class theorem (Breiman [4, p. 391]), $\sigma(E) \subseteq E$. But $E \subseteq \sigma(E)$ always, so $E = \sigma(E)$.

Step 2. Show any $f \in M$ is $E$ measurable. Take any $f \in M$. Since $\chi_\Omega \in M$ and $M$ is a subspace, $f - a\chi_\Omega \in M$ for $a \in R$. Since $M$ is a sublattice, $\max(f - a\chi_\Omega, 0) \in M$. Using the lattice property again, $\min[b_n, \max(f - a\chi_\Omega, 0), \chi_\Omega] \in M$ for $b_n \in R$. But

$$\lim_{b_n \to \infty} \min[b_n, \max(f - a\chi_\Omega, 0), \chi_\Omega] = \chi_{\{\omega \in \Omega: f(\omega) > a\}} \in M$$

since $M$ is closed under monotone sequences. Then by the definition of $E$, $\{\omega \in \Omega: f(m) > a\} \in E$ for all $a \in R$.

Step 3. Since $\sigma(M)$ by definition is the smallest $\sigma$-algebra for which all $f \in M$ are measurable, $\sigma(M) \subseteq E$. 

Theorem 1 can be viewed as a generalization of Ross’s [16, Theorem 5] result that, with a finite state space, call options are sufficient to complete markets. A subspace $M$ of $R^\Omega$ is a sublattice if it is closed under the $\max(\cdot, 0)$ operator. (See Cotlar and Cignoli [6, p.176].) A call option on $m \in M$, with exercise price $K \in R$, has a payoff of $\max(m - K, 0)$. Thus, conditions (i) and (ii) imply that $M$ contains all call options on the asset payoffs in $M$. Indeed, since $K$ can be zero, $M$ is a sublattice if it contains all calls. Alternatively, if $M$ is a subspace, a sublattice, and if $\chi_\Omega \in M$, then for $m \in M$, $m - K\chi_\Omega \in M$ and $\max\{m - K\chi_\Omega, 0\} \in M$.

When considering the case where $|\Omega| = +\infty$, however, another closure condition is required beyond (i) and (ii). Loosely speaking, the lattice structure must be extended so that the space of functions is closed under suitable limits. Condition (iii) ensures this for the $\sigma(M)$-measurable
functions. Weaker, but analogous, conditions will serve for $L^q$ spaces (see footnotes 4 and 5).

Thus, to be $\sigma(M)$-complete $M$ must include options with different exercise prices on the basis assets, on portfolios of these assets, on other options, and on portfolios of options. This is potentially a very rich set. The next result shows, however, that the process of compounding, or writing options on portfolios of options, may eventually become redundant.

**Theorem 2.** Let $M_0$ be the linear space of portfolios generated by a set of basis assets with $\dim(M_0) = K$ and $\chi_\Omega \in M_0$. Augment this set with all call options written directly on these portfolios, and then take the span of the resulting set. Repeat this process $K - 1$ times. The resulting portfolio space is dense in the space of $\sigma(M_0)$-measurable functions, with the product topology on $\mathbb{R}^2$ (i.e., the topology of pointwise convergence).

**Proof.** The proof is by induction. For $K = 2$, let $M_0 = \text{span}\{\chi_\Omega, f_1\}$ where $\dim(M_0) = 2$. Define $M$ to be $M_0$ augmented once by compounding with calls. Let $\tilde{M}$ be its closure under monotone pointwise convergence. Define $\psi = \{A \subset \Omega: A = \{f_i(\omega) > a\} \text{ for all } a \in \mathbb{R} \cup \{\Omega\}\}$. $\psi$ is closed under finite intersections. We claim that $\chi_A \in \tilde{M}$ for all $A \in \psi$. Indeed, for $b_n > 0$ define

$$y_n(f_1) = b_n \max(f_1 - a\chi_\Omega, 0) - \max(b_n, f_1 - (b_n a + 1)\chi_\Omega, 0)$$

$$= \min(b_n \max(f_1 - a\chi_\Omega, 0), \chi_\Omega).$$

Since $y_n(f_1) \in M$, $\lim_{n \to \infty} y_n(f_1) \in \tilde{M}$, and as Eq. (1) shows, this is the required indicator function. Kopp [12, 0.1.5(6)] modified to apply to pointwise convergence, implies $\tilde{M}$ contains all $\sigma(\psi)$ measurable functions, and hence the $\sigma(M_0)$ measurable functions.

For the induction step, let $M_0 = \text{span}\{\chi_\Omega, f_1, \ldots, f_K\}$ where $\dim(M_0) = K$. Construct $M$ by compounding calls $K$ times and let $\tilde{M}$ be its closure, as above. Define $\psi = \{A \subset \Omega: A = B \cap \{f_k(\omega) > a\} \text{ for all } B \in \psi_0, a \in \mathbb{R}\}$ where $\psi_0$ is the closure under finite intersections of the set \{\$\omega \in \Omega: f_i(\omega) > a_i, for a_i \in \mathbb{R}, i = 1, \ldots, K - 1\}. By the argument used in the previous step, $\chi_{B \cap \{f_k(\omega) > a\}} \in \tilde{M}$. By the induction hypothesis, $\chi_B, \chi_{B \cap \{f_k(\omega) > a\}} \in \tilde{M}$ for all $B \in \psi_0$ and they can be constructed as linear combinations of calls compounded at most $K - 1$ times. For given $A \in \psi$, compound once more as

$$\chi_A = \chi_{B \cap \{f_k(\omega) > a\}} \in \tilde{M}. $$

Applying again the theorem from Kopp [12], all $\sigma(M_0)$-measurable functions are in $\tilde{M}$. 

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1 Nachman [15] demonstrates that the sufficient condition in Theorem 2 can be weakened to a single compounding of options.
III. Efficient Funds and Finite State-Spaces

Previous literature on the spanning properties of options has emphasized the role of "efficient funds." An efficient fund is a portfolio with the characteristic that allowing options to be traded on it augments trading opportunities as fully as allowing them on all assets. These trading opportunities will correspond to the sublattice generated by the fund.

**Definition 2.** A set $A \in R^d$ generates a sublattice $B$ if $B$ is the smallest sublattice containing $A$.

**Definition 3.** A payoff $e \in M$ is a $\sigma(M)$-efficient fund if the closure under pointwise convergence of monotone sequences of the sublattice generated by $\text{span}(\mathcal{L}_0, e)$ is $\sigma(M)$-complete.

By Theorem 1 the closure of the sublattice generated by the span of $e$ and the riskless asset contains all $\sigma(e)$-measurable functions. To be $\sigma(M)$ complete it must also contain the $\sigma(M)$-measurable functions. This implies $\sigma(e) = \sigma(M)$, but since $e \in M$, $\sigma(e) = \sigma(M)$.

Efficient funds can be characterized through a fairly immediate generalization of Ross [16]. Define the equivalence relation $\equiv$ on $\Omega \times \Omega$ as $\omega_1 \equiv \omega_2$ if and only if $m(\omega_1) = m(\omega_2)$ for all $m \in M$. The associated equivalence classes form a disjoint partition of $\Omega$. (See Royden [17, p. 22]). Let $\bar{\Omega}$ be the set of equivalence classes. Define $\hat{e}: \bar{\Omega} \to R$ by $\hat{e}(\bar{\omega}) = e(\omega)$ for some $\omega \in \omega \in \bar{\omega}$. The function $\hat{e}$ is well-defined if and only if $e$ is constant on each $\omega \in \bar{\omega}$.

**Theorem 3.** A payoff $e \in M$ is $\sigma(M)$-efficient if and only if $\hat{e}$ is well-defined and injective.

**Proof.** By Breiman [4, Proposition A.21], $m \in R^d$ is $\sigma(e)$-measurable if and only if there is a Borel measurable function $h_m: R \to R$ such that $m(\omega) = (h_m \circ e)(\omega)$. For this to be the case for all $m \in M$ we must have, for any $\omega_1, \omega_2 \in \Omega$, $e(\omega_1) = e(\omega_2)$ implies $m(\omega_1) = m(\omega_2)$ for all $m \in M$. Then if $m(\omega_1) \neq m(\omega_2)$ for any $m \in M$ it must be the case that $e(\omega_1) \neq e(\omega_2)$. So, $\hat{e}$ is well-defined and one-to-one on the equivalence classes generated by $M$.

An efficient fund also eliminates the need for options on portfolios of options. If a subspace $M_0^\ast$ of $M_0$ exists with the property $\sigma(M_0^\ast) = \sigma(M_0)$ then the number of “layers” of compound options required to complete the market is, by Theorem 2, no greater than $\dim(M_0^\ast)$. With an efficient fund this dimension is one.

*See Ross [16], Arditti and John [1], and John [11].
When the state-space is finite, efficient funds always exist and are dense in the Euclidean space of portfolio weights, as is shown in Arditti and John [1]. This property will not generalize to models with continuously distributed asset payoffs even if the number of assets is finite. For example, choose $\Omega = \mathbb{R}$, $m_1(\omega) = \omega^2$, $m_2(\omega) = \omega^2 + 2\omega$, and $M = \{x_1m_1 + x_2m_2: (x_1, x_2) \in \mathbb{R}^2\}$. Here the portfolios on the line in $\mathbb{R}^2$, $x_1 = -x_2$, are the only efficient funds. Although efficient funds exist they are a “razor’s edge” occurrence since this line is a set with zero Lebesgue measure. This gives the “reverse” of the Arditti and John [1] result.

In fact, payoff spaces can be very rich and still have no efficient funds. Consider the countable state-space $\Omega = \{1, 2, \ldots\}$ and $\chi_i$, the indicator functions for the integers. Define the linear payoff space $M = \{m: m(\omega) = \sum_i x_i \chi_i(\omega), x_i \neq 0 \text{ for only a finite number of } i\}$. The closure of $M$ is a complete market in the Arrow–Debreu sense, and $\sigma(M)$ includes all subsets of $\Omega$. Yet it is obvious that $M$ contains no efficient portfolios since by construction any $m \in M$ has an infinite number of zero coefficients on the indicator functions.

IV. SPANNING DEBT AND EQUITY CLAIMS

In corporation finance the assumption is often made that the financial claims a firm supplies can be duplicated as portfolios of other securities. Demonstrations of capital structure irrelevance, separation between real and financial decisions, and unanimity among the firm’s shareholders generally exploit such assumptions. Here we ask “how complete” financial markets must be for such a “spanning” condition to hold when firms issue debt and equity?

Let $X(\omega) \geq 0$, $\omega \in \Omega$ denote the payoffs generated by a firm’s productive assets. If the firm issues debt with a total promised payment of $F$, then when $X(\omega) \geq F$, bondholders receive $F$. Otherwise, in default, bondholders share the remaining cash flow, $X(\omega)$. Equity holders receive whatever is left after the bondholders are paid, or $X(\omega) - \min\{X(\omega), F\} = \max\{X(\omega) - F, 0\}$. When the firm chooses its capital structure, it chooses $F$. Define the firm’s potential debt and equity claims to be set $Y = \{y(\omega): y(\omega) = \max\{X(\omega) - F, 0\} \text{ or } y(\omega) = \min\{X(\omega), F\} \text{ for } F \in (0, +\infty)\}$.

**Theorem 4.** Assume (a) the riskless asset is traded or (b) $X(\omega) \geq \varepsilon$.

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$\epsilon > 0$ for all $\omega \in \Omega$. Then if $Y \subset M$, $M$ is dense in the space $\sigma(X)$-measurable functions.\footnote{This result is similar in spirit to a theorem proved by Hellwig [10], who shows that particular set of marketed assets is sufficient to ensure (a) there is capital structure irrelevance because of spanning, and (b) any measurable contingent claim can be duplicated. Theorem 4 clarifies the sense in which (b) is necessary for (a).}

**Proof.** The set of all call options written on $X$ is contained in $Y$. Either (a) $\chi_D \in M$ or (b) $\chi_D \in Y$. By Theorem 2, span($\chi_D$, $Y$) includes the sublattice generated by span($\chi_D$, $X$), which is a dense (in the product topology) subset of the $\sigma(X)$-measurable functions. To span $Y$, $M$ must contain this dense subset.

Thus, the spanning condition generally invoked in corporation finance, while it falls short of Arrow–Debreu market completeness, still demands a very rich set of financial opportunities. Alternatively, this result suggests that a very rich set of trading opportunities is created when firms issue only debt and equity claims, and could be viewed as a weak rationalization for the predominance of these financial instruments. Of course, individual firms cannot issue all of their “potential” debt and equity claims, but they can issue subordinated debt; and this allows (in principle) investors to trade a dense subset of the $\sigma(X)$-measurable functions. Suppose the firm provides $K$ different debt “issues,” with face payments $F_k$, $k = 1, \ldots, K$. These can be ordered in terms of the priority of their claim on the firm, so that debt issue $k$ receives payment only if issues $1, \ldots, k - 1$ are paid in full. Then the debt which is $k$th in priority receives: $a_k = \min \{ \max \{ X - \sum_{j=1}^{k-1} F_j, 0 \}, F_k \}$. Choose $K$ distinct values of $X$, arranged in increasing magnitude, $x_1, \ldots, x_K$, and set $F_k = x_k - x_{k-1}$. Then with $x_0 \equiv 0$, $a_k = \min \{ \max \{ X - x_{k-1}, 0 \}, x_k - x_{k-1} \}$ and the $a_k$’s are linearly independent. As the firm issues more and more priorities of debt, it creates a set of linearly independent claims which in the limit span a dense subset of the $\sigma(X)$-measurable functions, because

$$
\lim_{x_k \to x_{k-1}} \min \left( \max \left( \frac{X - x_{k-1}}{x_k - x_{k-1}}, 0 \right), \frac{x_k - x_{k-1}}{x_k - x_{k-1}} \right)
= \lim_{x_k \to x_{k-1}} \min \left( \max \left( \frac{X - x_{k-1}}{x_k - x_{k-1}}, 0 \right), 1 \right)
= I \{ x > x_{k-1} \}.
$$

Thus, the firm could, in principle, complete the market for the contingencies associated with its payoffs by issuing only subordinated debt.
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