BELIEFS AND ARBITRAGE PRICING

Robert JARROW

Cornell University, Ithaca, NY 14853, USA

Received 18 February 1987
Accepted 18 May 1987

This paper investigates the necessary restriction on investor beliefs implied by a no-arbitrage price system. In addition, Stone's theorem is invoked to study the existence of equivalent probability measures, and to relate our approach to the existing literature.

1. Introduction

Pricing arguments using the absence of arbitrage opportunities have played a central role in financial economics. The primary reason underlying this trend is that arbitrage pricing techniques are robust; indeed, instead of imposing assumptions directly upon preferences and the equilibrium concepts employed, arbitrage pricing theory imposes its assumptions on the price functional. Common to arbitrage pricing, however, are assumptions upon the homogeneity of beliefs [see Harrison and Kreps (1979), Harrison and Pliska (1981) and Sethi (1984)].

The purpose of this paper is to relax the assumptions on investor beliefs underlying arbitrage pricing theory. We do this by defining a no-arbitrage opportunity price system solely in terms of the properties of prices. We then derive the necessary restrictions this economy imposes on beliefs. The implied homogeneity is shown to be weaker than that commonly invoked in the literature. Using Stone's theorem, we next study the existence and characterization of our no-arbitrage opportunity price functional in terms of probability measures. This study links our approach to the existing literature.

2. The model

We will restrict our attention to a single-period model with times 0 and 1. The state space for the economy at date 1 is represented by the set \( \Omega \). Assets are identified by state dependent cash flows at time 1; hence, an asset is a function \( x: \Omega \rightarrow R \). The space of all possible assets is \( R^\Omega \), but not all elements in \( R^\Omega \) represent the set of marketed assets which trade in frictionless markets. By frictionless, we mean that there are no transaction costs, no taxes, no restrictions on short sales, and assets are divisible. Consistent with this, the set of marketed assets \( M \) is assumed to be a linear subspace of \( R^\Omega \), i.e., it is closed under addition and scalar multiplication.

The limited liability assets are defined as

\[
M^+ = \{ m \in M : m(w) \geq 0 \text{ for all } w \in \Omega \}.
\]
Each asset in $M^+$ has non-negative cash flows at time 1, so the maximum loss to purchasing it is the initial purchase price.

We are given a price functional, $\pi: M \rightarrow R$, where $\pi(m)$ is the time 0 price of the asset $m$. Investors are also assumed to be price takers.

A pair $[M, \pi]$ with the above properties will be called a price system.

A price system $[M, \pi]$ is said to have no arbitrage opportunities if $\pi$ satisfies:

(a) given any $m_1, m_2 \in M$ and $\alpha, \beta \in R$, $\pi(\alpha m_1 + \beta m_2) = \alpha \pi(m_1) + \beta \pi(m_2)$, and

(b) given any $m \in M^+$, $\pi(m) \geq 0$.

Condition (a) requires that the package consisting of $\alpha$ shares of asset $m_1$ and $\beta$ shares of asset $m_2$ sells for the same price as the self-constructed portfolio consisting of purchasing $\alpha$ and $\beta$ separate shares of $m_1$ and $m_2$, respectively. Condition (b) requires that any limited liability asset has a non-negative price. An asset $m \in M$ which violates (a) or (b) is called an arbitrage opportunity. This definition is independent of investors' probability beliefs.

3. Beliefs and arbitrage opportunities

This section investigates the restrictions imposed by a no-arbitrage opportunity price system on investors' probability beliefs. The economy consists of $I$ investors. Each investor $i \in \{1, \ldots, I\}$ is endowed with a portfolio $m_i \in M$ and a (distinct) probability belief $Q_i$ defined over the event set $\mathcal{M}(M)$, where $\mathcal{M}(M)$ is the smallest $\sigma$-algebra generated by the assets in $M$. Let $L(Q_i) = \{ x \in R^I : \int \mathbb{X}^dQ_i < + \infty \}$.

Assumption 1. The $i$th investor has a von Neumann–Morgenstern utility function $U_i: R \rightarrow R$ which is differentiable, strictly increasing, and strictly concave.

Assumption 1 implies that the investor's optimization problem is

$$\max_{m \in M} \int_U U_i(m) \, dQ_i \quad \text{subject to} \quad \pi(m) = \pi(m_i).$$

Assumption 2. Differentiation under the integral in expression (1) is a valid operation.

Sufficient conditions for the satisfaction of Assumption 2 can be found in Bartle (1966, p. 46).

Assumption 3. A solution to the constrained optimization problem (1) exists.

Necessary and sufficient conditions for the satisfaction of Assumption 3 are in Bertsekas (1974).

The following theorem is the key result of this paper.

Theorem 1 (necessary restrictions on the homogeneity of beliefs). Given Assumptions 1–3, given a price system $[M, \pi]$ with no arbitrage opportunities, $\chi_{M^c} \in M$, and $\pi(\chi_{M^c}) > 0$, then for all $A \in D$, $Q_i(A) = 0$ if and only if $Q_j(A) = 0$ where $i, j$ represent distinct investors, and $D = \{ A \in 2^\Omega: A = m^{-1}((0, + \infty)) \}$ for some $m \in M^+$.

Proof. See the appendix.
This theorem states that in a no-arbitrage opportunity price system where the riskless asset \( \chi_D \) trades, all investors must agree on the probability zero events contained in the set \( D \). They can disagree on the magnitudes of the probabilities for non-zero probability events, and they can disagree on zero probability events in \( \sigma(M) \) and not in \( D \). \(^1\) Indeed, only the events in the set \( D \) can be 'bet upon' with the assets \( m \in M^+ \). For event sets outside \( D \), but within \( \sigma(M) \), investors cannot bet upon differences in opinions.

This homogeneity of beliefs is the weakest necessary condition consistent with a no-arbitrage opportunity price system \([M, \pi]\). If this homogeneity did not exist, a no-arbitrage opportunity price system would not exist. Furthermore, since a necessary condition of equilibrium is that the economy contains no arbitrage opportunities, this theorem also provides a necessary condition on the homogeneity of beliefs in equilibrium.

If the market is complete in the sense that all \( \sigma(M) \)-measurable functions trade [see Green and Jarrow (1987)], then \( D = \sigma(M) \). \(^2\) In this economy investors must agree on the zero probability events contained in \( \sigma(M) \). This stronger condition on beliefs is the restriction normally imposed in the existing arbitrage pricing literature [see Harrison and Kreps (1979), Harrison and Pliska (1981), and Sethi (1984)].

4. Existence of equivalent probability measures

Given a price system \([M, \pi]\) containing no arbitrage opportunities, an important issue is whether the price functional can be characterized as the discounted expected value using some probability measure. This alternative characterization of \( \pi \) is called the 'risk-neutrality' valuation technique [see Cox and Ross (1976) or Harrison and Kreps (1979)]. This section applies Stone's theorem to obtain sufficient conditions for the existence of such a measure.

Given a price system \([M, \pi]\) satisfying no arbitrage opportunities with \( \chi_D \in M \) and \( \pi(\chi_D) > 0 \), define an equivalent probability measure as a probability measure \( \mu \) defined on \((\Omega, \sigma(M))\) such that

\[
\pi(m) = \pi(\chi_D) \int_\Omega m \, d\mu \quad \text{for} \quad m \in M. \tag{2}
\]

The price of asset \( m \) at time 0, \( \pi(m) \), is the expected value of \( m \) using the measure \( \mu \) at time 1, discounted to time 0 by the riskless factor \( \pi(\chi_D) \).

**Theorem 2 (existence of a unique equivalent probability measure).** Let \([M, \pi]\) be a price system with no arbitrage opportunities. If \( \chi_D \in M, \pi(\chi_D) > 0 \), M is a sublattice of \( \mathbb{R}^\Omega \), and \( \pi \) is a \( \sigma \)-additive on \( M \), then there exists a unique equivalent probability measure.

**Proof.** Define \( \Pi: M \to R \) by \( \Pi(m) = \pi(m)/\pi(\chi_D) \) for all \( m \in M \). By construction, \( \Pi(\chi_D) = 1 \), and \( \Pi \) is also linear, positive, and \( \sigma \)-additive. A direct application of Stone's theorem to \( \Pi \) [Royden (1968, p. 297)] gives the theorem. \( \square \)

The first two sufficient conditions relate to the space \( M \) of marketed assets. It is required that the riskless asset \( \chi_D \) be marketed and that \( M \) be a sublattice. These conditions are sufficient for \( M \) to be

---

\(^1\) \( D \subset \sigma(M) \) since \( \sigma(M) = \{ A \in 2^\Omega : A = m^{-1}(B) \text{ for some } B \in \mathcal{B} \text{ and } m \in M \} \) where \( \mathcal{B} \) is the Borel \( \sigma \)-algebra on \( \mathbb{R} \).

\(^2\) If \( M \) is complete then \( \chi_A \in M^+ \) for all \( A \in \sigma(M) \), hence, \( \sigma(M) \subset D \).
dense in the set of $\sigma(M)$ measurable functions (using the product topology) [see Green and Jarrow (1987)].

The remaining hypothesis is that $\pi$ is $\sigma$-additive, i.e., given a decreasing sequence $\{m_n\} \in M$ with $\lim_{n \to \infty} m_n(w) = 0$ for all $w \in \Omega$, then $\lim_{n \to \infty} \pi(m_n) = \pi(0) = 0$. A violation of this condition would be an 'arbitrage opportunity' in the limit. Indeed, consider the decreasing sequence $\{m_n\}$ described above. This translates into the sequence $\{\pi(m_n)\}$. Since $\pi$ is positive and linear, this sequence is bounded below by zero and is monotone decreasing. Hence, $\lim_{n \to \infty} \pi(m_n) = \pi^* \geq 0$ exists. The content of the $\sigma$-additivity assumption is that $\pi^* = 0$.

This theorem has two immediate corollaries.

**Corollary 1 (partial converse of Theorem 2).** Let $[M, \pi]$ be a price system with no arbitrage opportunities. If there exists an equivalent probability measure $\mu$ on $(\Omega, \sigma(M))$, then $\pi$ is $\sigma$-additive.

**Proof.** Suppose there exists an equivalent probability measure satisfying (2). Define $\psi: L(\mu) \to R$ by

$$\psi(x) = \int_{\Omega} x \, d\mu.$$  \quad (3)

This functional $\psi$ restricted to $M$ equals $\Pi$. $\psi$ is $\sigma$-additive by the monotone convergence theorem [Royden (1968, p. 227)], hence, $\Pi$ and $\pi$ are $\sigma$-additive on $M$. \qed

**Corollary 2 (existence of unique extensions).** Let $[M, \pi]$ be a price system with no arbitrage opportunities. If $x_\Omega \in M$, $\pi(x_\Omega) > 0$, $M$ is a sublattice of $R^\Omega$, and $\pi$ is a $\sigma$-additive on $M$, then there exists a unique positive, linear, $\sigma$-additive extension $\phi: L(\mu) \to R$ of $\pi$.

**Proof.** The proof of Corollary 1 gives the existence of $\phi$ by setting $\phi(x) = \pi(x_\Omega)\psi(x)$. By Theorem 2, the measure defining $\psi$ is unique which guarantees the uniqueness of $\psi$. \qed

This last corollary gives sufficient conditions for the existence of a unique extension of $\pi$ from $M$ to $L(\mu)$. This corollary is useful when pricing contingent claims, like options, on the marketed assets in $M$.

Given a probability measure $P$ on $\sigma(M)$, a price system $[M, \pi]$ with $\pi$ linear, is said to have no *strong arbitrage opportunities with respect to $P$* if given any $m \in M$ satisfying $P(m \geq 0) = 1$, $P(m > 0) > 0$, then $\pi(m) > 0$. Such a linear functional $\pi$ is said to be *strictly* positive with respect to $L(P)$.

It follows from expression (2) that given the existence of an equivalent probability measure $\mu$ as in Theorem 2, $[M, \pi]$ has no strong arbitrage opportunities with respect to $P$ where $P$ is any probability measure mutually absolutely continuous with respect to $\mu$. This assertion links this paper's approach to that utilized in the existing literature. The existing literature, for a fixed probability measure $P$ on $\sigma(M)$, defines an arbitrage opportunity in the strong sense given above. If $P$ is mutually absolutely continuous to $\mu$, then the two approaches are equivalent.

**Appendix: Proof of Theorem 1**

Consider maximizing $V(m) = \int_{\Omega} L_j(m) \, dQ_j$ subject to $\{m \in M: \pi(m) = \pi(m_j)\} \equiv \phi$. Since $\phi$ is convex and $V(m)$ is strictly concave, the solution is unique.
Necessary conditions for \( m^* \in M \) being a maximum are

\[
\pi(m^*) = \pi(m_i) \quad \text{and} \\
\frac{\partial}{\partial \epsilon} \int_{\Omega} U_i(m^* + \epsilon(m - \pi(m)\chi_{\Omega}/\pi(\chi_{\Omega}))) \, dQ_i \bigg|_{\epsilon=0} = 0 \quad \text{for} \quad m \in M.
\]

This simplifies to

\[
\int_{\Omega} U_i'(m^*) m \, dQ_i / \lambda = \pi(m) \quad \text{for all} \quad m \in M \quad \text{where} \quad \lambda = \int_{\Omega} U_i'(m^*) \, dQ_i / \pi(\chi_{\Omega}) > 0.
\]

Consider \( A \in D \). There exists \( x \in M + \text{s.t.} \ A = x^{-1}((0, \infty)) \), so \( \pi(x) \geq 0 \). Note that \( U_i'(m^*) x > 0 \) on \( A \) for all \( i \). So, if \( \pi(x) > 0 \), then \( Q_i(A) > 0 \) for all \( i \). Conversely, if \( \pi(x) = 0 \), then \( Q_i(A) = 0 \) for all \( i \). \( \square \)

References