THE PRICING OF COMMODITY OPTIONS WITH STOCHASTIC INTEREST RATES

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I. INTRODUCTION

With the recent introduction of traded commodity options on futures contracts, there has been renewed academic interest in the pricing of these securities; for example, see Brenner, Courtdon, and Subrahmanyam [5], Ramaswamy and Sundaresan [26], and Ball and Torous [1]. Almost without exception, these studies and their predecessors (Black [4], Hoag [19], and Garman and Kohlhagen [11]) use the arbitrage pricing approach to price commodity options under deterministic interest rates. Under deterministic interest rates, futures prices equal forward prices, so that commodity options on the spot commodity and the futures contract are "insignificantly" different. Commodity options on futures contracts have been priced under stochastic interest rates, but in general equilibrium models involving

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preferences or the expectations hypothesis; see Cox, Ingersoll, and Ross [7], Richard and Sundaresan [27], and Ramaswamy and Sundaresan [26]. The purpose of this paper is to provide an arbitrage pricing model for commodity options on futures with stochastic interest rates in complete markets.

The approach followed utilizes the “risk neutrality” arbitrage pricing techniques pioneered by Cox and Ross [6], Harrison and Kreps [16], and Harrison and Pliska [17]. In this literature, questions exist concerning the inability to incorporate Merton’s [25] stochastic interest rate economy; see Harrison and Kreps [16; p. 403] or Harrison and Pliska [17; p. 248]. This paper resolves these questions by combining the two approaches. This is done through the statement and proof of a new lemma. In the process, new insights into the distinctions between forward prices and futures prices are obtained. New (and simpler) proofs of existing results are provided, as well as a new characterization of forward prices in terms of futures prices.

Section II presents the initial assumptions and notation underlying the stochastic interest rate economy. Section III reviews the probabilistic approach to pricing contingent claims. Section IV and V review forward contracts and futures contracts, respectively. Section VI prices commodity options on the spot commodity, while section VII prices commodity options on futures contracts. Section VIII, a summary, completes the paper.

II. THE ECONOMY

For expositional clarity, we consider an economy consisting of only three assets (a spot commodity, a long-term bond, and a short-term bond) with two independent sources of uncertainty. This economy could be easily generalized to \( (n + 1) \) assets with \( n \) independent sources of uncertainty. We assume that:

\[
(A.1) \quad \text{The assets trade continuously in frictionless markets over } t \geq 0.
\]

Let \( \{W_1(t), W_2(t): t \geq 0\} \) represent two independent Brownian motions defined on the probability space \( (\Omega, \mathcal{H}, Q) \), where \( (\mathcal{H}_t)_{t \geq 0} \) represents the Brownian filtration.\footnote{We denote expectation with respect to the probability \( Q \) by \( E(\cdot) \).} The spot commodity’s price \( \{S(t): t \geq 0\} \) satisfies the following assumption:

\[
(A.2) \quad dS(t) = \alpha S(t) \, dt + \sigma_1 S(t) \, dW_1(t) + \sigma_2 S(t) \, dW_2(t),
\] (1)
where \( S(0), \alpha, \sigma_1, \) and \( \sigma_2 \) are constants with \( \sigma_1 \geq 0, \sigma_2 \geq 0 \). In addition, the spot commodity has a cash flow of \( DS(t) \, dt \) per unit time, where \( D \) is a constant.

The parameters \( \alpha, \sigma_1, \sigma_2, \) and \( D \) are assumed constant in assumption (A.2). This restriction is for expositional clarity and is easily relaxed.\(^3\)

The long-term bond price process \( \{ P(t): t \geq 0 \} \) satisfies

\[
\begin{align*}
\text{(A.3)} \quad dP(t) &= \beta(t, P(t))P(t) \, dt + \gamma_1(t)P(t) \, dW_1(t) \\
&\quad + \gamma_2(t)P(t) \, dW_2(t)
\end{align*}
\]

where \( \beta(t, P(t)) \) is bounded, and \( \gamma_1(t), \gamma_2(t) \) are bounded deterministic functions with \( \gamma_1(t) \geq 0, \gamma_2(t) \geq 0 \). Furthermore, \( P(t) > 0 \) for all \( t \in [0, T] \) and \( P(T) = 1 \) a.e. \( Q \).

The long-term bond is assumed to be default free and to mature at time \( T \). It has a cash flow of 1 dollar at this time.

Last, the short-term bond process is modeled by \( \{ B(t): t \geq 0 \} \), where \( B(t) \) is interpreted as the time \( t \) value from a time 0 investment of 1 dollar in a short-term bond that is continuously rolled over.

\[
\begin{align*}
\text{(A.4)} \quad dB(t) &= r(t)B(t) \, dt,
\end{align*}
\]

where \( B(0) = 1 \) and \( \{ r(t): t \geq 0 \} \) is a bounded predictable stochastic process with \( r(t) \geq 0 \) for all \( t \geq 0 \).\(^4\)

Expression (3) can be rewritten

\[
B(t) = \exp \left[ \int_0^t r(y) \, dy \right], \quad \text{for} \quad t \geq 0.
\]

Since the spot rate \( r(t) \) is stochastic, the bond’s price \( B(t) \) is stochastic as well. The key restriction implied by expression (3) is that \( B(t) \) is a process of bounded variation over \([0, T]\). This is in contrast to the spot commodity and the long-term bond, which are not of bounded variation due to the Brownian motion terms in expressions (1) and (2).

The stochastic process for the spot rate is very general; for example, it is consistent with the following stochastic process:\(^5\)

\[
\begin{align*}
dr(t) &= \psi_1(r(t)) \, dt + \psi_2(r(t)) \, dW_1(t) + \psi_3(r(t)) \, dW_2(t),
\end{align*}
\]

where \( \psi_1(\cdot), \psi_2(\cdot), \) and \( \psi_3(\cdot) \) are bounded Lipschitz continuous functions mapping the real line \( \mathbb{R} \) into \( \mathbb{R} \).

We do not assume that the spot rate \( r(t) \) satisfies expression (5). We only use expression (5) as an illustration.
Finally, we assume complete markets:

(A.5) There exists a constant $\varepsilon > 0$ such that the instantaneous variance matrix

$$\begin{pmatrix} \sigma_1 & \gamma_1(t) \\ \sigma_2 & \gamma_2(t) \end{pmatrix} \geq \varepsilon I > 0, \quad \text{for all } t \geq 0,$$

where $I$ is the $(2 \times 2)$ identity matrix. This assumption states that the variance matrix involving $(\sigma_1, \sigma_2, \gamma_1, \gamma_2)$ is invertible for all $t \geq 0$. It essentially allows, with continuous trading, all "regular" contingent claims on $S(t)$, $P(t)$, and $B(t)$ to be duplicated with dynamic portfolio strategies in $S(t)$, $P(t)$, and $B(t)$ alone.

III. THE ARBITRAGE PRICING METHODOLOGY

For pedagogical purposes, this section briefly reviews the relevant results needed from the risk neutrality approach to arbitrage pricing pioneered by Cox and Ross [6], Harrison and Kreps [16], and Harrison and Pliska [17]. Two lemmas are presented. Although lemma 1 has its origin in Harrison and Pliska [17; Section 5], the explicit form presented here is new. Lemma 2 resolves an outstanding question involving the risk neutrality approach and Merton's stochastic interest rate economy. It gives sufficient conditions under which the two approaches are consistent. Lemma 2 is new to the literature. Both lemmas are not proven (or stated) in their most general form; however, the level of generality is sufficient for the subsequent analysis.

First, we define the class of continuous trading strategies allowed in this economy. Continuous trading strategies are continuous time-stochastic processes $\{N_1(t), N_2(t), N_3(t): t \in [0, T]\}$ that satisfy the following three conditions:

1. $N_1(t), N_2(t),$ and $N_3(t)$ are predictable;

2. $\int_0^T N_1(t)^2 S(t)^2 \, dt < +\infty$ a.e. $Q$,

3. $\int_0^T N_2(t)^2 P(t)^2 \, dt < +\infty$ a.e. $Q$,

4. $\int_0^T N_3(t)^2 B(t)^2 \, dt < +\infty$ a.e. $Q$;
The value of the portfolio,
\[ V(t) = N_1(t)S(t) + N_2(t)P(t) + N_3(t)B(t), \]
can be written as
\[ V(t) = V(0) + \int_0^t N_1(y) \, dS(y) + \int_0^t N_1(y) DS(y) \, dy + \int_0^t N_2(y) \, dP(y) \]
\[ + \int_0^t N_3(y) \, dB(y), \quad \text{for any} \quad t \in [0, T]. \]

Condition (1) restricts trading strategies to those that at time \( t \) use only information available at time \( t \) (or earlier). Condition (2) is a technical condition implying that the stochastic integrals in (3) are well defined. Last, condition (3) states that the portfolio \( V(t) \) determined by the trading strategy \( \{N_1, N_2, N_3\} \) is self-financing, that is, the portfolio has no cash inflows or outflows over the time period \([0, T]\). In particular, the cash flows due to the spot commodity \( DS(t) \, dt \) are reinvested within the portfolio. This is without loss of generality since one can always reinvest cash flows using the short-term bond process.

An arbitrage opportunity over \([0, T]\) is defined to be a continuous trading strategy \( \{N_1, N_2, N_3; t \in [0, T]\} \) such that
\[ Q(w \in \Omega: V(T) \geq 0) = 1, \quad Q(w \in \Omega: V(T) > 0) > 0, \]
\[ V(0) \leq 0 \]
where \( V(t) = N_1(t)S(t) + N_2(t)P(t) + N_3(t)B(t) \). This definition states that a trading strategy is an arbitrage opportunity if it has nonnegative cash flows at time 0, nonnegative cash flows at time \( T \), and strictly positive cash flows at time \( T \) with positive probability. No cash inflows or outflows occur over \((0, T)\) since a trading strategy is self-financing.

To price contingent claims using the absence of arbitrage opportunities, we need to introduce a seemingly unrelated topic. Lemmas 1 and 2 below, however, demonstrate the relevance of this topic. The topic is the existence of "equivalent martingale" probability measures. This topic is rather technical in nature, and the reader uninterested in the details can skip immediately to the statement of lemma 1 and the subsequent discussion.

Consider the relative prices \( Z_1(t) = S(t)/B(t) \) and \( Z_2(t) = P(t)/B(t) \) over \([0, T]\). We want to find a probability measure \( \tilde{Q} \) on \((\Omega, \mathcal{H})\), mutually absolutely continuous with respect to \( Q \), such that \( Z_1 \) (plus its cash flows) and \( Z_2 \) are \( \tilde{Q} \)-martingales. The existence and construction of the measure \( \tilde{Q} \) is a direct application of Girsanov's theorem.
The measure $\tilde{Q}$ is defined by
\[
\frac{d\tilde{Q}}{dQ} = \exp\left\{ -\frac{1}{2} \int_0^T (\theta_1^2(y) + \theta_2^2(y)) \, dy - \int_0^T \theta_1(y) \, dW_1(y) - \int_0^T \theta_2(y) \, dW_2(y) \right\} > 0,
\]
where
\[
\theta_1(t) = \frac{[\gamma_2(t)(\alpha + D - r(t)) - \sigma_2(\beta(t, P(t)) - r(t))]}{[\sigma_1 \gamma_2(t) - \gamma_1(t) \sigma_2]},
\]
\[
\theta_2(t) = \frac{[-\gamma_1(t)(\alpha + D - r(t)) + \sigma_1(\beta(t, P(t)) - r(t))]}{[\sigma_1 \gamma_2(t) - \gamma_1(t) \sigma_2]}.
\]
and $E(d\tilde{Q}/dQ)^2 < +\infty$. This measure makes $Z_1(t) + \int_0^T DZ_1(y) \, dy$ and $Z_2(t)$ $\tilde{Q}$-martingales, and it makes $V(t)/B(t)$ into a local $\tilde{Q}$-martingale, where $V(t) = N_1(t)S(t) + N_2(t)P(t) + N_3(t)B(t)$ and $\{N_1, N_2, N_3\}$ is a continuous trading strategy. Expectation with respect to the $\tilde{Q}$ measure will be denoted by $\tilde{E}(\cdot)$.

For the subsequent analysis, it is convenient to define
\[
\tilde{W}_1(t) = W_1(t) + \int_0^t \theta_1(y) \, dy, \quad \tilde{W}_2(t) = W_2(t) + \int_0^t \theta_2(y) \, dy.
\]
By Levy's theorem (see Durrett [8: p. 75]), it is easy to show that $\{\tilde{W}_1, \tilde{W}_2\}$ are independent Brownian motions with respect to $\tilde{Q}$ on the filtration $\{H_t\}_{t \geq 0}$. Using the definition of $Z_1, Z_2$ along with Ito's lemma and expression (9) gives
\[
\begin{align*}
    dZ_1(t) + DZ_1(t) \, dt &= \sigma_1 Z_1(t) \, d\tilde{W}_1(t) + \sigma_2 Z_1(t) \, d\tilde{W}_2(t), \\
    dZ_2(t) &= \gamma_1(t)Z_2(t) \, d\tilde{W}_1(t) + \gamma_2(t)Z_2(t) \, d\tilde{W}_2(t),
\end{align*}
\]
on the probability space $(\Omega, \{H_t\}_{t \geq 0}, \tilde{Q})$.

The measure $\tilde{Q}$ has the interpretation of generating a risk-neutral valuation operator. To understand this interpretation, we can rewrite expression (10), using Ito's lemma, as
\[
\begin{align*}
    Z_1(T) &= Z_1(t) \exp\left\{ -D(T-t) - \frac{1}{2}(\sigma_1^2 + \sigma_2^2)(T-t) \\
    &\quad + \int_t^T \sigma_1 \, d\tilde{W}_1(y) + \int_t^T \sigma_2 \, d\tilde{W}_2(y) \right\} \\
    Z_2(t) &= Z_2(t) \exp\left\{ -\frac{1}{2} \int_t^T (\gamma_1(y)^2 + \gamma_2(y)^2) \, dy \\
    &\quad + \int_t^T \gamma_1(y) \, d\tilde{W}_1(y) + \int_t^T \gamma_2(y) \, d\tilde{W}_2(y) \right\}.
\end{align*}
\]
Taking expectations of expression (11) yields

\[ S(t) = B(t) \mathbb{E}\left( S(T) \exp[D(T - t)] \middle| B(T) \right) = \mathbb{E}\left( S(T) \exp\left[ D(T - t) - \int_t^T r(y) \, dy \right] \middle| H_t \right), \]  

(12a)

\[ P(t) = B(t) \mathbb{E}(1/B(T) \middle| H_t) = \mathbb{E}\left( \exp\left[ -\int_t^T r(y) \, dy \right] \middle| H_t \right). \]  

(12b)

Expression (12a) says that the present value of receiving \( S(T) e^{D(T-t)} \) at time \( T \) is obtained by first discounting by the spot rate, and then taking the conditional expectation with respect to \( \tilde{Q} \). Expression (12b) is the expectations hypothesis under \( \tilde{Q} \). It states that the long-term bond price is equal to the expected value under \( \tilde{Q} \) of receiving 1 dollar at time \( T \) and continuously discounting at the spot rate. We now state Lemma 1.

**Lemma 1.** Consider the continuous trading strategies \( \{N_1, N_2, N_3: t \in [0, T]\} \) such that \( V(t) = N_1(t)S(t) + N_2(t)P(t) + N_3(t)B(t) \) and \( \mathbb{E}[(V(T)/B(T))^2] < +\infty \).

There are no arbitrage opportunities in this class of trading strategies if and only if \( \{V(t)/B(t): t \in [0, T]\} \) is a \( \tilde{Q} \)-martingale.

**Proof.** In the appendix. ///

This lemma is very useful in pricing contingent claims. It will be used in the following manner. Consider the contingent claim with no cash flows over \((0, T)\) and a payoff at time \( T \) of \( h(T) = h(S(T), P(T), B(T)) \) where the function \( h \) satisfies \( \mathbb{E}(h(T)/B(T))^2 < +\infty \). By standard arguments, there exists a continuous trading strategy \( \{N_1(t), N_2(t), N_3(t): t \in [0, T]\} \) that duplicates \( h(T) \) at time \( T \), i.e., such that \( V(T) = h(T) \). Under the assumption of no arbitrage opportunities, by lemma 1, we have that

\[ V(t) = \mathbb{E}(h(T)/B(T) \middle| H_t)B(t) \]  

(13)

represents the value of the contingent claim at time \( t \).

A subtle issue needs to be mentioned here. We exogenously impose the assumption of no arbitrage opportunities in the derivation of expression (13). This assumption effectively restricts the set of continuous trading strategies so that the doubling strategy identified by Harrison and Kreps [16] is not allowed. An alternative approach is to restrict the trading strategies directly, either through a positive wealth constraint (see Latham [24], Dybvig [9]) or through margin requirements (see Heath and Jarrow [18]).
The above lemma utilizes all three traded assets. An analogous lemma can be stated and proven considering only the subsystem \(\{S(t), P(t)\}\). This subsystem corresponds to Merton's [25] stochastic interest rate economy. Again, we pause to consider some technical details. The noninterested reader can skip directly to lemma 2 and the subsequent discussion.

Consider the relative price \(X(t) = S(t)/P(t)\) over \([0, T]\). We want to find a probability measure \(\hat{Q}\) on \((\Omega, H)\), mutually absolutely continuous with respect to \(Q\), such that \(dX(t) + DX(t)\, dt\) is a \(\hat{Q}\)-martingale. By Girsanov's theorem,

\[
\frac{d\hat{Q}}{dQ} = \exp\left\{\frac{1}{2} \int_0^T \left( \hat{\theta}_1(y) + \hat{\theta}_2(y) \right) dy \right. \\
\left. - \int_0^T \hat{\theta}_1(y) \, dW_1(y) - \int_0^T \hat{\theta}_2(y) \, dW_2(y) \right\},
\]

where

\[
\hat{\theta}_1(t) = \begin{cases} 
((\alpha + D - \beta(t, P(t))) \\
-(\gamma_1(t)\sigma_1 + \gamma_2(t)\sigma_2 + \gamma_1^2(t) + \gamma_2^2(t))/[\sigma_1 - \gamma_1(t)] \\
\text{if } \sigma_1 \neq \gamma_1(t), \quad 0 \text{ otherwise,}
\end{cases}
\]

\[
\hat{\theta}_2(t) = \begin{cases} 
0 \text{ if } \sigma_1 \neq \gamma_1(t), \\
((\alpha + D - \beta(t, P(t))) \\
-(\gamma_1(t)\sigma_1 + \gamma_2(t)\sigma_2 + \gamma_1^2(t) + \gamma_2^2(t))/[\sigma_2 - \gamma_2(t)] \\
\text{otherwise,}
\end{cases}
\]

and \(E(d\hat{Q}/dQ)^2 < +\infty\). Note that by assumption (A.4), either \(\sigma_1 \neq \gamma_1(t)\) or \(\sigma_2 \neq \gamma_2(t)\). This measure makes \(X(t) + \int_0^t DX(y)\, dy\) a \(\hat{Q}\)-martingale. Expectation with respect to \(\hat{Q}\) will be denoted by \(\hat{E}(\cdot)\).

Define

\[
\hat{W}_1(t) = W_1(t) + \int_0^t \hat{\theta}_1(y) \, dy, \quad \hat{W}_2(t) = W_2(t) + \int_0^t \hat{\theta}_2(y) \, dy.
\]

By Levy's theorem, \(\{\hat{W}_1, \hat{W}_2\}\) are independent Brownian motions with respect to \(\hat{Q}\) on \(\{H_t\}_{t=0}\). Given (15), it can be shown that

\[
dX(t) + DX(t)\, dt = (\sigma_1 - \gamma_1(t))X(t) \, d\hat{W}_1(t) \\
+ (\sigma_2 - \gamma_2(t))X(t) \, d\hat{W}_2(t)
\]

on the probability space \((\Omega, \{H_t\}_{t=0}, \hat{Q})\). We now state lemma 2.

**Lemma 2.** Consider the continuous trading strategies \(\{N_1, N_2, N_3: t \in [0, T]\}\) such that \(N_3(t) = 0, \ V(t) = N_1(t)S(t) + N_2(t)P(t), \) and
V(T) = h(S(T)) for any function h: R → R twice continuously differentiable except at a finite set of points with E|h(S(T))| < +∞ and |h(x)| + |h'(x)| + |h''(x)| ≤ L(1 + |x|^m) for some constants L, m > 0 and for all x ∈ R (at the excluded points, interpret these derivatives as left- and right-hand derivatives).

There are no arbitrage opportunities in this class of trading strategies if and only if V(t)/P(t) is a ̂Q-martingale.

Proof. In the appendix. //

This lemma provides a valuation formula for pricing contingent claims. Consider the contingent claim with no cash flows over (0, T) and a payoff at time T of h(S(T)) where the function h satisfies the hypothesis of the lemma. Under the assumption of no arbitrage opportunities, the value of this claim at time t is

\[ V(t) = \hat{E}(h(S(T)) | H_t) P(t). \]  

(17)

To calculate the expectation in (17), the following stochastic differential equation for S(t) is useful. It is derived from the definition of S(t) = X(t)P(t), equation (16), and Ito's lemma:

\[
d(\log S(t)) = d(\log P(t)) - D \, dt - \frac{1}{2} \left( \sigma_1 - \gamma_1(t) \right)^2 + (\sigma_2 - \gamma_2(t))^2 \right] dt \\
+ (\sigma_1 - \gamma_1(t)) \, d\hat{W}_1(t) + (\sigma_2 - \gamma_2(t)) \, d\hat{W}_2(t).
\]  

(18)

Given P(T) = 1, expression (18) states that S(T)/S(t) is log-normally distributed (with respect to the ̂Q-measure), with parameters

\[ \hat{E}(S(T) | H_t) = S(t) e^{-D(T-t)} / P(t) \]  

(19)

\[ \hat{Var}(\log(S(T)/S(t)) | H_t) = \int_t^T \left[ (\sigma_1 - \gamma_1(y))^2 + (\sigma_2 - \gamma_2(y))^2 \right] dy. \]

Expressions (17) and (18) will prove useful when pricing commodity options in subsequent sections.

IV. FORWARD CONTRACTS

This section briefly reviews forward contracts. Detailed reviews can be found in Jarrow and Oldfield [22], French [10], and Cox, Ingersoll, and Ross [7]. A forward contract with maturity T is a financial asset that obligates its owner to buy one unit of the spot commodity at time T for a predetermined price, the forward price, set at the initiation date of the contract. At the initiation date, the forward price is set such that the forward contract has zero value. No cash flows from the forward contract occur until the maturity date T.
Define

\[ A(t) = \text{the time } t \text{ value of the forward contract maturing} \]
\[ \text{at time } T \text{ and initiated at time } 0, \]
\[ F(t) = \text{the forward price at time } t \text{ on the forward contract} \]
\[ \text{initiated at time } t \text{ and maturing at time } T. \]

This notation implies that the forward price of the forward contract
whose value is \( A(t) \) is \( F(0) \), since \( A(t) \) is initiated at time 0. The
maturity date \( T \) is held fixed.

By construction, the value of the forward contract at maturity is
the spot price minus the forward price on the contract, i.e.,

\[ A(T) = S(T) - F(0). \tag{20} \]

It is easy to show that there exists a continuous trading strategy
\( \{N_1, N_2, N_3\} \) such that at time \( T \), \( V(T) = A(T) \). Buying the spot
commodity and borrowing at the long-term bond rate to finance the
purchase provides the appropriate trading strategy. Since \( E(A(T)^2) < +\infty \), we can apply lemma 1 to get

**Proposition 1.** Given no arbitrage opportunities,

\[ F(t) = S(t) e^{-D(T-t)}/P(t). \tag{21} \]

**Proof.** By lemma 1,

\[ A(t) = \bar{E}((S(T) - F(0))/B(T)\mid H_t)B(t) \]
\[ = \bar{E}(S(T)/B(T)\mid H_t)B(t) - F(0)\bar{E}(B(t)/B(T)\mid H_t) \]
\[ = S(t) e^{-D(T-t)} - F(0)P(t), \tag{22} \]

by expression (12).

By construction of the contract, \( A(0) = 0 \); hence \( F(0) = S(0) e^{-DT}/P(0) \). \( \|\|

Proposition 1 gives the cash and carry model for the forward price
(see Jarrow and Oldfield [21]). It says that the discounted forward
price \( F(t)P(t) \) equals the spot price minus storage costs \( S(t) e^{-D(T-t)} \).
From proposition 1, we see that the value of a forward contract is
equal to the discounted change in the forward prices, i.e.,

\[ A(t) = [F(t) - F(0)]P(t). \tag{23} \]

Second, theorem 1 also implies, by the definition of \( \hat{Q} \), that \( F(t) e^{D(T-t)} \)
is a continuous \( \hat{Q} \)-martingale. This is useful in that it characterizes
the forward-price stochastic process, i.e.,

\[ dF(t) = [(\sigma_1 - \gamma_1(t)) d\hat{W}_1(t) + (\sigma_2 - \gamma_2(t)) d\hat{W}_2(t)] F(t). \]
This expression implies (along with (15))

\[
\operatorname{Var}(dF(t)/F(t)) = \hat{\operatorname{Var}}(dF(t)/F(t)) = [(\sigma_1 - \gamma_1(t))^2 + (\sigma_2 - \gamma_2(t))^2] dt. \tag{24}
\]

Given that as the long-term bond matures \((t \approx T)\) we have \(\gamma_1(t) \to 0\) and \(\gamma_2(t) \to 0\), the variance of the forward price must increase as the maturity date of the contract approaches. This result is consistent with the empirical evidence; see Ball and Torous [1].

V. FUTURES CONTRACTS

This section reviews futures contracts, providing new proofs of existing results as well as a new characterization of forward prices in terms of futures prices. A futures contract with maturity \(T\) is a financial asset that obligates its owner to buy one unit of the spot commodity at time \(T\) for a predetermined price, the futures price, set at the initiation date of the contract. At the initiation date, the futures price is set such that the futures contract has zero value. In addition, the difference between the spot price at time \(T\) and the futures price is distributed as a continuous cash flow over the life of the contract. The cash flow distributed is equal to the change in the futures price, thereby, making the futures contract's value identically zero.

Define

\[
a(t) = \text{the time } t \text{ value of the futures contract that matures at time } T \text{ and initiated at time } 0,
\]

\[
f(t) = \text{the futures price at time } t \text{ on a futures contract that matures at time } T,
\]

\[
g(t) = \text{the cash flow at time } t, \text{ per unit time, to the futures contract that matures at time } T.
\]

The cash flow \(g(t)\) is defined by

\[
f(t) = f(0) + \int_0^t g(y) \, dy \quad \text{for all } t \in [0, T], \tag{25}
\]

where \(f(T) = S(T)\). We assume

(A.5) \(\{g(t) : t \geq 0\}\) is a bounded predictable stochastic process.

The purpose of assumption (A.5) is to ensure that the cash flow at time \(t\) only depends on information available prior to time \(t\), and to make \(f(t)\) representable as a stochastic integral. The boundedness condition is to ensure finiteness of expected values.
A futures contract's value is identically equal to zero, i.e., \( a(t) = 0 \) for all \( t \in [0, T] \). Its cash flow is \( g(t) \) dt per unit time. We can transform this continuous cash flow \( g(y) \) to time \( T \) by investing it in the short-term bond \( B(y) \) at time \( y \). The accumulated cash flow at time \( T \) is therefore

\[
\int_1^T [g(y)/B(y)]B(T) \, dy.
\]

Since \( E(\int_1^T [g(y)/B(y)]B(T) \, dy)^2 < +\infty \), by standard techniques,\(^9\) it can be shown that there exists a continuous trading strategy \( \{N_1, N_2, N_3; t \in [0, T]\} \) with \( V(t) = N_1(t)S(t) + N_2(t)P(t) + N_3(t)B(t) \) that generates \( V(T) = \int_1^T [g(y)/B(y)]B(T) \, dy \) at time \( T \). Under the assumption of no arbitrage opportunities, the value of this trading strategy at time \( 0 \), \( V(0) \), equals the value of the futures contract, \( a(0) = 0 \). Given this insight, we now prove:

**Proposition 2.** Given no arbitrage opportunities, \( f(t) \) is a continuous \( \bar{Q} \)-martingale, i.e., \( f(t) = \bar{E}(f(T)|H_t) \) a.e.

**Proof.** By Lemma 1,

\[
a(t) = \bar{E}\left( \int_1^T \frac{(g(y)/B(y))B(T) \, dy}{B(T)} \right| H_t) \bigg|_{B(t)} = \bar{E}\left( \int_1^T (g(y)/B(y)) \, dy \ \bigg| \ H_t \right).
\]

By construction \( a(t) = 0 \) for all \( t \in [0, T] \). We use this fact to show that

\[
\eta_t = \int_0^t (g(y)/B(y)) \, dy
\]

is a \( \bar{Q} \)-martingale. Note

\[
\bar{E}(\eta_t|H_t) = \bar{E}\left( \int_1^T (g(y)/B(y)) \, dy + \int_0^t (g(y)/B(y)) \, dy \ \bigg| \ H_t \right) = 0 + \eta_t.
\]

By Durrett [8; p. 86], \( \eta_t \) is continuous a.e. \( \bar{Q} \).

Next define \( h(t) = f(0) + \int_0^t B(y) \, d\eta_y \). We have that \( h(t) \) is a continuous local \( \bar{Q} \)-martingale. By the associative law, Durrett [8; p. 62],

\[
h(t) = f(0) + \int_0^t B(y)(g(y)/B(y)) \, dy = f(0) + \int_0^t g(y) \, dy = f(t).
\]

Since \( g(y) \) is bounded, \( \sup_{t \in [0, T]} \bar{E}(f(t))^2 < +\infty \) and \( f(t) \) is a \( \bar{Q} \)-martingale. \( \square \)

This theorem generates numerous insights into futures contracts. First, because \( S(T) = f(T) \), it implies that

\[
f(t) = \bar{E}(S(T)|H_t) \quad \text{a.e.} \quad (26)
\]
This simple observation implies that the futures price \( f(t) \) can be viewed as the time \( t \) value of a contract paying \( S(T)B(T)/B(T) \) at time \( T \). This follows by lemma 1 since the value of this contract is equal to

\[
\hat{E}(S(T)B(T)/B(t)B(T)|H_t)B(t) = \hat{E}(S(T)|H_t) = f(t).
\]

This result has been previously proven by Cox, Ingersoll, and Ross \cite{7} using an alternative approach. It is significant since it demonstrates that \( f(t) \) can be viewed as the value of a traded asset, i.e., a dynamic self-financing portfolio involving \( S(t) \), \( P(t) \), and \( B(t) \). We shall use this observation when pricing commodity options on futures contracts.

Proposition 2 also implies that there exists a predictable stochastic processes \( \{\eta_1(t)f(t), \eta_2(t)f(t): t \in [0, T]\} \) such that

\[
\hat{E}\left( \int_0^T \eta_1^2(t)f(t)^2 \, dt \right) < +\infty, \quad \hat{E}\left( \int_0^T \eta_2^2(t)f(t)^2 \, dt \right) < +\infty,
\]

and

\[
df(t) = \left[ \theta_1(t)\eta_1(t) + \theta_2(t)\eta_2(t) \right] f(t) \, dt
+ \eta_1(t)f(t) \, dW_1(t) + \eta_2(t)f(t) \, dW_2(t). \tag{27}
\]

Expression (27) gives the price dynamics followed by the futures price.

Given more information about the stochastic process for the spot rate \( \{r(t): t \geq 0\} \), it is possible to give an alternative representation of \( f(t) \).

**Corollary 1.** If \( r(t) \) satisfies condition (5), and \( f(t) \) is a twice continuously differentiable function in \( (S(t), r(t), t) \) then \( f(t) = \phi(S(t), r(t), t) \) is the solution to

\[
\phi_t + rS\phi_s + \psi_1\phi_r + \frac{1}{2}\phi_{ss}S^2(\sigma_1^2 + \sigma_2^2)
+ \phi_{sr}(\psi_1 \sigma_1 + \psi_2 \sigma_2)
+ \frac{1}{2}\phi_{rr}(\psi_1^2 + \psi_2^2) = 0 \tag{28}
\]

subject to \( \phi(S(T), r(T), T) = S(T) \), where subscripts denote partial differentiation.

**Proof.** Define

\[
\phi(x, y, t) = x\hat{E}\left( \exp\left[ -D(T - t) + \int_t^T r(\tau) \, d\tau + \int_t^T \sigma_1 \, d\tilde{W}_1
+ \int_t^T \sigma_2 \, d\tilde{W}_2 - \frac{1}{2} \int_t^T (\sigma_1^2 + \sigma_2^2) \, d\tau \right] \bigg| r(t) = y \right). 
\]
Then
\[
f(t) = \tilde{E}(S(T)|H_t) = \tilde{E}\left(S(t) \exp\left[-D(T-t) + \int_t^T r(\tau) \, d\tau + \int_t^T \sigma_1 \, d\tilde{W}_1 \right. \right.
\]
\[
\left. + \int_t^T \sigma_2 \, d\tilde{W}_2 - \frac{1}{2} \int_t^T (\sigma_1^2 + \sigma_2^2) \, d\tau \right| H_t \right)
\]
\[
= \phi(S(t), r(t), t),
\]
where the last equality uses the independence of \(\tilde{W}_1(T) - \tilde{W}_1(t)\) and \(\tilde{W}_2(T) - \tilde{W}_2(t)\) from \(H_t\) and the strong Markov property of \(r(t)\); see Durrett [8; p. 288].

By Ito's lemma,
\[
f(t) - f(0) = \int_0^t \phi_s \, dS + \int_0^t \phi_r \, dr + \int_0^t \phi_t \, dt + \frac{1}{2} \int_0^t \phi_{ss} \, (dS)^2
\]
\[
+ \int_0^t \phi_{sr} \, dS \, dr + \frac{1}{2} \int_0^t \phi_{rr} \, (dr)^2
\]
\[
= (\text{local } \tilde{Q}-\text{martingale})
\]
\[
+ \int_0^t (\phi_s rS + \phi_r \psi_1 + \phi_t + \frac{1}{2} \phi_{ss} S^2 (\sigma_1^2 + \sigma_2^2)
\]
\[
+ \phi_{sr} S (\sigma_1 \psi_2 + \sigma_2 \psi_3) + \frac{1}{2} \phi_{rr} (\psi_2^2 + \psi_3^2)) \, d\tau.
\]

This implies that the last integral is a local \(\tilde{Q}\)-martingale. Since it is also of bounded variation (by Durrett [8; p. 54]), it is identically zero. Because the integrand is continuous, we get the stated result. ///

Given an explicit representation of the parameters for the spot process (e.g., a square root process),\(^{11}\) expression (28) can be used to estimate the futures price numerically. Finally, we give the last proposition of this section.

**Proposition 3.** Given no arbitrage opportunities,

\[
F(t) = f(t) + \tilde{\text{cov}}\left(S(T), \exp\left[-\int_t^T r(y) \, dy \right| H_t \right) / P(t). \tag{29}
\]
\textbf{Proof}

\begin{align*}
F(t) &= S(t) e^{-D(t-t)}/P(t) \\
&= \hat{E}\left(S(T) \exp\left[-\int_t^T r(y) \, dy \right] \bigg| H_t \right) / P(t) \\
&= \tilde{E}(S(T)|H_t) \tilde{E}\left(\exp\left[-\int_t^T r(y) \, dy \right] \bigg| H_t \right) / P(t) \\
&\quad + \text{cov}\left(S(T), \exp\left[-\int_t^T r(y) \, dy \right] \bigg| H_t \right)/P(t).
\end{align*}

Expressions (12b) and (26) yield the stated result. \(\Box\)

Proposition 3 gives the exact relationship between forward prices and futures prices. The forward price equals the futures price plus a term related to the covariance between spot commodity prices and spot interest rates.

A necessary and sufficient condition for the equivalence between forward and futures prices is

\[
\text{cov}\left(S(T), \exp\left[-\int_t^T r(y) \, dy \right] \bigg| H_t \right) = 0.
\]

For stochastic spot prices and arbitrary probability beliefs, this will be true if and only if \(\int_t^T r(y) \, dy \) is deterministic, i.e., spot rates are deterministic. This result generalizes a sufficient condition contained in Jarrow and Oldfield [21].

Expression (29) generates a testable implication, independent of the probability belief \(\tilde{Q}\). From the proof of expression (29) we see that \(F(t)P(t) = S(t) e^{-D(t-t)}\). By expression (12) this yields

\[
F(t)P(t) = \tilde{E}(S(T)B(t)/B(T)|H_t).
\]

If the spot rate \(r(t)\) is positive with positive probability, then \(B(t)/B(T) < 1\), so that

\[
F(t)P(t) < \tilde{E}(S(T)|H_t), \quad \text{or} \quad F(t)P(t) < f(t).
\]

The discounted forward price is seen to be strictly less than the futures price. This result is a direct implication of the preceding proposition.
VI. COMMODITY OPTIONS ON THE SPOT COMMODITY

This section of the paper prices call options on the spot commodity. A European-type call option on the spot commodity with maturity $T$ and exercise price $K$ gives its owner the option to buy one unit of the spot commodity at time $T$ for an exercise price of $K$. The value of the option at maturity is

$$C(T) = \max(S(T) - K, 0),$$

(30)

where $C(t)$ is the time $t$ value of the European call option on the spot commodity with an exercise price of $K$ and maturity date $T$. By standard arguments, it can be shown that the payoff given in (30) can be duplicated by a self-financing, continuous trading strategy. Hence, by lemma 1,

$$C(t) = \mathbb{E}\left( \max(S(T) - K, 0) \exp\left[ -\int_t^T r(y) \, dy \right] \big| H_t \right).$$

(31)

If interest rates were deterministic, this simplifies to Black’s [4] formula.

This valuation expression is often hard to calculate explicitly. It can be approximated, however, by adding the following hypotheses and then numerically solving the following partial differential equation.

If $r(t)$ satisfies condition (5) and $C(t)$ is a twice continuously differentiable function in $(S(t), r(t), t)$, then $C(t) = C(S(t), r(t), t)$ is the solution to

$$C_t + rSC_s + \psi_1 C_r + \frac{1}{2} C_{ss} \sigma_1^2 + \sigma_2^2
+ C_r \psi_2 \sigma_1 + \psi_3 \sigma_2
+ \frac{1}{2} C_{rr} \psi_3^2 + \psi_2^2 = 0$$

subject to $C(S(T), r(T), T) = \max(S(T) - K, 0)$. (32)

The proof of this assertion is identical to the proof of corollary 1. Given that the call’s payoff at time $T$ can be constructed using $S(t)$ and $P(t)$ alone (see the proof of Lemma 2), we can alternatively price the call using lemma 2,

$$C(t) = \mathbb{E}(\max(S(T) - K, 0) | H_t) P(t).$$

(33)

Using the explicit representation of the distribution for $S(T)$ as given in condition (19), direct calculation of (33) yields Merton’s [25] formula applied to the spot commodity:

$$C(t) = S(t) e^{-D(T-t)} N(h) - KP(t) N(h - \sqrt{q})$$

(34)
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where

\[ h = \frac{\log(S(t) e^{-D(t-t)}/K) - \log P(t) + q/2}{\sqrt{q}} \]

\[ q = \int_{t}^{T} [(\sigma_1 - \gamma_1(y))^2 + (\sigma_2 - \gamma_2(y))^2] dy \]

\[ N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \]

This formula differs from the Black–Scholes call option formula in two ways. First, the long-term bond’s price \( P(t) \) replaces the discount factor based on a constant spot rate. Second, the spot commodities volatility is replaced by \( q = \int_{t}^{T} [(\sigma_1 - \gamma_1(y))^2 + (\sigma_2 - \gamma_2(y))^2] dy \), which is the forward price’s volatility. Note that \( q = q(t, T) \), i.e., it depends on both the current time \( t \) and the maturity date \( T \). However, it is independent of the exercise price \( K \). Pricing calls using condition (34) can be obtained with the Black–Scholes formula using implied volatilities as long as both \( t \) and \( T \) are held fixed when calculating the implicit volatilities. Use of forward prices, as in expression (21), rather than spot prices dispenses with estimating the storage cost component \( Ds(t) \). This procedure is most often used for foreign currency options; see Garman and Kohlhagen [11].

This completes the discussion of pricing European-type call options on spot commodities. To price American-type calls, there are two alternatives. The most abstract generates a valuation expression as in expression (31), involving the supremum over the set of early exercise stopping times, as in Bensoussan [2]. This approach can be used with either lemma 1 or 2. The second approach, the most applied, generates a partial differential equation from these abstract formulas, as in expression (32), and appends an early exercise boundary condition. Numerical approximation techniques can then be applied.

VII. COMMODITY OPTIONS ON FUTURES CONTRACTS

This section prices European-type options on futures contracts. The pricing of American-type options is discussed at the close of this section. The results derived below are new to the literature. We consider a futures contract with maturity date \( T \) as in Section V. A European-type call option on this futures contract with an exercise price of \( K \) and maturity date \( \tau < T \) gives its owner the right to purchase the futures contract at time \( \tau \) for \( K \) dollars. At the maturity
date of the call, the futures contract has \((T - \tau)\) time units remaining before maturity. The value of the option at maturity is

\[ c(\tau) = \max(f(\tau) - K, 0), \]  

where \(c(t)\) is the time \(t\) value of the European call option on the futures contract with an exercise price of \(K\) and maturity \(\tau\), where \(t \leq \tau\).

The call option’s cash flow occurs at time \(\tau\). To make this financial claim compatible with lemma 1, we need to transform this cash flow to time \(T\). This is done by investing \(c(\tau)\) in the short-term bond at time \(\tau\); hence the value at time \(T\) is

\[ [\max(f(\tau) - K, 0) / B(\tau)]B(T). \]  

Using standard arguments (see the proof of lemma 1), this cash flow can be obtained by a continuous trading strategy involving \(S(t)\), \(P(t)\), and \(B(t)\). Applying lemma 1, we obtain

\[
c(t) = \tilde{E}([\max(f(\tau) - K, 0) / B(\tau)]B(T) / B(T) | H_t)B(t) \\
= \tilde{E}(\max(f(\tau) - K, 0) / B(\tau) | H_t)B(t) \\
= \tilde{E}\left( \max(f(\tau) - K, 0) \exp\left[ -\int_t^\tau \tau(y) \, dy \right] | H_t \right). \tag{37}
\]

This is the expression we would have obtained if we had applied lemma 1 with a termination date of \(\tau\) instead of \(T\).

Next, expression (26) implies that

\[
c(t) = \tilde{E}\left( \max(\tilde{E}(S(T) | H_t) - K, 0) \exp\left[ -\int_t^\tau \tau(y) \, dy \right] | H_t \right). \tag{38}
\]

Expression (38) gives an explicit representation of the option price on the futures contract in terms of the spot price and the interest rate process. We see that its value differs from that of a commodity option on the spot commodity in one important respect. The commodity option on the spot commodity with maturity \(\tau\) has a value equal to

\[
C(t, \tau) = \tilde{E}\left( \max(S(\tau) - K, 0) \exp\left[ -\int_t^\tau \tau(y) \, dy \right] | H_t \right). \tag{39}
\]

The futures option replaces \(S(\tau)\) in expression (39) with \(\tilde{E}(S(T) | H_t)\), a conditional expectation. If \(S(\tau)\) equalled the conditional expectation with respect to \(\tilde{Q}\), the values would be identical. Unfortunately, in general \(S(\tau)\) does not equal \(\tilde{E}(S(T) | H_t)\). Indeed, using expression (11) and Ito’s lemma, it can be shown that

\[
dS(t) = (\tau(t) - D)S(t) \, dt + \sigma_1 S(t) \, d\tilde{W}_1(t) + \sigma_2 S(t) \, d\tilde{W}_2(t). \tag{40}
\]
If storage costs are negative over \([\tau, T]\), i.e., \(-D > 0\), then \(r(t) - D > 0\) for all \(t\), and \(S(t)\) would be a strict submartingale with respect to \(\tilde{Q}\), i.e., \(\tilde{E}(S(T) | H_t) > S(\tau)\). This would imply that the option on the futures is strictly greater in value than the option on the spot. This is the case for most commodities.

If storage costs are positive over \([\tau, T]\), i.e., \(-D < 0\), as for example with a foreign currency (where \(D\) corresponds to the foreign spot rate), then if \(r(t) - D \geq 0\) a.e. and \(r(t) - D > 0\) with positive probability, \(\tilde{E}(S(T) | H_t) > S(\tau)\). Again, in this case, the options on the futures is strictly greater in value than the option on the spot.

These relationships are summarized in the following proposition.

**Proposition 4.** Given no arbitrage opportunities,

\[
c(t) \geq C(t, \tau) \quad \text{if and only if} \quad \tilde{E}\left(\int_\tau^T [r(y) - D]S(y) \, dy \, \bigg| \, H_t\right) \geq 0.
\]

The valuation formula for commodity options on futures contracts developed in expression (38) is very abstract. To utilize this model in applications, we need to impose additional restrictions on the spot rate process and then numerically approximate the solution.

If \(r(t)\) satisfies condition (5) and \(c(t)\) is a twice continuously differentiable function of \((S(t), r(t), t)\) then \(c(t) = c(S(t), r(t), t)\) is the solution to

\[
c_t + rS c_s + \psi_1 c_r + \frac{1}{2} c_{ss} S^2 (\sigma_1^2 + \sigma_2^2) \\
+ c_{sr} S (\psi_2 \sigma_1 + \psi_3 \sigma_2) + \frac{1}{2} c_{rr} (\psi_1^2 + \psi_2^2) = 0 \tag{41}
\]

subject to \(c(S(T), r(T), T) = \max(S(T) - K, 0)\). Expression (41) is a generalization of the result contained in Ramaswamy and Sundaresan [26; p. 1332, expression (10)]. It generalizes their result because it does not depend on the exogeneous assumption of the expectations hypothesis holding with respect to \(Q\). Expression (41) requires the assumption of complete markets, which enables us to construct a new probability \(\tilde{Q}\), for which the expectations hypothesis does hold!

If we restrict \(r(t)\) to satisfy a square root process, then expression (41) reduces exactly to the partial differential equation contained in Ramaswamy and Sundaresan [26].

Another approach to value the commodity option on the futures contract is to employ lemma 2, in a modified form. We now present the sequence of arguments required to obtain this alternative valuation model. First, we know that \(f(t)\) can be thought of as the value of a traded asset, the asset with a cash flow of \(S(T)B(T)/B(t)\) at time
T. Under this scenario, \( f(t) \)'s stochastic process is given by expression (27), i.e.,

\[
\begin{align*}
df(t) &= (\theta_1(t)e_1(t) + \theta_2(t)e_2(t))f(t) \, dt \\
&\quad + e_1(t)f(t) \, dW_1(t) + e_2(t)f(t) \, dW_2(t). 
\end{align*}
\]  

(42)

Now, consider a "new" economy consisting of assets \( f(t) \) and a long-term bond \( \bar{P}(t) \), where \( \bar{P}(t) \) matures at time \( \tau \). Lemma 2 applies to this subsystem as long as \( e_1(t) \) and \( e_2(t) \) are smooth functions of \( f(t) \) and suitably bounded. We specialize this approach in the following theorem.

**Proposition 5.** Under no arbitrage opportunities, if

(a) \[ df(t) = \eta_0(t)f(t) \, dt + \eta_1(t)f(t) \, dW_1(t) + \eta_2(t)f(t) \, dW_2(t), \]

where \( \eta_0(t), \eta_1(t), \eta_2(t) \) are bounded deterministic functions on \([0, \tau]\), and

(b) \[ d\bar{P}(t) = \bar{\beta}(t, \bar{P}(t))\bar{P}(t) \, dt + \bar{\gamma}_1(t)\bar{P}(t) \, dW_1(t) + \bar{\gamma}_2(t)\bar{P}(t) \, dW_2(t), \]

where \( \bar{\beta}(t, \bar{P}(t)) \) is bounded and \( \bar{\gamma}_1(t), \bar{\gamma}_2(t) \) are bounded deterministic functions on \([0, \tau]\), \( \bar{\beta}(t) > 0 \) for \( t \in [0, \tau] \), and \( \bar{\beta}(\tau) = 1 \), then

\[
\begin{align*}
c(t) &= f(t)N(m) - K\bar{P}(t)N(m - \sqrt{n}), 
\end{align*}
\]

where

\[
\begin{align*}
m &= [\log(f(t)/K) - \log \bar{P}(t) + n/2]/\sqrt{n}, \\
n &= \int_r^{\infty} \left[(\eta_1(t) - \bar{\gamma}_1(t))^2 + (\eta_2(t) - \bar{\gamma}_2(t))^2\right] dy, \\
N(x) &= \int_{-\infty}^{x} e^{-y^2/2} \, dy.
\end{align*}
\]

**Proof.** If \( \eta_0(t), \eta_1(t), \eta_2(t) \) satisfy the stated conditions, then lemma 2 holds with \( \bar{P} \) replacing \( P \), \( f(t) \) replacing \( S(t) \), \( D = 0 \), \( \sigma_1 = \eta_1(t) \), and \( \sigma_2 = \eta_2(t) \). Applying expression (34) of Section VI gives the result. //

This theorem provides a valuation formula for options on futures contracts when futures prices follow a slightly modified geometric Brownian motion. It differs from the Black–Scholes formula in two ways. First, the long-term bond price \( \bar{P}(t) \) is identified with the interest rate used in the model. Second, the simple volatility for \( f(t) \) is replaced by

\[
\begin{align*}
n &= \int_t^T \left[(\eta_1 - \bar{\gamma}_1)^2 + (\eta_2 - \bar{\gamma}_2)^2\right] dy.
\end{align*}
\]
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For European-type calls, using the Black–Scholes formula with the implicit volatility determined keeping \( t \) and \( \tau \) fixed provides an estimate of expression (43). For many applications, this approximation will be superior to solving the p.d.e. contained in expression (41).

To price American calls, early exercise is a distinct possibility; see Ball and Torous [1], Brenner, Courtedon, and Subrahmanyam [5], or Jarrow and Oldfield [22]. In this case, to price the calls, one can utilize the technique employed in corollary 1 to obtain a partial differential equation subject to an early exercise boundary condition. This gives the standard partial differential equation for pricing American calls under stochastic interest rates as contained in Merton [25]. This p.d.e. can be numerically solved using standard techniques; see Geske and Shastri [12].

VIII. SUMMARY

This paper studies the pricing of commodity options under stochastic interest rates utilizing complete markets and the risk neutrality approach. The major contribution of the paper is a simple valuation formula for European commodity options on futures contracts under stochastic interest rates (expression (43)). The formula is analogous to Merton’s [25] model. A secondary contribution of this paper is a cataloging of new and existing results concerning forward and futures contracts. The model in this paper can be significantly generalized; it is kept simple, however, for expositional clarity.

APPENDIX

Proof of Lemma 1

STEP 1. Suppose for every strategy, \( V(t)/B(t) \) is a \( \tilde{Q} \)-martingale. Then \( V(0) = V(0)/B(0) = \tilde{E}(V(T)/B(T)) < +\infty \). Consider \( V(T) \) such that \( Q(V(T) \geq 0) = 1 \) and \( Q(V(T) > 0) > 0 \). This implies \( \tilde{Q}(V(T) \geq 0) = 1 \) and \( \tilde{Q}(V(T) > 0) > 0 \) since \( Q \ll \tilde{Q} \) and \( Q \gg \tilde{Q} \). Hence, \( \tilde{E}(V(T)) > 0 \) and \( \tilde{E}(V(T)/B(T)) = V(0) > 0 \). There are no arbitrage opportunities.

STEP 2. This is proved by contradiction. Suppose there are no arbitrage opportunities in this class of strategies, and that there is some strategy \( \{N_1, N_2, N_3\} \) such that \( V(t)/B(t) \) is not a \( \tilde{Q} \)-martingale. Since \( V(t)/B(t) \) is a local \( \tilde{Q} \)-martingale over \([0, T]\), this implies that \( \tilde{E}(V(T)/B(T)) \neq V(0)/B(0) \). Suppose without loss of generality that \( \tilde{E}(V(T)/B(T)) < V(0)/B(0) \).
Define \( Y_t = \tilde{E}(V(T)/B(T)|H_t) \). Given that \( \tilde{E}(|V(T)/B(T)|) \leq E((V(T)/B(T))^2)^{1/2}E((d\tilde{Q}/d\mathcal{Q}))^{1/2} < +\infty \), \( Y_t \) is a martingale. It is continuous since \( H_t \) is a Brownian filtration; see Durrett [8, p. 86]. By the martingale representation theorem (Ikeda and Watanabe [20; p. 80]), there exist predictable processes \( \{M_1(t), M_2(t); t \in [0, T]\} \) such that

\[
Y_t = Y_0 + \int_0^t M_1 \, d\tilde{W}_1 + \int_0^t M_2 \, d\tilde{W}_2,
\]

where

\[
\int_0^T M_1^2 \, dt < +\infty \text{ a.e. } \tilde{Q} \quad \text{and} \quad \int_0^T M_2^2 \, dt < +\infty \text{ a.e. } \tilde{Q}.
\]

Next, we find a trading strategy that duplicates \( Y_t \), i.e., find \( \{\tilde{N}_1, \tilde{N}_2, \tilde{N}_3; t \in [0, T]\} \) such that \( \tilde{V}(t)/B(t) = Y_t \), where \( \tilde{V}(t) = \tilde{N}_1(t)S(t) + \tilde{N}_2(t)P(t) + \tilde{N}_3(t)B(t) \). Define

\[
\begin{bmatrix}
\tilde{N}_1(t) \\
\tilde{N}_2(t)
\end{bmatrix} = \begin{bmatrix}
Z_1(t)\sigma_1 & Z_2(t)\gamma_1(t)
\end{bmatrix}^{-1} \begin{bmatrix}
M_1(t) \\
M_2(t)
\end{bmatrix},
\]

and \( \tilde{N}_3(t) = Y_t - \tilde{N}_1(t)Z_1(t) - \tilde{N}_2(t)Z_2(t) \). The inverse exists by assumption (A.4). This strategy has \( \tilde{V}(t)/B(t) = Y_t \); hence \( \tilde{V}(t)/B(t) \) is a \( \tilde{Q} \)-martingale.

Finally, we now show there exists an arbitrage opportunity. Consider the strategy:

\[
\begin{align*}
N_1^\dagger(t) &= \tilde{N}_1(t) - N_1(t), \\
N_2^\dagger(t) &= \tilde{N}_2(t) - N_2(t), \\
N_3^\dagger(t) &= \tilde{N}_3(t) - N_3(t) - (\tilde{E}(Y_T) - V(0)/B(0)).
\end{align*}
\]

By construction,

\[
\begin{align*}
V^\dagger(0) &= N_1^\dagger(0)S(0) + N_2^\dagger(0)P(0) + N_3^\dagger(0)B(0) \\
&= [\tilde{E}(Y_T) - V(0)] - [\tilde{E}(Y_T) - V(0)] = 0.
\end{align*}
\]

There are no cash flows over \((0, T)\), and at time \(T\),

\[
\begin{align*}
N_1^\dagger(T)S(T) + N_2^\dagger(T)P(T) + N_3^\dagger(T)B(T) \\
&= -\tilde{E}(V(T)/B(T)) - V(0)B(T) > 0 \\
\text{a.e. } Q. \text{ This is an arbitrage opportunity, and therefore a contradiction.} \quad \Box
\end{align*}
\]
Proof of Lemma 2

STEP A. Identical to step 1 in the proof of Lemma 1.

STEP 2. This is proved by contradiction. Suppose there are no arbitrage opportunities in this class of strategies, and that there is some strategy \{N_1, N_2: t \in [0, T]\} such that \(V(t)/P(t)\) is not a \(\hat{Q}\)-martingale. Then there is some \(t \in [0, T]\) such that \(\hat{E}(V(T)/P(T)|H_t) \neq V(t)/P(t)\). Without loss of generality, suppose that \(\hat{E}(V(T)/P(T)|H_T) > V(t)/P(t)\).

Consider \(V(T) = h(S(T))\). We want to find a continuous trading strategy that is initiated at time \(t\), is self-financing, and duplicates \(h(S(T))\) at time \(T\). Following Merton [25], suppose \(\bar{N}_1(t) = \bar{N}_1(S(t), P(t), t)\) and \(\bar{N}_2(t) = \bar{N}_2(S(t), P(t), t)\) are twice continuously differentiable functions such that \(\bar{V}(t) = \bar{N}_1(t)S(t) + \bar{N}_2(t)P(t)\) and \(d\bar{V}(t) = \bar{N}_1(t)dS(t) + \bar{N}_2(t)dP(t) + \bar{N}_1(t)DS(t)dt\). It can be shown (see Bergman [3] or Jarrow and Rudd [23]) that any \(\bar{N}_1(t), \bar{N}_2(t)\), \(\bar{V}(t)\) satisfying the above conditions imply that

\[
\bar{N}_1(t) = \bar{V}_S(S(t), P(t), t), \quad \bar{N}_2(t) = \bar{V}_P(S(t), P(t), t), \quad (A.1)
\]

where the subscript denotes partial differentiation, and \(\bar{V}(S(t), P(t), t)\) solves

\[
\bar{V}_t + \frac{1}{2}\bar{V}_{SS}S^2(\sigma_1^2 + \sigma_2^2) + \frac{1}{2}\bar{V}_{PP}P^2(\gamma_1^2 + \gamma_2^2) + \bar{V}_PS(\sigma_1\gamma_1 + \sigma_2\gamma_2) = 0, \quad (A.2)
\]

subject to \(\bar{V}(S(T), P(T), T) = h(S(T))\). By Gielet [14; p. 82], there exists a unique twice continuously differentiable solution \(\bar{V}(S(t), P(t), t)\) of (A.2). Hence, there exist \(\bar{N}_1(t), \bar{N}_2(t)\) satisfying the required conditions.

Next, we show that \(\bar{V}(t)/P(t)\) is a \(\hat{Q}\)-martingale. First, by (A.1),

\[
\bar{V}(t) = \bar{N}_1(t)S(t) + \bar{N}_2(t)P(t) = \bar{V}_S(S(t)) + \bar{V}_P(t)P(t).
\]

This implies that \(\bar{V}\) is homogeneous of degree 1 in both \(S(t)\) and \(P(t)\), i.e.,

\[
\bar{V}(S(t), P(t), t) = P(t)\bar{V}(S(t)/P(t), 1). \quad (A.3)
\]

Thus \(\bar{V}(t)/P(t) = \bar{V}(X(t), 1)\). But (A.2) implies \(\bar{V}_t + \frac{1}{2}((\sigma_1 - \gamma_1)^2 + (\sigma_2 - \gamma_2)^2X^2\bar{V}_{XX} = 0\).

By Gihman and Skorohod [13; Theorem 1, p. 73], \(\bar{V}(X(t), 1) = \hat{E}(h(T)|H_t)\) is the solution to this expression. Since \(\hat{E}|h(S(T))| < +\infty, \bar{V}(X(t), 1)\) is a \(\hat{Q}\)-martingale. Summarizing,

\[
\bar{V}(t)/P(t) = \hat{E}(h(T)|H_t) = \hat{E}(V(T)|H_t). \quad (A.4)
\]
Finally, we construct the arbitrage opportunity. Consider

\[ N^*_1(y) = \begin{cases} 
0, & \text{if } y < t, \\
N_1(y) - \bar{N}_1(y), & \text{if } t \leq y \leq T, \\
0, & \text{if } y < t,
\end{cases} \]

\[ N^*_2(y) = \begin{cases} 
N_2(y) - \bar{N}_2(y) + (E(V(T)|H_t) - V(t)/P(t))/P(t), & \text{if } t \leq y \leq T,
\end{cases} \]

and

\[ V^*(y) = N^*_1(y)S(y) + N^*_2(y)P(y). \]

By construction, \( V^*(0) = 0 \). At time \( t \),

\[ V^*(t) = (N_1(t) - \bar{N}_1(t))S(t) + (N_2(t) - \bar{N}_2(t))P(t) \]
\[ - (\hat{E}(V(T)|H_t) - V(t)/P(t)) \]
\[ = V(t)/P(t) - \hat{V}(t)/P(t) + [\hat{E}(V(T)|H_t) - V(t)/P(t)]. \]
\[ = 0, \]

by (A.4). Finally, at time \( T \),

\[ V^*(T) = [\hat{E}(V(T)|H_T) - V(T)/P(T)]/P(T) > 0 \quad \text{a.e. } Q. \]

This is an arbitrage opportunity, and therefore a contradiction. //

NOTES

1. One exception is that commodity options on the spot commodity have been priced under stochastic interest rates; see Grabbe [15].
2. A filtration is an increasing sequence of \( \sigma \)-fields. \( (H_t)_{t \geq 0} \) satisfies \( H_0 = \{ \emptyset, \Omega \} \), \( \lim_{t \to \infty} H_t = H, \) it is right continuous, and includes all subsets of \( Q \) null sets; see Durrett [8; p. 19] for further elaboration.
3. The interested reader is referred to Bensoussan [2] or Durrett [8] for details. Essentially, we only require equation (1) to be a well-defined stochastic integral and the cash flows to be predictable stochastic processes.
4. The stochastic process \( r(t) \) is predictable if it is \( H_t \) measurable for all \( t \geq 0 \) and if \( r(t, w) \) is measurable with respect to the \( \sigma \)-field on \( [0, \infty) \times \Omega \) generated by the left-continuous, \( H_t \) measurable processes; see Durrett [8, p. 49]. \( r(t) \) is bounded if there exists a constant \( k \geq 0 \) such that \( |r(t)| \leq k \) for all \( t \). This assumption is easily generalized and it is only imposed to avoid technical difficulties.
5. This statement is formally true only if we omit the bounded hypothesis contained in (3) with respect to \( r(t) \).
6. Use the martingale representation theorem, under assumption (A.4) (see Durrett [8; p. 88] or the standard finance construction of a hedged portfolio with boundary condition \( b(T) \) (see Merton [25]).
7. To derive expression (18), using $S = X \cdot P$ and Ito's lemma gives $dS/S = dP/P + dX/X + dX dP/XP$. However, $d(\log S) = dS/S - \frac{1}{2}(dS)^2/S^2$ and $d(\log P) = dP/P - \frac{1}{2}(dP)^2/P^2$, and so substitution yields

$$d(\log S) = d(\log P) - \frac{1}{2}(dS)^2/S^2 + \frac{1}{2}(dP)^2/P^2 + dX/X + (dX/X)(dP/P).$$

Using (15), $dP/P = (\beta - \gamma_1 \tilde{W}_1 - \gamma_2 \tilde{W}_2) dt + \gamma_1 d\tilde{W}_1 + \gamma_2 d\tilde{W}_2$. Substituting this and expression (16) into the above gives the result.

8. To see this, note that $F(t) = X(t) e^{-DT^{-t}}$. By Ito's lemma,

$$dF = \frac{dX}{X} e^{-DT^{-t}} X + X e^{-DT^{-t}} (+D dt)$$

which gives the result using expression (16).

9. Either use the martingale representation theorem given (A.4), as in the proof of lemma 1, or construct the hedge portfolio directly as in the proof of lemma 2.

10. To see this, note that, given proposition 2 and since $f(t) > 0$ for all $t$, by the martingale representation theorem, Durrett [8; p. 88], there exists $\{\eta_1(t)f(t), \eta_2(t)f(t) : t \in [0, T]\}$ satisfying the conclusions of expression (27), such that

$$df(t) = \eta_1(t)f(t) d\tilde{W}_1(t) + \eta_2(t)f(t) d\tilde{W}_2(t).$$

Condition (9) yields expression (27).

11. If

$$\psi_1(r(t)) = \kappa (\mu_1 - r(t)), \quad \psi_2(r(t)) = \mu_2 \sqrt{r(t)}, \quad \psi_3(r(t)) = \mu_3 \sqrt{r(t)},$$

where $\kappa, \mu_1, \mu_2, \mu_3$ are constants, we get a square root process.

REFERENCES


The Pricing of Commodity Options with Stochastic Interest Rates:
Corrigendum

pages 29 and 30: \( g(y)dy \) can be replaced by \( dg(y) \) everywhere and the analysis generalizes.

page 31: There is a missing phrase. The 9th line should read "\( S(t), P(t), \text{ and } B(t) \) but necessarily paying a cash flow (dividend) of \( r(t)f(t)\,dt \) per unit time."

page 31, equation (28) is missing a term. The missing term is \( -(\theta_1\psi_2 + \theta_2\psi_3)\phi_r \).

page 32, the equation after the phrase (local \( \tilde{Q} \)-martingale) is missing the term \( -(\theta_1\psi_2 + \theta_2\psi_3)\phi_r \).

page 34, equation (32) is missing a term. The missing term is \( -(\theta_1\psi_2 + \theta_2\psi_3)C_r \).

page 37, equation (41) is missing a term. The missing term is \( -(\theta_2\psi_2 + \theta_3\psi_3)c_r \).

page 37, 10 lines from the bottom, the word "complete" should be replaced by "arbitrage free".

page 38, proposition 5 is missing a statement after the word "and." The statement is "\( f(t) \) pays a dividend of \( r(t)f(t) \) per unit time."

page 38, expression (43), "\( f(t) \)" should be replaced by "\( f(t)\bar{P}(t) \)".

page 38, in the proof, second line, the expression "\( D \equiv 0 \)" should be "\( D \equiv r(t) \)".

page 41, equation (A.4), both \( h(T) \) and \( V(T) \) should be divided by \( P(T) \).

page 42, top of page, \( V(T) \) should be divided by \( P(T) \) wherever it appears.