Preferences, Continuity, and the Arbitrage Pricing Theory

Robert A. Jarrow
Cornell University

This article investigates the structure on preferences required to derive Ross's arbitrage pricing theory (APT). It is shown that only ordinal preferences are required. In particular, the APT does not require that agents possess preferences representable as risk-averse expected utility functions. This characteristic of the APT is not shared by the standard equilibrium-based capital asset pricing models.

The two leading models in financial economics that attempt to explain the relationships among asset returns are the capital asset pricing model (CAPM) and the arbitrage pricing theory (APT). These two paradigms are obtained under differing sets of assumptions. The CAPM imposes significant structure upon preferences, at least enough to imply that preferences are representable by an expected utility function and that traders are risk-averse. Ross’s APT, on the other hand, imposes more structure upon returns by postulating that they must satisfy a linear factor model. Additional differences are identifiable, perhaps the most significant of which is that the CAPM characterizes prices resulting from a competitive equilibrium, whereas the classical APT characterizes prices resulting from only the absence of arbitrage opportunities.

Since the APT relies only on the absence of arbitrage, it seems that the APT should also require less structure on preferences than the CAPM, but how much less? Do
preferences need to be characterizable by an expected utility function or, at a minimum, to satisfy the strong independence axiom? Do investors need to be risk-averse or to have bounded risk aversion? From counterexamples provided by Ross (1976, p. 345) and Huberman (1982, p. 190), we know that more structure is needed than simply the requirement that preferences be strictly increasing. This additional structure is needed because the APT involves an infinite number of assets, and the pricing argument takes place in the limit. In contrast, for economies consisting of only a finite number of linearly independent assets, strictly increasing preferences would (almost) be enough.

The purpose of this article is to provide joint sufficient conditions on preferences and market structures for the APT pricing argument to hold. To study this issue, I specialize Krep's (1981) results to the Chamberlain and Rothschild (1983) economy consisting of an infinite number of assets. This differs from Ross's original derivation, which considered the limit of a sequence of finite-asset economies. I show that only mild restrictions on preferences and market structure are needed to obtain the APT. Two sets of sufficient conditions are derived, corresponding to different assumptions about the market structure. In complete markets, the APT can be derived if there exists at least one investor with strictly increasing preferences and an optimal portfolio. In incomplete markets, however, it is required in addition that his preferences be continuous.

The intuition underlying these results can be obtained by considering a finite-asset economy. In a finite-asset economy it is easy to show that the existence of an investor with an optimal portfolio and strictly increasing preferences is sufficient to exclude arbitrage opportunities. Next, consider an infinite-asset economy. If markets are complete, then all limit portfolios trade (where a limit portfolio is the limit of a sequence of portfolios each consisting of only a finite number of assets); in this case the argument mimics the finite-asset economy and no additional restrictions are required. In incomplete markets, however, some limit portfolios do not trade. Nonetheless, the additional restriction that the investor possess continuous preferences implies that they “almost” trade. Here again, the reduction to the finite-asset case occurs, giving the stated results.

It is significant that these sufficient conditions require neither the strong independence axiom nor the expected utility hypothesis. The APT’s pricing argument, therefore, can be based on ordinal utility theory rather than on the cardinal utility theory that underlies the CAPM. Given the experimental evidence indicating a systematic violation of the strong independence axiom [see Machina (1982) for a review] and the growing concern that this systematic violation underlies observed market anomalies from the CAPM [see Shefrin and Statman (1984), DeBondt and Thaler (1985)], the fact that the APT is independent of the expected utility hypothesis is significant. In theory, at least, the APT provides a means to test the hypothesis that the observed market anomalies relative to the CAPM are due to this type of irrationality on the part of investors.
Section 1 of this article presents the details of the Chamberlain and Rothschild (1983) economy. Section 2 studies the relationship between arbitrage opportunities and the price functional defined over asset payoffs. Section 3 presents two sets of sufficient conditions on preferences and market structures which imply that prices are continuous. Finally, a summary section completes the article. All proofs are contained in the Appendix.

1. The Economy

This section presents the details of the economy and definitions.

1.1 Beliefs

The economy lasts for a single time period, starting at date 0 and terminating at time 1. The uncertainty in the economy at time 1 is characterized by a probability space with \( P \) the probability beliefs.\(^2\) The probability belief \( P \) should be interpreted as the probability belief of an arbitrary trader in an economy populated by multiple traders with "heterogeneous" beliefs.\(^3\) Expectation with respect to the probability \( P \) will be denoted by \( E(\cdot) \).

1.2 The marketed assets

An asset is defined as a random variable with a finite mean and variance. The interpretation is that an asset has a random payoff at time 1, the distribution of which is determined by the probability \( P \). The set of assets that trade in a frictionless, competitive market at time 0 is denoted by the symbol \( M \), the set of marketed assets. This set of assets is assumed to be closed under portfolios consisting of a finite number of assets; that is, if \( x_1 \) and \( x_2 \) trade, then \( (\alpha_1 x_1 + \alpha_2 x_2) \) trades, where \( (\alpha_1 x_1 + \alpha_2 x_2) \) represents a portfolio consisting of \( a_1 \) shares of asset \( x_1 \) and \( \alpha_2 \) shares of asset \( x_2 \).\(^4\)

We isolate an infinite number of assets that trade, denoted by \( \{x_i\}_{i=0}^\infty \). A finite-dimensional asset space is one where only a finite number of the marketed assets \( M \) are linearly independent. At this point in the analysis, we do not assume that the \( x_i \) are linearly independent. Hence, this description still includes finite- as well as infinite-dimensional asset spaces.

A subset of the traded assets, the limited-liability assets, will play a significant role in the subsequent analysis. A limited-liability asset is defined to be a traded asset whose payoffs at time 1 are always nonnegative and

---

\(^2\) A probability space is a triplet \((\Omega, F, P)\) where \( \Omega \) is a set, \( F \) is a field on \( \Omega \), and \( P \) is a countably additive probability measure. Without loss of generality, it will be assumed that \( P \) is complete.

\(^3\) To be precise, let \( P_i \) be the probability measures for investors \( i = 1, \ldots, I \) defined over \((\Omega, F)\). First, let us assume that \( P_i \) and \( P_j \) for \( i \neq j \) are mutually absolutely continuous. Second, assume that there exist constants \( \alpha, \beta \) (depending on \( i, j \)) such that \( aP_i \leq P_j \leq \beta P_i \) for \( i \neq j \). The two conditions are imposed to ensure that all investors agree upon arbitrage opportunities (both finite and infinite), as defined below.

\(^4\) Formally, \( M \) is identified with a linear subspace of \( L^2(P) \), where \( L^2(P) = \{ x \in R^n : E(x^2) < +\infty \} \).
strictly positive with positive probability. Formally, $\mathbf{x} \in \mathcal{M}$ is a limited-liability asset if $P(\mathbf{x} \geq 0) = 1$ and $P(\mathbf{x} > 0) > 0$.

Since we have assumed that all the marketed assets in $\mathcal{M}$ have finite second moments, we can study mean-squared convergence. Let $\overline{\mathcal{M}}$ denote the set of traded assets in $\mathcal{M}$ augmented by the limits of sequences of portfolios in $\mathcal{M}$ under mean-squared convergence.\(^5\) We now add one additional restriction upon the marketed assets. We assume that the riskless asset can be obtained as the limit of a sequence of traded limited-liability portfolios. The riskless asset is defined as the constant random variable that equals 1 across all states.

This assumption is satisfied if the riskless asset trades, but it does not imply that the riskless asset trades. Furthermore, it is stronger than assuming both that the set of limited-liability assets is nonempty and that the riskless asset is obtainable as the limit of a sequence of traded assets (which may not be of limited liability).\(^6\) It is a weak restriction, however; its relaxation is explored in Section 3. This assumption should be interpreted as saying that the riskless asset “almost” trades.

The set of marketed assets together with the previous structure is called an incomplete market. This terminology makes sense once what is meant by a complete market is defined. A complete market is a set of marketed assets $\mathcal{M}$ that satisfies three additional conditions:

1. The riskless asset trades.
2. For any traded asset $\mathbf{x} \in \mathcal{M}$, call options on $\mathbf{x}$ with all exercise prices trade (i.e., $\max(\mathbf{x} - k, 0) \in \mathcal{M}$ for all real numbers $k$).
3. The traded assets contain all limits of sequences of portfolios in $\mathcal{M}$ (i.e., $\mathcal{M} = \overline{\mathcal{M}}$)

The meaning of saying that an asset is “traded” is that there exists a portfolio of assets in $\mathcal{M}$ equal to the “traded”-asset. Green and Jarrow (1987) show that this definition of market completeness is equivalent to assuming that all contingent claims on individual assets in $\mathcal{M}$ trade, where a contingent claim is any function (meeting mild regularity conditions) mapping the real line to the real line. Since an uncountably infinite number of states are possible, this definition is much weaker than the more familiar definition of Arrow-Debreu completeness.

1.3 Prices
Let us represent the prices of the traded assets at time 0 by a function $\pi$, mapping the set of traded assets $\mathcal{M}$ into the set of real numbers. For example, if $\mathbf{x}$ is a traded asset, then $a(\mathbf{x})$, a real number, is the time 0 price of asset $\mathbf{x}$. The function $\pi$ is often called a price functional.

\(^{5}\) Given a sequence $\mathbf{m}_{j, \pi} \rightarrow \mathbf{m}$, it converges in mean square to $\mathbf{m} \in \overline{\mathcal{M}}$ if $E(m_{j} - m)^2 \rightarrow 0$ as $j \rightarrow \infty$. $\overline{\mathcal{M}}$ represents the closure of $\mathcal{M}$ in the topology induced by the $L^2(\mathcal{P})$ norm.

\(^{6}\) Define $\mathcal{M}^* = \{\mathbf{x} \in \mathcal{M} : x_1$ is limited liability$.\}$. The assumption states that $\mathbf{x}_a \in (\overline{\mathcal{M}})$, where $\mathbf{x}_a(u) = 1$ for all $u \in \Omega$. Note that $\mathbf{x}_a \in \mathcal{M}$ implies this condition but that $\mathcal{M}^* \neq \emptyset$ and $\mathbf{x}_a \in \overline{\mathcal{M}}$ do not.
Let us assume that the price functional \( \pi \) satisfies the property of \textit{value additivity} (or linearity); that is, if \((a_1x_1 + a_2x_2)\) represents the portfolio consisting of \(a_1\) shares of \(x_1\), and \(a_2\), shares of \(x_2\), then
\[
\pi(a_1x_1 + a_2x_2) = a_1\pi(x_1) + a_2\pi(x_2).
\]

Given that the assets \(\{x_j\}_{j=0}^\infty\) trade, an \textit{n-asset portfolio} of \(\{x_j\}_{j=0}^\infty\) will be denoted as an infinite vector \((a_0(n), a_1(n), a_2(n), \ldots)\) where only \(n\) of the elements of this vector are nonzero and where \(a_i(n)\) represents the number of shares of asset \(i\) in the portfolio. The payoff to this portfolio at time 1 is therefore \(\sum_{j=0}^\infty a_j(n)x_j\). Although the sum involves an infinite number of terms, it is always well-defined since only a finite number of the terms are nonzero. By value additivity, the time 0 value of this portfolio is \(\sum_{j=0}^\infty a_j(n)\pi(x_j)\). The reason for introducing this notation is that a limiting portfolio of these assets can now be defined without changing the summation indices (or rearranging terms) by just letting \(n \to \infty\).

For convenience, let us also assume that \(\pi(x_j) \neq 0\) for the assets \(\{x_j\}_{j=0}^\infty\). This assumption ensures that the returns on these assets are well-defined. Without loss of generality, we can now normalize these assets so that \(\pi(x_j) = 1\) for all \(j\). This normalization will give \(\sum_{j=0}^\infty a_j(n)x_j\) the additional interpretation of being the rate of return on the portfolio \((a_0(n), a_1(n), \ldots)\) at time 1.

Finally, we say that prices are continuous on the marketed assets (i.e., \(\pi\) is continuous on \(M\)) if, for every sequence of traded portfolios \(\{m_j\}_{j=0}^\infty \in M\) that converge in quadratic mean to a traded asset \(m \in M\), the prices of the portfolios in this sequence, \(\pi(m_j)\), converge to the price of asset \(m\). In symbols, \(\lim_{j \to \infty} \pi(m_j) = \pi(m)\). This concept of price continuity is closely related to a key hypothesis underlying Ross’s APT.

### 1.4 Preferences

Investors are assumed to have preferences defined over the traded assets. Choosing an arbitrary investor, his preferences are represented by a \textit{complete} and \textit{transitive} relation over \(M\), denoted by \(\rho\). \textit{Complete} means that given any two traded assets \(x\) and \(y\), either \(x \rho y\) or \(y \rho x\). The symbol \((x \rho y)\) should be read “\(x\) is preferred to or indifferent to \(y\).” \textit{Transitive} means that for traded assets \(x, y, \text{ and } z\), if \(x \rho y\) and \(y \rho z\), then \(x \rho z\).

The preference relation \(\rho\) is said to be \textit{strictly increasing} if for any traded limited liability asset \(m\) and any traded asset \(y\),
\[
(y + m) \rho y.
\]

This statement reads that the portfolio \((y + m)\) is strictly preferred to the portfolio \(y\). This condition is justified by noting that since \(m\) is of limited liability,
\[ P(y + m \geq y) = 1 \quad \text{and} \quad P(y + m > y) > 0 \]

That is, the portfolio \( y + m \) dominates the portfolio \( y \) in terms of the asset’s payoffs at time 1.

The preference relation \( \rho \) is said to be continuous if the following condition is satisfied: \(^8\)

Given a sequence of traded assets \( \{m_j\} \in M \) converging in quadratic mean to the traded asset \( m \in M \), and given a fixed traded asset \( x \in M \), if \( m_j \rho x \) for all \( j \), then \( m \rho x \) (and conversely for \( x \rho m \) and \( x \rho m \)).

In other words, preferences are “smooth” in the sense that if a portfolio \( m_j \) is almost the same as \( m \), and \( m_j \) is preferred or indifferent to another portfolio \( x \), then \( m \) is also preferred or indifferent to portfolio \( x \).

As of yet, it is not assumed that the trader’s preferences are either strictly increasing or continuous. The above discussion merely defined these terms. It is assumed, however, that the investor is endowed with a portfolio \( M \) at time 0. A portfolio \( y \) is said to be optimal for \( \rho \) (or maximal) if

\[ y \rho x \text{ for all traded } x \in M \]

such that \( \pi(x) \leq \pi(e) \). That is, \( y \) maximizes the investor’s preferences over the set of traded assets that satisfy his budget constraint. \(^9\)

2. Arbitrage Opportunities and Prices

This section relates the concept of arbitrage opportunities in the limit to the continuity of prices. This is the content of Proposition 1. Prior to the statement of the theory, however, some preliminary remarks are needed.

We need to distinguish between two types of arbitrage opportunities: a finite-asset arbitrage opportunity and an infinite-asset arbitrage opportunity. A finite-asset arbitrage opportunity is defined to be a traded asset \( m \in M \) such that \( m \) is of limited liability and the price of asset \( m \) is less than or equal to zero; that is, \( \pi(m) \leq 0 \). Such an asset would represent a probabilistic money pump. It has a positive or zero cash flow at time 0, non-negative cash flows at time 1, and strictly positive cash flows at time 1 with positive probability. The absence of finite-asset arbitrage opportunities implies that \( \pi \) is strictly positive on the set of limited-liability assets. It also

---

\(^8\) Equivalently, for all \( x \in M \), \( \{m \in M: m \rho x\} \) and \( \{m \in M: x \rho m\} \) are closed in the relative topology on \( M \) induced by \( L^p(P) \).

\(^9\) Sufficient conditions on preferences and market structures to obtain an optimal portfolio are:

1. Preferences \( \rho \) can be represented by a utility function.

2. The utility function is concave, upper semicontinuous, and \( \{x \in M: \pi(x) \leq \pi(e)\} \) is restricted to being a bounded subset of \( L^p(P) \); see Aubin and Ekeland (1984, Proposition 6, p. 12).

For this set of sufficient conditions, the key insight is that the existence of an integral representation for the utility function is not required.
implies that if the probability is 1 that an asset has a zero payoff at time 1, then the price of the asset is zero.\(^{10}\)

To derive the APT, we need to exclude infinite-asset arbitrage opportunities as well. Following Chamberlain and Rothschild (1983, p. 1287), let us define an infinite-asset arbitrage opportunity as any sequence of portfolios \((\alpha_0(n), \alpha_1(n), \ldots)\) for \(n = 1, \ldots, \infty\) that violates the following condition.

**Condition 1. If**

\[
\lim_{n \to \infty} \text{var}\left( \sum_{j=0}^{\infty} \alpha_j(n)x_j \right) = 0
\]

\[\text{and}\]

\[
\lim_{n \to \infty} \pi\left( \sum_{j=0}^{\infty} \alpha_j(n)x_j \right) = 1,
\]

then

\[
\lim_{n \to \infty} E\left( \sum_{j=0}^{\infty} \alpha_j(n)x_j \right) = \delta > 0,
\]

where \(\text{var}(x) = E(x - E(x))^2\).

To interpret Condition 1, consider the limit portfolio, \(\lim_{n \to \infty} \sum_{j=0}^{\infty} \alpha_j(n)x_j\), which is in \(\bar{M}\). The first and second hypotheses state that this portfolio has zero variance and a unit “price.” Given these hypotheses, the implication is that the portfolio’s expected value is positive.

To understand this condition, note that since the variance of this portfolio is zero, the portfolio’s value at time 1 is \(\delta\) for sure. If the limit portfolio trades and \(\delta \leq 0\), then it represents an arbitrage opportunity of the type previously defined. If the limit portfolio does not trade, it is only an arbitrage opportunity in the limit.

Chamberlain and Rothschild (1983, p. 1287) also define another type of limiting arbitrage opportunity with zero investment at time 0. It can be such that

\[
\pi(0) = 0.
\]

Note that if \(\pi(m) = 0\) and \(m \in M^*\). Contradiction. Conversely, if \(\pi(x) > 0\), repeat the argument for \(m = +\pi(x)x_0/\pi(x) - z\). In such a case, there exists an arbitrage opportunity. The proof is in the Appendix.

\(^{10}\) The linearity of \(\pi\) implies that \(\pi(y) = \pi(w) = 0\) if \(\pi(w) = 0\) for all \(w \in \Omega\); however, it does not imply that \(\pi(x) = 0\) for \(x \in M\) such that \(P(x = 0) = 1\). This last condition follows from the strict positivity of \(\pi\), given \(M^* \neq 0\). Here is the proof. Let \(P(x = 0) = 1\) for \(x \in M\). Let \(x_0 \in M^*.\) If \(\pi(x) > 0\), consider \(m = -\pi(x)x_0/\pi(x) + z\). Note that \(\pi(m) = 0\) and \(m \in M^*\). Contradiction. Conversely, if \(\pi(x) > 0\), repeat the argument for \(m = +\pi(x)x_0/\pi(x) - z\). \(\blacksquare\)

\(^{11}\) According to Chamberlain and Rothschild (1983, p. 1287), this other type of arbitrage opportunity is defined as a sequence of portfolios \((\alpha_0(n), \alpha_1(n), \ldots)\) for \(n = 1, \ldots, \infty\) such that it violates

\[
\text{var}\left( \sum_{j=0}^{\infty} \alpha_j(n)x_j \right) \rightarrow 0 \quad \pi\left( \sum_{j=0}^{\infty} \alpha_j(n)x_j \right) \rightarrow 0
\]

\[
E\left( \sum_{j=0}^{\infty} \alpha_j(n)x_j \right) \rightarrow 0
\]

Condition 1 implies the exclusion of this type of arbitrage opportunity. The proof is in the Appendix.
shown that the satisfaction of Condition 1 excludes this other type of limiting arbitrage opportunity.\footnote{This follow from the fact that linear functionals on finite-dimensional spaces are always continuous in the norm topology; see Cotlar and Cignoli (1974, p. 80).}

Chamberlain and Rothschild (1983) show that given an approximate \textit{k-factor structure} for returns, Condition 1 yields the APT. Consequently, conditions on preferences and market structures that imply Condition 1 are sufficient to derive the APT. The following proposition provides a characterization of Condition 1 in terms of the continuity of prices. This proposition, for the Chamberlain and Rothschild economy, equates the definition of an infinite-asset arbitrage opportunity as used in the APT to the corresponding concept employed by Kreps \cite[Condition (4.2), p. 22]{Kreps1981}.

\textbf{Proposition 1.} Given that there are no finite-asset arbitrage opportunities, prices are continuous if and only if there are no arbitrage opportunities in the limit that is, Condition 1 holds.

The proof of this proposition is contained in the Appendix. The intuition underlying the proof can be obtained by considering the case in which the limit portfolio \(\lim_{n \to \infty} \sum_{j=0}^{\infty} \alpha_j(n) x_j\) satisfying Condition 1 trades. It was argued earlier that in this case \(\lim_{n \to \infty} \sum_{j=0}^{\infty} \alpha_j(n) x_j\) is equal to the constant 6. Next suppose that the riskless asset, denoted \(x_0\), also trades. Now if prices are continuous, then \(\pi(\lim_{n \to \infty} \sum_{j=0}^{\infty} \alpha_j(n) x_j) = \delta \pi(x_0)\), which by hypothesis is 1; therefore, \(\delta > 0\). Conversely, suppose that Condition 1 holds but that prices are not continuous. It is then possible by rescaling to find a sequence of portfolios converging to the asset whose payoff is identically zero but whose price converges to 1. This implies (by Condition 1), however, that this limit portfolio is the constant \(\delta > 0\), generating a contradiction.

The sufficiency part of Proposition 1 is new. The necessity part is due to Chamberlain and Rothschild (1983). The next section states sufficient conditions on preferences and market structures for prices to be continuous, and hence by Proposition 1 to obtain the APT. Note that Proposition 1 is really only significant when the marketed assets contain an infinite number of linearly independent assets, \(\{x_j\}_{j=0}^{\infty}\). For the finite-dimensional case, prices are always continuous,\footnote{This follow from the fact that linear functionals on finite-dimensional spaces are always continuous in the norm topology; see Cotlar and Cignoli (1974, p. 80).} and Condition 1 adds no additional structure to the economy.

3. Preferences and Arbitrage Opportunities

This section states two propositions giving sufficient conditions for prices to be continuous. The first proposition relates to complete markets, and
the second to incomplete markets. This observation is formalized as a well-known lemma.

**Lemma (no finite-asset arbitrage opportunities).** If there exists an investor with strictly increasing preferences who holds an optimal portfolio, then there are no finite-asset arbitrage opportunities.

This lemma follows directly from the fact that if a finite-asset arbitrage opportunity existed, it would be feasible and make the investor better off. This would contradict the existence of an optimal portfolio for this investor.

**Proposition 2 (complete markets).** If markets are complete, and if there exists an investor with strictly increasing preferences who holds an optimal portfolio, then prices are continuous.

The proof of this proposition is contained in the Appendix. The idea underlying the proof is simple. If markets are complete, then any infinite-asset arbitrage opportunity would trade (and therefore would be a finite-asset arbitrage opportunity). Given that there are no finite-asset arbitrage opportunities, the proposition follows by the lemma.

**Proposition 3 (incomplete markets).** If markets are incomplete, and if there exists an investor with strictly increasing and continuous preferences who holds an optimal portfolio, then prices are continuous.

This proposition is due to Kreps (1981, Theorem 2). The intuition underlying this proposition is that under continuous preferences, for a sequence of traded assets \( \{m_j\}_{j=0}^{\infty} \) approaching the traded asset \( m \), for large \( j \), \( m \), is approximately equal to \( m \). Hence, the price of asset \( m_j \) should be close to the price of asset \( m \); otherwise, an approximate finite-asset arbitrage opportunity would exist. This would contradict the existence of an optimal portfolio for this investor.

Contrasting Proposition 2 with Proposition 3, we see that there is a trade-off between structures on markets and structures on preferences that imply continuous prices. In a complete market, no additional structures on preferences are needed beyond those sufficient to exclude finite-asset arbitrage opportunities. In an incomplete market, however, continuity of preferences is also needed. This trade-off is even more apparent if we remove the maintained assumption that the riskless asset is the limit of a sequence of limited-liability assets. In this circumstance, we need even more structure on preferences to derive the result. First, preferences must be defined over \( \tilde{M} \) and not \( M \). Second, preferences must also be convex, which implies risk aversion.

---

The preference \( \succ \) is said to be convex if given any \( x, y, z \in M \) where \( x \succ y \succ z \), and \( \lambda \in [0, 1] \), then \( \lambda x + (1 - \lambda) y \succ z \).
Proposition 3 includes as a special case the conditions on preferences contained in Ross (1976). Ross assumes that preferences are representable as an expected utility function whose underlying utility function is strictly increasing (strictly increasing preferences), concave, and bounded below. The joint conditions of concavity and boundedness below imply that preferences are continuous; in fact, this is the content of Ross (1976, Appendix 2, p. 358). The conditions in Proposition 3 are weaker since the existence of the expected utility representation is not required. Hence, preferences need not satisfy numerous restrictive axioms, one of which is the strong independence axiom. We see that at most only ordinal utility theory is needed to yield the APT. Further implications of this single observation are pursued in the next section.

4. Conclusion

The derivation of the APT requires three basic assumptions: an infinite number of linearly independent assets, an approximate $K$ linear factor structure, and no arbitrage opportunities in the limit. In having explored the structure on preferences and markets required to exclude limiting arbitrage opportunities, this article has shown that only very weak conditions are required. In complete markets, it is required only that preferences be strictly increasing and that there exists an optimal portfolio. These conditions are identical to those sufficient to exclude finite-asset arbitrage opportunities. Second, in incomplete markets, an additional restriction on preferences needs to be imposed. Preferences need to be continuous.

The APT is thus an ordinal utility theory, independent of the strong independence axiom and the expected utility hypothesis. Of the three hypotheses underlying the APT, the structure on preferences required to exclude limiting arbitrage opportunities is probably the least restrictive. Given the number of financial assets that trade on organized exchanges, the infinite-asset assumption is probably a reasonable approximation; hence, the factor hypothesis is the most crucial. The truth or falsity of the APT probably depends on the truth or falsity of this hypothesis.

In contrast, the CAPM depends on cardinal utility theory and presumes the existence of a utility function, the strong independence axiom, and the expected utility hypothesis. Experimental evidence, however, is consistent with the systematic violation of the strong independence axiom [see Machina (1982) for a review]. There is a related hypothesis that the market anomalies associated with the CAPM may be due to this type of systematic irrationality on the part of investors; see Shefrin and Statman (1984) and DeBondt and Thaler (1985). In principle, since the APT is an ordinal utility theory, it offers the possibility of testing this hypothesis.

Appendix

This appendix formally proves the propositions in the text. A list of notations is provided for easy reference:
\( M \subset L^2(\Omega, F, P) \) are the traded assets.
\( M^+ = \{ x \in M : P(x \geq 0) = 1, P(x > 0) > 0 \} \).
\( x_0(\omega) = 1 \) for all \( \omega \in \Omega \).
\( \tilde{M} \) is the closure of \( M \) in \( L^2(\Omega, F, P) \).
\( \pi : M \rightarrow R \) is the linear price functional,
\( \rho \) is a complete, transitive binary relation on \( M \).

The maintained assumption is:

Assumption A1 (marketed assets).

a. \( M \) is a linear subspace of \( L^2(\Omega, F, P) \).
b. \( M^+ \neq \emptyset \).
c. \( \{ x_i \}_{i=0}^{\infty} \subset M \).
d. \( x_0 \in (M^+) \).


To prove the propositions, we need the following lemmas.

Lemma 1. \( \pi \) continuous on \( M \) implies that there exists a unique \( \tilde{\pi} : \tilde{M} \rightarrow R \) such that \( \tilde{\pi} \) is continuous, \( \tilde{\pi} \) is linear, and \( \tilde{\pi}(m) = \pi(m) \) for \( m \in M \).

This lemma is a standard extension theorem for Hilbert spaces. By Lemma 1, we have the existence of a unique continuous linear extension \( \tilde{\pi} : \tilde{M} \rightarrow R \). This extension is used below.

Definition. \( \pi \) strictly positive means that if \( m \in M^+ \), then \( \pi(m) > 0 \).

Lemma 2. Given that \( \pi \) is strictly positive, then \( \tilde{\pi}(x_0) \geq 0 \).

Proof: By Assumption A1 d, there exists \( \{ m_j \}_{j=1}^{\infty} \in M^+ \) such that \( m_j \rightarrow x_0 \). But \( \pi(m_j) > 0 \) for all \( j \); hence, by the continuity of \( \tilde{\pi} \), \( \lim \tilde{\pi}(m_j) = \tilde{\pi}(x_0) \geq 0 \).

Definition. An n-asset portfolio is a vector in \( R^\infty \) where \( (\alpha_0(n), \alpha_1(n), \ldots) \) has only \( n \) of the \( \alpha_j(n) \neq 0 \).

We write \( \Sigma_{j=0}^{\infty} \alpha_j(n)x_j = \Sigma \alpha_j(n)x_j \).

Proposition 1. If \( \pi \) is strictly positive, then \( \pi \) is continuous on \( M \) if and only if Condition I of the text holds.

Proof: Suppose that \( \pi \) is continuous. Let \( \text{var}(\Sigma \alpha_j(n)x_j) \rightarrow 0, \Sigma \alpha_j(n)x_j \rightarrow 1 \), and \( E(\Sigma \alpha_j(n)x_j) \rightarrow \delta \) for some sequence of portfolios \( (\alpha_0(n), \alpha_1(n), \ldots) \) for \( n = 1, 2, \ldots \). I claim that \( \Sigma \alpha_j(n)x_j \rightarrow \delta x_0 \) in \( L^2(\Omega, F, P) \). This, note that
Next, \( \pi(\Sigma \alpha_j(n) x_j) \rightarrow 1 \) by hypothesis. Using \( \tilde{\pi} \) from Lemma 1 gives \( \tilde{\pi}(\delta x_0) = 1 \). Since \( \tilde{\pi}(x_0) \geq 0 \) by Lemma 2, we get \( \delta > 0 \) and \( \tilde{\pi}(x_0) > 0 \). Conversely, suppose that Condition 1 holds. The continuity of \( \pi \) was proved by Chamberlain and Rothschild (1983, p. 1288).

**Corollary.** Under the hypotheses of Proposition 1, Condition 1 implies that if there exists a sequence of portfolios \( (\alpha_0(n), \alpha_1(n), \ldots) \) \( n = 1, \ldots, \) such that \( \text{var}(\Sigma \alpha_j(n) x_j) \rightarrow 0, \pi(\Sigma \alpha_j(n) x_j) \rightarrow 0 \), then \( E(\Sigma \alpha_j(n) x_j) \rightarrow 0 \).

**Proof:** I show that the continuity of \( \pi \) on \( M \) implies this condition. The proof is completed by using Proposition 1. Suppose that \( \pi \) is continuous on \( M \). Consider a sequence of portfolios \( (\alpha_0(n), \alpha_1(n), \ldots) \) \( n = 1, \ldots, \), such that \( \text{var}(\Sigma \alpha_j(n) x_j) \rightarrow 0 \) and \( \pi(\Sigma \alpha_j(n) x_j) \rightarrow 0 \).

- **Case 1.** Suppose that \( \lim \inf E(\Sigma \alpha_j(n) x_j) = \beta < 0 \). Take the subsequence \( n_k \) such that \( E(\Sigma \alpha_j(n_k) x_j) \rightarrow \beta \). By the same argument as in the proof of Proposition 1, we get \( \Sigma \alpha_j(n_k) x_j \rightarrow \beta x_0 \) in \( L^2(\Omega, F, P) \). For \( \tilde{\pi} \) in Lemma 1, \( \tilde{\pi}(\Sigma \alpha_j(n_k) x_j) \rightarrow \tilde{\pi}(\beta x_0) \). But \( \tilde{\pi}(x_0) > 0 \) by Proposition 1; therefore, \( \tilde{\pi}(\beta x_0) = \beta \tilde{\pi}(x_0) < 0 \). This is a contradiction; hence, \( \lim \inf E(\Sigma \alpha_j(n) x_j) \geq 0 \).

- **Case 2.** A similar argument shows that \( \lim \sup E(\Sigma \alpha_j(n) x_j) \leq 0 \).

Combined, these cases prove the desired result.

**Remark.** Proposition 1 and its corollary used Assumption Al d to get \( \tilde{\pi}(x_0) \geq 0 \) in Lemma 2.

**Definition.** \( \varrho \) is said to be strictly increasing if \( m \in M^+ \) implies \( (y + m) \varrho \) \( y \) for all \( y \in M \).

**Definition.** \( y \) is said to be an optimal portfolio for \( \varrho \) if \( y \) is \( \varrho \) maximal in \( \{x \in M: \pi(x) \leq \pi(e)\} \).

**Lemma 3.** If \( \varrho \) is strictly increasing and there exists an optimal portfolio \( y \in M \) for \( \varrho \), then \( \pi \) is strictly positive.

**Proof.** Suppose not; that is, suppose that there exists \( m \in M^+ \) such that \( \pi(m) \leq 0 \). Consider \( y + m \in M \). We have \( (y + m) \varrho \) \( y \) by \( \varrho \) strictly increasing, and \( (y + m) \leq \pi(y) \leq \pi(e) \). This contradicts the maximality of \( y \).

**Proposition 2.** If \( (a) M = \tilde{M}, M \) is a sublattice of \( L^2(\Omega, F, P) \) (i.e., \( \max(x-k, 0) \in M \) for all \( x \in M, k \in R \), and \( x_0 \in M \)) and if \( (b) \varrho \) is strictly increasing and there exists an optimal portfolio \( y \in M \) for \( \varrho \), then \( \pi \) is continuous on \( M \).

**Proof.** Condition (b) implies that \( \pi \) is strictly positive. This proposition
then follows directly from Schaefer (1980, p. 228) because of condition (a). It says that every positive, linear functional on a complete, Banach lattice is continuous; ■

Definition. \( \rho \) is continuous if \( \{ x \in M : x \rho y \} \) and \( \{ x \in M : y \rho x \} \) are closed in the relative topology on \( M \) as a subspace of \( L^2(\Omega, F, P) \) for all \( y \in M \).

Proposition 3. If \( \rho \) is strictly increasing and continuous and there exists an optimal portfolio \( y \in M \) for \( \rho \), then \( \pi \) is continuous on \( M \).

Proof. This follows from Kreps (1981, Theorem 2), specialized to the economy of this article. ■

Definition. \( \rho \) is convex on \( \tilde{M} \) if \( x \rho z, y \rho z, \lambda \in [0, 1] \) imply \( \lambda x + (1 - \lambda)y \rho z \) for all \( x, y, z \in \tilde{M} \).

Proposition 4. If under the hypotheses of Proposition 3 we remove \( x_0 \in (\tilde{M}^+) \), but add

a. \( \rho \) strictly positive on \( \tilde{M} \), \( \rho \) continuous on \( \tilde{M} \)

b. \( \rho \) convex on \( \tilde{M} \)

the natural \( \tilde{\pi} \) is continuous and strictly positive on \( \tilde{M} \).

Proof. This follows from Kreps (1981, Theorem 1, p. 21), using the uniqueness of \( \tilde{\pi} \) on \( \tilde{M} \). ■

Remark. Proposition 4 implies that if \( x_0 \in \tilde{M} \), then \( \tilde{\pi}(x_0) > 0 \). This is what \( x_0 \in (\tilde{M}^+) \) is used for in the proof of Proposition 1. Hence, the hypotheses of Proposition 4 implies Condition 1 without \( x_0 \in (\tilde{M}^+) \).

References


