FORWARD OPTIONS AND FUTURES OPTIONS

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I. INTRODUCTION

American forward and futures options are different types of contracts. If exercised early, a forward option's value equals the discounted difference between the striking price and the current forward price. A futures option's value, if exercised early, equals the simple difference between the striking price and the current futures price. This slight discrepancy is enough to prevent early forward option exercise and to allow early futures option exercise. Thus, American forward options can be treated like European options whereas American futures options cannot.

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Our purpose is to study the distinctions between American forward and futures options given stochastic interest rates. Futures options are considered in several recent papers: Ball and Torous [1], Ramaswamy and Sundaresan [14], and Brenner, Courtdon, and Subrahmanyan [4]. Our paper extends many of the results contained therein. For example, forward and futures prices and contracts are frequently treated as practically identical. From an empirical standpoint, this appears reasonable since forward and futures prices appear similar (see French [10] and Elton, Gruber, and Rentzler [9]). Yet, under stochastic interest rates, this practical equivalence cannot be extended to forward and futures options. The early exercise potential adds value to futures options. Moreover, this exercise potential depends critically on the statistical properties of forward and futures prices. For example, we show that a futures option's value depends upon the covariance between the spot price and a transform of the instantaneous reinvestment rate.

The next section describes forward options, shows that an American forward option can be treated like a European option, and describes a pricing technique. Section III presents some interesting special features of Treasury bill options. In Section IV, we define and analyze futures options. Section V presents an example, and a summary is given in the final section.

II. FORWARD OPTIONS

An agreement to trade immediately is settled at the cash or spot price. An agreement to trade in the future for a fixed price set now is a forward contract. Several types of financial forward contracts trade frequently. Banks commonly quote forward currency prices for commercial customers. Forward contracts on Treasury securities, called WI or "when issued" contracts, trade in the week prior to a Treasury auction. In addition, synthetic forward Treasury positions can be created with cash bills. Finally, packagers of mortgage-backed securities periodically short mortgage pools forward. These TBA or "to be allocated" GNMA, FNMA, and FHLMC mortgage-backed securities can be sold several months forward. Options on WIs and TBAs trade over the counter daily among wholesale dealers in the underlying cash securities.

We have many prices and contracts to differentiate so we begin with some definitions and symbols. Define the cash or spot price of a security at time \( s \) as \( p(s, s) \) where \( s \in [t, t^*] \). We assume that \( p(s, s) > 0 \) for all \( s \). The forward price at time \( s \) for delivery and payment of a security at time \( t^* \) is \( p(s, t^*) \). In this framework, a cash contract is a zero-term forward contract.

The forward price adjusts continually to make a new forward contract's value equal to zero. We denote the forward contract's value at the initiation
date $t$ as $f[p(t, t^*); k(t), t^*] = 0$ in which $k(t) = p(t, t^*)$ is the delivery price of the contract. For a long contract, the maturity value at time $t^*$ is $f[p(t^*, t^*); k(t), t^*] = p(t^*, t^*) - k(t)$. It is easy to show (see Jarrow and Oldfield [12] and Cox, Ingersoll, and Ross [6]) that the current forward and spot prices are linked through a discount factor:

$$p(s, s) = p(s, t^*)b(s, t^*) \quad \text{for all } s \in [t, t^*].$$  \hspace{1cm} (1)

In Equation (1), $b(s, t^*)$ denotes the time $s$ price of a default free bill that matures and pays one dollar at time $t^*$. For convenience, to obtain expression (1) we assume that there are no storage costs for the spot asset. This assumption can be easily relaxed in the subsequent analysis. The relationship among the spot price, the forward price, and the discount factor is crucial for the subsequent pricing of forward options.

A forward call option is exercised at a stated exercise price in favor of the underlying forward contract. If a call is exercised, its owner receives a long forward contract written at the stated exercise price. This means that a forward call option’s exercise value at time $s$ is

$$c[p(s, t^*); K, \tau] = f[p(s, t^*); K, t^*] = [p(s, t^*) - K]b(s, t^*).$$  \hspace{1cm} (2)

We label the call price $c[p(s, t^*); K, \tau]$ in which $p(s, t^*)$ is the current forward price, time $s$ is between times $t$ and $\tau$, $K$ is the stated exercise forward price, and $\tau$ is the expiration date ($t \leq \tau < t^*$). In words, Equation (2) means that an exercised call is worth the delivered forward contract. Note that $K$ is not the price that a call’s owner must pay to exercise. There is no cost to exercise the option.

Consider two alternative investment strategies at time $s \in [t, t^*]$. One can buy a European forward option that matures at $\tau$ and $K$ bills that mature at $t^*$ or one can buy the spot security. Suppose the spot security makes no interest or dividend payments prior to the forward option’s expiration at time $\tau$. At time $\tau$, the value of the spot asset investment is $p(\tau, t^*)$. For the first investment strategy, if $p(\tau, t^*) < K$ (with positive probability), the call option’s value is $c = 0$ and the investment’s worth is $Kb(\tau, t^*)$. Since $p(\tau, t^*)b(\tau, t^*) = p(\tau, \tau)$, we get $Kb(\tau, t^*) > p(\tau, \tau)$. Conversely, if $p(\tau, t^*) > K$, the value of the first strategy involving the option and $K$ bills is $[p(\tau, t^*) - K]b(\tau, t^*) + Kb(\tau, t^*) = p(\tau, \tau)$ and the second strategy’s value is $p(\tau, \tau)$ too. This means that given $p(\tau, t^*) < K$ with positive probability then the initial value of the forward options and $K$ bills strategy must exceed the value of the spot commodity. Thus,

$$c[p(s, t^*); K, \tau] > p(s, s) - Kb(s, t^*).$$  \hspace{1cm} (3)

The value of an American call option is at least as great as a European call option’s price (see Merton [13]). Thus, Equation (3) holds for an American call option too.
It is easy to show that an American option and a European option have the same price. Indeed, we simply establish that there is no incentive for early option exercise. If a forward call option is exercised before \( \tau \), say at time \( s \), its rational value is \( \max \{ p(s, t^*) - K, 0 \} \), or equivalently, \( p(s, s) - K b(s, t^*) \). But from Equation (3), \( c > p(s, s) - K b(s, t^*) \). The option's value alive is greater than its exercised value. Thus, an American forward call option can be treated like a European call option.

To price a forward contract option, we rely upon the equivalent martingale framework described by Harrison and Pliska [11]. This is a generalization of the arbitrary preference, direct option pricing technique introduced by Cox and Ross [5]. In brief, given certain completeness, regularity, and no arbitrage conditions, an arbitrary stochastic price process can be transformed into a risk-neutral process. For example, the "pseudo-probabilities" in binomial option pricing are a discrete time representative of the general transformation (see Bartter and Rendleman [2] and Cox, Ross, and Rubinstein [7]). We assume that the spot, forward, and discount rate processes are imbedded in an economy that has an equivalent martingale transformation. An explicit example of such an economy is given in Section V.

Let us define the interest rate process by

\[
\begin{align*}
b(s, t^*) &= \mathbb{E}_s[\exp \left\{ - \int_s^t r(y) \, dy \right\}]. \tag{4}
\end{align*}
\]

The subscript \( s \) in the \( \mathbb{E}_s(\cdot) \) operator reflects the fact that the expectation is conditional upon the information available at time \( s \). Equation (4) means that in an equivalent risk-neutral world, arbitrage assures that all yields adjust on the basis of expected values alone.

It is convenient to specify an accumulation equation for instantaneous reinvestment. Define \( B(\tau, t) \) by

\[
B(\tau, t) = \exp \left\{ \int_t^\tau r(s) \, ds \right\} \quad \text{where } B(t, t) = 1. \tag{5}
\]

This gives the value at time \( \tau \) of one dollar continuously reinvested at \( r(s) \) from time \( t \) to time \( \tau \). The discount factor for continuous reinvestment is the expected value of \( 1/B(\tau, t) \) as in expression (4).

A forward call option can be priced by using the equivalent martingale measure. Since the American call option's value equals the European call option's value, its price is its discounted expected value at expiration.

\[
\begin{align*}
c \{ p(s, t^*); K, \tau \} &= \mathbb{E}_s[\max \{ p(\tau, t^*) - K b(\tau, t^*), 0 \}] / B(\tau, s) \\
&= \mathbb{E}_s[\max \{ p(\tau, t^*) - K, 0 \}] / B(t^*, s) \tag{6}
\end{align*}
\]
This expression illustrates the present value operator in this economy. It corresponds to the generalization of expression (4) where the known cash flow of 1 dollar at time $t^*$ was valued.

III. TREASURY BILL OPTIONS

Options on 13, 26, and 52 week U.S. Treasury bills trade over the counter among spot bill dealers. These options are tailored to a customer's specifications. Some standardization is effected, to provide liquidity in secondary trading. A common bill call option has some interesting features. It is written against $25,000,000 face value of 26 week bills and has a 6-month term until expiration. A bill call option can be exercised any time during its life for new 26 week bills delivered on Thursday of the next auction's delivery week (new 26 week bills are auctioned every Monday with payment and delivery on the following Thursday). Once exercised, the option's owner must pay the fixed delivery price when the bills are delivered. The delivery specifications introduce two unusual features. First, exercise is in terms of forward delivery. Second, the underlying forward contract changes every Friday. If a call option is exercised on or before Friday of the auction week, delivery and payment (at the exercise price) are effected on Thursday of the next week. In other words, delivery and payment always occur on Thursday after the next auction. The option expires on Friday of the last week. This scheme means that a call writer always has an auction for new bills in which to bid if delivery must be made.

The contract terms for a bill call option mean that the correct hedging instrument is a forward 26 bill. This forward bill can be a WI contract that settles on the next Thursday or a synthetic 26 week forward bill assembled from existing cash bills. A call is in the money when the 26 week forward price for the next delivery day exceeds the striking price. Over a weekend, the forward position switches to one that matures a week later. In effect the common type of Treasury bill call option comprises a compound option on a sequence of short duration forward or WI bills.

A recursive technique can be used to price a Treasury bill call option. Consider a call's owner on Friday afternoon 1 week before expiration. If the call is exercised immediately, the underlying forward price is $p(t, t^*)$. If the option is not exercised, the relevant forward price is $p(t, t^* + 1)$, in which time is measured in weeks. Thus, the owner must compute both the exercise value, $[p(t, t^*) - K]b(t, t^*)$ and the unexercised value, $c[p(t, t^* + 1); K, t^*]$. During the week prior to Friday, the option is a standard forward call and the contract should stay alive. The change in underlying forward price on Friday is like a dividend payment. In this case, the exercise alternative must be considered.
Pricing the option on Friday 1 week before it expires is straightforward. Solve Equation (6) given \( p(t, t^*) + 1 \) and compare the solution to \([p(t, t^*) - K]b(t, t^*)\). The larger value tells whether to hold or exercise. On Thursday or before, pricing involves a more complicated analysis. Again, one relies on the equivalent martingale framework to get an answer. In effect, the value on Thursday is the call's expected equivalent value on Friday, discounted 1 day. This reasoning applies for the whole week before Friday because before Friday the option is a normal forward call that will not be prematurely exercised. Thus, for any day \( t \) during the week, when Friday is denoted day \( s \) after day \( t \), the call's value is:

\[
c[p(t, t^*); k, \tau] = \bar{E}_t [\max \{c[p(s, t^*) + 1]; K, \tau, [p(s, t^*) - K]b(t, t^*)\}] / B(s, t).
\] (7)

This recursive relationship gives forward call prices at any time before expiration at \( \tau \). The forward price resetting means that early exercise potential is a factor in pricing. The call is free to exercise at any time and \( K \) is paid on delivery of the spot bills.

IV. FUTURES OPTIONS

Futures contracts are a lot like forward contracts. The main difference is that forward contracts have zero cash flows but change value, whereas futures contracts have continuous cash flows (equal to the change in the current futures price) but the contract's value is constant at zero. Prices adjust continually so that this is the case. We call the futures price \( P(t, t^*) \) for a contract opened at time \( t \) for delivery at time \( t^* \). The futures price for immediate delivery is the spot price so \( P(s, s) = p(s, s) \).

As with forward contract options, the futures price–spot price relationship is essential for pricing futures options. It can be shown that the current futures price equals the equivalent expected spot price.

\[
P(t, t^*) = \bar{E}_t [p(t^*, t^*)]
\] (8)

The proof of this futures-spot relationship comes from Cox, Ingersoll, and Ross [6]. They show that the futures price is the present value of a security that pays \( p(t^*, t^*)B(t^*, t) \) at time \( t^* \). This security trades since we assume that markets are complete. In effect, a title to one unit of the spot commodity collateralizes a loan continuously reinvested at \( r(s) \). The present value of this loan is \( \bar{E}_t [p(t^*, t^*) B(t^*, t) / B(t^*, t)] \), which gives the result in Equation (8). In words, the futures price represents the price of a synthetic traded asset. This observation will be significant in the subsequent analysis.

We can use the futures-spot relationship to show how forward and futures prices relate to one another. From Equation (1) and the valuation
operator we write \( p(t, t^*) b(t, t^*) = p(t, t) \) and \( p(t, t) = \tilde{E}_t [p(t^*, t^*)/B(t^*, t)] \). We also know that the accumulation factor, \( B(t^*, t) \), is greater than unity since \( r(s) > 0 \) with positive probability. Thus, the present value of the future spot price is less than the current futures price, i.e., \( E_t [p(t^*, t^*)/B(t^*, t)] < \tilde{E}_t [p(t^*, t^*)] \). Given \( \tilde{E}_t [p(t^*, t^*)] = P(t, t^*) \) from Equation (8), we can rearrange the arguments above to write \( p(t, t^*) b(t, t^*) < P(t, t^*) \), i.e., the discounted value of the current forward price is strictly less than the current futures price for the same commodity and delivery day. This is a testable implication of the above model.

A parallel argument relates the current futures price to the futures price at a later date. The futures price at a later date, from the discussion after expression (8), represents the value of a traded asset. Hence, it has a present value. Utilizing the valuation operator, the present value is \( \tilde{E}_t [P(\tau, t^*)/B(\tau, t)] \), where \( \tau \) is before \( t^* \). We claim that

\[
P(t, t^*) > \tilde{E}_t [P(\tau, t^*)/B(\tau, t)].
\] (9)

The current futures price exceeds the present value of the subsequent futures price. The proof is straightforward. First, \( \tilde{E}_t [P(\tau, t^*)/B(\tau, t)] = \tilde{E}_t [\tilde{E}_t [P(t^*, t^*)]/B(\tau, t)] \). Since \( B(\tau, t) > 1 \) with positive probability, this last expression is less than \( \tilde{E}_t [P(t^*, t^*)] \). Finally, from expression (8), \( \tilde{E}_t [p(t^*, t^*)] = P(t, t^*) \) and this gives Equation (9). This result shows that the futures price today, \( P(t, t^*) \), exceeds the present value of the futures price received at a later date, \( \tilde{E}_t [P(\tau, t^*)/B(\tau, t)] \). Hence, the futures price represents the price of a deteriorating asset, whose value declines over time by discounting at the accumulated spot rate.

With the spot, forward, and futures price relationship formalized, we can move to futures option pricing. We label the price of an American futures call option at time \( s \) by \( C(P(s, t^*); K, \tau) \) in which \( C(\cdot) \) denotes the call price, \( P(s, t^*) \) the current futures price, \( K \) the exercise price, and \( \tau \) the expiration date where \( t \leq s \leq \tau < t^* \). The first concept to explore is the potential for early futures call exercise. We know that forward calls are not exercised early. This is not true for a futures call. From expression (9) we know that a futures price effectively has a "continuous dividend outflow" due to its deteriorating present value. Thus, early exercise of futures options is quite possibly optimal in some circumstances. We want to characterize these circumstances.

To find the conditions in which early exercise is optimal, we compare the exercised value of an American call option at time \( s \) to the unexercised value of a European call option at time \( s \). If \( P(s, t^*) - K \) exceeds the European call's price for some \( P(s, t^*) \), then early exercise is optimal. Formally, early exercise is optimal with positive probability if and only if for some time \( s \) and some set of possible prices,

\[
P(s, t^*) - K \geq \pi + \tilde{E}_s [P(\tau, t^*)/B(\tau, s)] - Kb(s, \tau)
\] (10)
where $\pi$ is the price of a European futures put option. The left side of Equation (10) is the futures call's exercised value at time $s$. The right side, by the standard put–call parity relationship, is the European call's value at time $s$ since $\tilde{E}_s[P(\tau, t^*)/B(\tau, s)]$ represents the present value of the futures price, the underlying asset, at time $s$.

We restate Equation (10) by adding $K$ to both sides and rearranging terms. Then early exercise is optimal with positive probability if and only if with positive probability

$$P(s, t^*) - \tilde{E}_s[P(\tau, t^*)/B(\tau, s)] \geq \pi + K[1 - b(s, \tau)].$$  \hspace{1cm} (11)

The forward, future price expression in Equation (8) relates to the left side of Equation (11). Restated, the left-hand side is $\tilde{E}_s[p(t^*, t^*) - pt^*/B(\tau, s)]$. Factoring inside the brackets gives the present value of the product, i.e., $\tilde{E}_s[p(t^*, t^*)[1 - 1/B(\tau, s)]]$. This can be rewritten as the product of expected values plus a covariance term. Substituting this transformation and rearranging Equation (11) shows early exercise is optimal with positive probability if and only if with positive probability

$$P(s, t^*) \geq [\pi + K[1 - b(s, \tau)] - \text{cov}_s[p(t^*, t^*), [1 - 1/B(\tau, s)]]]/\tilde{E}_s[1 - 1/B(\tau, s)].$$  \hspace{1cm} (12)

Roughly speaking, early exercise is optimal under some circumstances if the covariance term is positive and large enough so that the right side is below the futures price $P(s, t^*)$. Otherwise, early exercise need not be optimal. In particular, if the $\text{cov}_s[p(t^*, t^*), [1 - 1/B(\tau, s)]]$ is sufficiently negative, it is quite possible that early exercise is never optimal. Thus, for each commodity, knowledge of this covariance term's magnitude is essential for identifying instances in which the American option pricing formula and the European option pricing formula coincide.

In some cases, a European formula can be used. In other cases, ones in which the relationship in expression (12) is satisfied, early exercise is a consideration. For example, if the spot rate process is statistically independent of the commodity's spot price process, then the covariance term is identically zero and expression (12) is satisfied with positive probability for large futures prices where $\pi \equiv 0$. This example generalizes a result contained in Ball and Torous [1]. They consider deterministic interest rates where the statistical independence condition is trivially satisfied.

Next, to price futures call options, we consider two cases. The European case is first. This case occurs when expression (12) is never satisfied. Here, using the present value operator, the futures call value is

$$C[P(s, t^*); K, \tau] = \tilde{E}_s[\max[P(\tau, t^*) - K, 0]/B(\tau, s)].$$  \hspace{1cm} (13)
It is revealing to contrast expression (13) with the value for the forward call option contained in expression (6). To make the comparison, we need to specify the relationship between forward prices and futures prices. It is easy to show that

$$P(\tau, t^*) = p(\tau, t^*) + c \delta v_r[p(t^*, t^*), 1 - 1/B(t^*, \tau)]/b(\tau, t^*).$$

(14)

Substitution into expression (13) gives

$$C[P(s, t^*); K, \tau] = \bar{E}_s[\max\{p(\tau, t^*) + c \delta v_r[p(t^*, t^*), 1 - 1/B(t^*, \tau)]/b(\tau, t^*)\} - K, 0]/B(\tau, s)].$$

(15)

The two call option values in expression (6) and (15) differ due to the covariance term within the maximum operator of expression (15) and the discounting factor. If the covariance term is zero or positive, then the European futures call option price exceeds the forward call's price. Conversely, if the covariance term is sufficiently negative, the forward call's price exceeds the European futures call's price. This situation corresponds to the case in which the forward price exceeds the futures price.

Finally, when expression (12) is satisfied with positive probability, early exercise is a consideration. To value the American futures call option in this case, one must first prespecify an early exercise policy (which depends on the information available at each date), and then value the European call under the value maximizing policy. This procedure is detailed in Benoussan [3].

Futures price limit moves are an institutional reality in futures contract markets. It is easy to show that they do not influence futures option prices in our framework. The reason is simple. We can construct a synthetic security with the futures price value using the spot commodity and the discount factor [see Equation (8)]. Neither the spot commodity nor the discount factor is subject to exchange-defined limit moves. Thus, standard option pricing can proceed with the synthetic futures prices despite an arbitrary closing of the futures contract market.4

V. EXAMPLE

This section presents a simple example to illustrate the conclusions obtained in the previous sections. We study an economy lasting only two periods (t = 0, 1, 2 = t*). The stochastic structure of the economy is given in Figure 1. There are four states {1, 2, 3, 4}, each occurring with positive probability. At time 1, either event {12} or {34} occurs.
The spot commodities price follows a simple binomial process:

\[ p(0, 0) = 1 \]
\[ p_{12}(1, 1) = e^{\alpha_u}, \quad p_{34}(1, 1) = e^{\alpha_d} \]
\[ p_1(2, 2) = e^{2\alpha_u} \]
\[ p_2(2, 2) = p_3(2, 2) = e^{\alpha_u + \alpha_d} \]
\[ p_4(2, 2) = e^{2\alpha_d} \]

where \( \alpha_u > \alpha_d \). The subscripts on \( p(t, i) \) represent the state of the economy. For example, \( p_{12}(1, 1) \) is the spot price at time 1 in state [12]. The spot commodities price is normalized to be 1 at time 0. If state [12] occurs, then the price jumps "up" \( (e^{\alpha_u} - 1) \) percent, while if state [34] occurs, the price jumps "down" \( (e^{\alpha_d} - 1) \) percent. A similar pattern occurs from time 1 to time 2.

The accumulation factor (or spot rate) satisfies the following equations:

\[ B(0, 0) = 1 \]
\[ B_{12}(1, 0) = B_{34}(1, 0) = e^{r(0)} \]
\[ B_1(2, 0) = B_2(2, 0) = e^{r(0) + r_u(1)} \]
\[ B_3(2, 0) = B_4(2, 0) = e^{r(0) + r_d(1)} \]

where \( \alpha_u > r_u(1) > r_d(1) > \alpha_d \) and \( \alpha_u > r(0) > \alpha_d \). Subscripts on \( B(t, 0) \) represent the state of the economy at the appropriate time \( t \). The spot rate
over the first period is known to be \( r(0) \). At time 1, if state \([12]\) occurs, then the spot rate is \( r_{12}(1) \). If state \([34]\) occurs, however, the spot rate is \( r_{34}(1) \) where \( r_{34}(1) > r_{12}(1) \). So, by construction spot rates are negatively correlated to the spot commodity's price.

The long-term bond process is given by

\[
\begin{align*}
    b(0,2) &= e^{-i_u - r_u(1)} = e^{-i_u - r_{12}(1)} \quad (18a) \\
    b_{12}(1,2) &= e^{-r_{12}(1)} \\
    b_{34}(1,2) &= e^{-r_{34}(1)} \quad (18b) \\
    b_1(2,0) &= b_2(2,0) = b_3(2,0) = b_4(2,0) = 1 \quad (18c)
\end{align*}
\]

where \( i_u > i_d \). Again, the subscripts on \( b(t,s) \) represent the state of the economy at time \( t \). By construction the bond's price is 1 at time 2. Since its value, \( b(0,2) \), is known at time 0, expression \((18a)\) expresses the constraint imposed on its returns over the next two periods. It earns the spot rate over the last period, and either a return of \( i_u \) over the first period if state \([12]\) occurs, or a return of \( i_d \) if state \([34]\) occurs where \( i_u > i_d \).

Given there are only two possibilities at each time period, trading in the spot commodity and the accumulation factor guarantees that the market is complete. This completeness property means that any contingent claim related to these two traded assets can be duplicated by some dynamic portfolio strategy involving both assets. It also implies that an equivalent martingale measure exists. This measure is denoted by \( \bar{q}(w) \) for \( w \in \{1, 2, 3, 4\} \) and can be shown to equal

\[
\begin{align*}
    \bar{q}((12)) &= \frac{[e^{r_0} - e^{i_d}]}{[e^{i_u} - e^{i_d}]} \\
    \bar{q}((34)) &= 1 - \bar{q}((12)) \quad (19a) \\
    \bar{q}((1) | (12)) &= \frac{[e^{r_{12}(1)} - e^{i_d}]}{[e^{i_u} - e^{i_d}]} \\
    \bar{q}((2) | (12)) &= 1 - \bar{q}((1) | (12)) \\
    \bar{q}((3) | (12)) &= \bar{q}((4) | (12)) = 0 \\
    \bar{q}((3) | (34)) &= \frac{[e^{r_{34}(1)} - e^{i_d}]}{[e^{i_u} - e^{i_d}]} \\
    \bar{q}((4) | (34)) &= 1 - \bar{q}((3) | (34)) \\
    \bar{q}((1) | (34)) &= \bar{q}((2) | (34)) = 0. \quad (19b)
\end{align*}
\]

These are the "psuedo-probabilities" discussed in Section II. In addition, it is a straightforward exercise to show that \([p(t,S)/B(t,0)]\) is a martingale under the probability \( \bar{q} \).

Since the long-term bond is redundant in this economy, to avoid
arbitrage, its value must equal

\[ b(0, 2) = \mathbb{E}[1/B(2, 0)]. \]

This condition is equivalent to

\[ q((12))e^{i_1} + q((34))e^{i_2} = e^{r(0)}. \] (20)

Together with condition (18a), condition (20) completely determines the parameters of the long-term bond process \( i_1 \) and \( i_2 \) as functions of \( [\alpha_2, \alpha_n, r(1), r(2), r(0)]. \) In particular, expression (20) implies that \( i_u > r(0) > i_d. \) Using conditions (1) and (8) from the previous sections (which can be directly proven in this simple economy), it can be shown that

\[ p(0, 2) = e^{i_u + r(1)} \] (21a)

and

\[ P(0, 2) = p(0, 2) + [e^{r(1)} + r(0) - e^{r(1)} + i_u] \]
\[ + [1 - q(12)]e^{r(1)} + a[e^{i_1} - e^{i_u} - 1]. \] (21b)

The last two terms in expression (21b) are of different signs. The first is negative since \( i_u > r(0), \) while the second is positive since \( i_u > i_d. \) For different values of the parameters, therefore, the forward price \( p(0, 2) \) can exceed the futures price \( P(0, 2), \) or be less than it. The sum of these two terms corresponds to the covariance component in expression (14).

For call options, using expression (15) and (6), the valuation formulas are (if the exercise price \( K \) is chosen so that there is a positive probability of exercise at time 1, \( q[p(1, 2) > K] > 0),

\[ c[p(0, 2); K, 1] = \mathbb{E}[\max[p(1, 2) - K, 0]/B(2, 0)] \]
\[ < \mathbb{E}[\max[p(1, 2) - K, 0]/B(1, 0)] = C[p(0, 2), K, 1] \] (22)

where \( \text{cov}_1[p(2, 2), 1/B(2, 1)] = 0 \) because \( B(2, 1) = e^{r(1)} \) is known at time 1. This inequality verifies the results of the last section.

VI. SUMMARY

In this paper, we analyze the properties of forward options and futures options. Stochastic interest rates are assumed throughout. We derive several important results. First, in cases in which the spot asset makes no interim payments, a forward option will not be exercised before its expiration date. Thus, an American forward option can be treated like a European forward option. Second, an over the counter Treasury bill option can be treated like a compound forward option. This allows a recursive price technique based upon a series of European option pricing equations to be used. Finally,
futures options may be exercised early. The futures option value depends critically on the covariance between the spot price and the expression $[1 - 1/B(t^*, \tau)]$. Since the term $B(t^*, \tau)$ represents an accumulation factor at time $t^*$ for continuous reinvestment at the instantaneous risk free rate over $[\tau, t^*]$, the stochastic interest rate process is seen to influence the price of any futures option.

Options on spot, forward, and futures fixed income securities are widely traded. This paper gives a comprehensive analytic method based upon an equivalent martingale framework. Although the mathematic foundation is abstract, the direct pricing applications are based largely on algebraic relationships. The approach allows us to derive pricing equations for call options on forward and futures contracts with stochastic interest rates. An example is also provided to illustrate the main conclusions.

NOTES

1. The economy we consider is contained in Harrison and Pliska [11]. The market is complete and the trading strategies are admissible, allowing no arbitrage opportunities. Under these circumstances, Harrison and Pliska prove the existence of an equivalent martingale measure.

2. The proof of this statement follows. Suppose it is never optimal to exercise early. Then the American call's value unexercised is always strictly larger than its exercised value. But, the European call's value equals the American call's value in this case. Conversely, if early exercise is optimal with positive probability then there is some time $s$ and some set of prices where $P(s, t^*) = K$ exceeds or equals the American call's value unexercised, which in turn exceeds or equals the European call's value. This completes the proof of this statement.

3. The proof is

$$ p(r, t^*) = \frac{p(r, \tau)}{B(r, t^*)} = \mathbb{E}_r\left[p(t^*, r)|B(t^*, \tau)\right]/b(r, t^*) $$
$$ = \mathbb{E}_r[p(t^*, t^*)]\mathbb{E}_r[1/B(t^*, \tau)]/b(r, t^*) $$
$$ - \text{cov}_r[p(t^*, t^*), 1 - 1/B(t^*, \tau)]/b(r, t^*) $$

which by expression (4) and (8) gives (14).

4. One can now value the futures option, based on the synthetic futures price. Here, the exchange set limit move restrictions on the futures price, influences the value of the option when exercised. These restrictions can be viewed as "new" boundary conditions on the option. The same technique as discussed in Section III can be applied.

REFERENCES


