A CHARACTERIZATION OF COMPLETE SECURITY MARKETS ON A BROWNIAN FILTRATION

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This paper provides a characterization theorem for a complete securities market when security prices follow Itô processes on a multidimensional Brownian filtration. This characterization theorem is a special case of Harrison and Pliska (1983), and it clarifies a counterexample provided by Müller (1989).

KEYWORDS: complete markets, multidimensional Brownian filtrations, unique martingale measures

1. INTRODUCTION

Harrison and Pliska (1981, p. 241) conjecture that a securities market model with an equivalent martingale measure is complete if and only if the set of equivalent martingale measures is a singleton. In Harrison and Pliska (1983) they provide a proof of their conjecture. Crucial to an understanding of their conjecture and proof is the meaning of the word complete.

Using the definition of completeness as given in Harrison and Pliska (1981), Müller provides a valid counterexample. Müller’s counterexample is for a securities market where stock prices follow Itô processes on a two-dimensional Brownian filtration. Further, Müller also provides a sufficient condition for the satisfaction of Harrison and Pliska’s conjecture. This condition is that the cross-variation of security prices must be zero for all distinct pairs of primary traded securities (Müller 1989, condition (7), p. 40). Unfortunately, although easy to verify, this condition is not satisfied by most models used in applications (e.g., Heath, Jarrow, and Morton 1987).

Using the implicit definition of completeness as given in Harrison and Pliska (1983), however, Müller’s counterexample fails. Although Harrison and Pliska’s proof is correct, the statement of their theorem is imprecise.

The purpose of this paper is to clarify this distinction between the different definitions of completeness, and to illustrate this difference for the class of models where security prices follow Itô processes on multidimensional Brownian filtrations. In this case, the proof of Harrison and Pliska’s (1983) theorem is less abstract. Furthermore, this class of models includes most of those used in practical applications. We show that crucial to the characterization theorem is the definition of the class of admissible self-financing trading strategies employed. If the class is taken to be that defined by Harrison and Pliska (1981), then their conjecture is false, because an additional condition is needed. However, if the class of admissible self-financing trading strategies is expanded to the largest over which multi-

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dimensional stochastic integrals can be defined (as implicitly done in their 1983 paper), then their conjecture is correct.

2. THE SECURITIES MARKET MODEL

This securities market model is similar to that contained in Harrison and Pliska (1981, Section 5). We consider a continuous trading economy with the trading interval $\tau = [0, T]$ for $T < +\infty$. There is a probability space $(\Omega, F, P)$ and an augmented\(^2\) filtration $(F_t: t \in \tau)$ with $F_T = F$ generated by a $d$-dimensional standard Brownian motion \(\{W_1(t), \ldots, W_d(t): t \in \tau\}\) initialized at $(0, \ldots, 0)$. There are $d + 1$ securities trading. The securities prices satisfy the following stochastic processes:

\[
S_0(t, \omega) = 1 \quad \text{for all } (t, \omega) \in \tau \times \Omega,
\]

\[
S_i(t, \omega) = S_i(0) \exp \left\{ \int_0^t \alpha_i(s, \omega) \, ds - \frac{1}{2} \sum_{j=1}^{d} \int_0^t \sigma_{ij}^2(s, \omega) \, ds \right. \]
\[
\left. + \sum_{j=1}^{d} \int_0^t \sigma_{ij}(s, \omega) \, dW_j(s) \right\}
\]

for all $(t, \omega) \in \tau \times \Omega$ and all $i = 1, \ldots, d$, where for all $i = 1, \ldots, d, S_i(0) > 0$ is a positive constant, $\{\alpha_i(t): t \in \tau\}$ is an adapted stochastic process with

\[
\int_0^T |\alpha_i(s, \omega)| \, ds < +\infty \quad \text{a.e. } P,
\]

and $\{\sigma_{ij}(t): t \in \tau\}$ are adapted stochastic processes with

\[
\int_0^T \sigma_{ij}^2(s, \omega) \, ds < +\infty \quad \text{a.e. } P \quad \text{for } j = 1, \ldots, d.
\]

The zeroth asset (2.1) is interpreted as a money market account. Without loss of generality, its price is taken to be unity for all times and states (i.e., it serves as the numeraire). The securities $i = 1, \ldots, d$ (2.2) are risky and follow exponential Itô processes on a multidimensional Brownian filtration. It is the restriction to a Brownian filtration which distinguishes our class of models from the general class of semimartingales (with discontinuous sample paths) studied by Harrison and Pliska (1981) and Müller (1989).

For future reference, we note that (2.2) implies\(^4\) that

\[
P \left( \inf_{0 \leq t \leq T} S_i(t) > 0 \quad \text{for } i = 1, \ldots, d. \right)
\]

\(^2\)It is augmented to include all $P$-null events; hence, it is right-continuous (see Karatzas and Shreve 1988, p. 89).

\(^3\)A stochastic process $\{x(t): t \in \tau\}$ is a mapping $: \tau \times \Omega \to R$ which is jointly measurable from $B(\tau) \times F \to B$, where $B$ denotes the Borel $\sigma$-algebra and $B(\tau)$ is the Borel $\sigma$-algebra restricted to $\tau$.

\(^4\)This follows from $\int_0^T |\alpha_i| \, ds < +\infty$ a.e. $P$ for $i = 1, \ldots, d$ and Karatzas and Shreve (1988, problem 3.10, p. 155).
To facilitate the reader's understanding of this economy, we record the following characteristics for the existence and uniqueness of equivalent martingale measures. These characteristics and their proofs can be found in Heath, Jarrow, and Morton (1987).

Let \( \Sigma: \tau \times \Omega \to R^{d \times d} \) be defined as the \( d \times d \) dispersion matrix

\[
\Sigma(t, \omega) = \begin{bmatrix}
\sigma_{11}(t, \omega) & \cdots & \sigma_{1d}(t, \omega) \\
\vdots & & \vdots \\
\sigma_{d1}(t, \omega) & \cdots & \sigma_{dd}(t, \omega)
\end{bmatrix}.
\]

(2.4)

**THEOREM 2.1 (Existence of Equivalent Martingale Measures).** There exists a probability measure \( \tilde{P} \) on \( (\Omega, F) \) equivalent\(^5\) to \( P \) making \( \{S_i(t): t \in \tau\} \tilde{P}\)-martingales with respect to \( \{F_i: t \in \tau\} \) for \( i = 1, \ldots, d \) if and only if there exist unique\(^6\) \( \phi_j: \tau \times \Omega \to R \) for \( j = 1, \ldots, d \) jointly measurable and adapted such that

\[
\sum_{j=1}^d \int_0^T \phi_j^2(s, \omega) \, ds < +\infty \quad \text{a.e. } P,
\]

\[
E \left[ \exp \left( \sum_{j=1}^d \int_0^T \phi_j(s, \omega) \, dW_j(s) - \frac{1}{2} \sum_{j=1}^d \int_0^T \phi_j^2(s, \omega) \, ds \right) \right] = 1,
\]

\[
E \left( \exp \left( \sum_{j=1}^d \int_0^T \left[ \sigma_{jj}(s, \omega) + \phi_j(s, \omega) \right] \, dW_j(s) - \frac{1}{2} \sum_{j=1}^d \int_0^T \left[ \sigma_{jj}(s, \omega) + \phi_j(s, \omega) \right]^2 \, ds \right) \right) = 1 \quad \text{for } i = 1, \ldots, d,
\]

and

\[
P \left( -\Sigma(t, \omega) \begin{bmatrix}
\phi_1(t, \omega) \\
\vdots \\
\phi_d(t, \omega)
\end{bmatrix} = \begin{bmatrix}
\alpha_1(t, \omega) \\
\vdots \\
\alpha_d(t, \omega)
\end{bmatrix} \quad \text{a.e. } t \right) = 1.
\]

The \( d \)-dimensional stochastic processes \( (\phi_1(t), \ldots, \phi_d(t): t \in \tau) \) identified in Theorem 2.1 are known as the *market prices for risk*.

**THEOREM 2.2 (Uniqueness of the Equivalent Martingale Measure).** Suppose there exists an equivalent probability measure \( \tilde{P} \) on \( (\Omega, F) \) making \( \{S_i(t): t \in \tau\} \tilde{P}\)-martingales with respect to \( \{F_i: t \in \tau\} \) for \( i = 1, \ldots, d \). Then \( \tilde{P} \) is unique if and only if \( P(\Sigma(t) \text{is nonsingular a.e. } t) = 1 \).

It is easy to see that the example given by Müller (1989, (2), (3)) satisfies the hypotheses of both of these theorems. Indeed, the example is the specialization of the above

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\(^5\)Two measures are *equivalent* if they are mutually absolutely continuous with respect to each other.

\(^6\)They are unique in the sense that if \( \phi_j \) for \( j = 1, \ldots, d \) are another set of mappings satisfying the conditions of the theorem, then \( \int_0^T [\phi_j(t, \omega) - \phi_j(t, \omega)]^2 \, dt = 0 \) a.e. \( P \) for \( j = 1, \ldots, d \). This is equivalent to \( P(\phi_j(t, \omega) = \phi_j(t, \omega) \text{ a.e. } t) = 1 \).
economy to $d = 2$, $\alpha_1(t) = \alpha_2(t) = 0$, and

$$
\Sigma(t) = \begin{bmatrix} 1 & 0 \\ u(t) & 1 - u(t) \end{bmatrix},
$$

where

$$
u(t) = \begin{cases} 1 - t & \text{if } 0 < t < 1, \\ 1/2 & \text{if } t = 0, \quad t \geq 1. \end{cases}
$$

Theorem 2.2 is satisfied since $\Sigma^{-1}(t)$ exists for all $t \in \tau$. We have uniquely $\phi_1(t) = \phi_2(t) = 0$. Also, $\Sigma(t)$ is uniformly bounded above by 1, so Novikov’s condition (Karatzas and Shreve 1988, p. 198) gives the hypothesis of Theorem 2.1.

We remark that if the hypotheses of Theorems 2.1 and 2.2 hold, then by Girsanov’s theorem $\{\hat{W}_1(t), \ldots, \hat{W}_d(t): t \in \tau\}$ is a standard $d$-dimensional Brownian motion on the filtered probability space $(\Omega, F, \tilde{P})$, $(F_t; t \in \tau)$, where $^7$

$$
(2.5) \quad \hat{W}_j(t) = W_j(t) + \int_0^t \phi_j(s, \omega) \, ds \quad \text{for all } t \in \tau \text{ and } j = 1, \ldots, d.
$$

In this filtered probability space, the risky asset prices satisfy

$$
(2.6) \quad dS_j(t, \omega) = \sum_{j=1}^d \sigma_{ij}(t, \omega) S_i(t, \omega) \, d\hat{W}_j(t) \quad \text{for all } t \in \tau \text{ and } j = 1, \ldots, d.
$$

3. THE CHARACTERIZATION THEOREM FOR MARKET COMPLETENESS

Prior to stating and proving the theorem, we need to introduce some terminology. For this section, we assume the existence of an equivalent probability measure $\tilde{P}$ on $(\Omega, F)$ making $\{S_i(t): t \in \tau\}$ $\tilde{P}$-martingales with respect to $(F_t; t \in \tau)$ for $i = 1, \ldots, d$. Define the $d \times d$ nonnegative symmetric matrix $C: \Omega \times \tau \to \mathbb{R}^{d \times d}$ by

$$
C(t, \omega) = \begin{bmatrix} C_{11}(t, \omega) & \cdots & C_{1d}(t, \omega) \\ \vdots & \ddots & \vdots \\ C_{d1}(t, \omega) & \cdots & C_{dd}(t, \omega) \end{bmatrix},
$$

where

$$
C_{ij}(t, \omega) = \sum_{k=1}^d \sigma_{jk}(t, \omega) S_i(t, \omega) \sigma_{jk}(t, \omega) S_j(t, \omega) \quad \text{for all } t \in \tau, \omega \in \Omega.
$$

To facilitate understanding, we purposefully deviate from Harrison and Pliska’s (1981) terminology to emphasize a distinction. We define a component self-financing trading strategy (component s.f.t.s.) as a $(d + 1)$-dimensional adapted stochastic process

$^7$In fact, the filtration $(F_t; t \in \tau)$ is the same as the one generated by $\{\hat{W}_1(t), \ldots, \hat{W}_d(t): t \in \tau\}$. 
\{\beta_0(t), \ldots, \beta_d(t): t \in \tau\} such that

\begin{align}
(3.1a) \quad & \sum_{i=1}^d \sum_{j=1}^d \int_0^T \beta_i^2(t, \omega) S_j^2(t, \omega) \sigma_{ij}^2(t, \omega) \, dt < +\infty \quad \text{a.e. } P, \\
(3.1b) \quad & \sum_{i=1}^d \int_0^T |\beta_i(t, \omega) \alpha_i(t, \omega)| S_i(t, \omega) \, dt < +\infty \quad \text{a.e. } P, \\
(3.1c) \quad & |\beta_0(t)| < +\infty \quad \text{for all } t \in \tau \text{ a.e. } P,
\end{align}

and we define

\begin{align}
(3.1d) \quad & \nu_p: \tau \times \Omega \to R \quad \text{by } \nu_p(t, \omega) = \beta_0(t, \omega) + \sum_{i=1}^d \beta_i(t, \omega) S_i(t, \omega).
\end{align}

Then

\begin{align}
(3.1e) \quad & \nu_p(t) = \nu_p(0) + \sum_{i=1}^d \int_0^t \beta_i(s) \, dS_i(s) \quad \text{for all } t \in \tau \quad \text{a.e. } P.
\end{align}

For simplicity of notation, define

\begin{align*}
\beta(i) &= (\beta_1(t), \ldots, \beta_d(t)), \\
S(t) &= (S_1(t), \ldots, S_d(t)),
\end{align*}

\begin{align*}
\beta(t) \cdot S(t) &= \sum_{j=1}^d \beta_j(t) S_j(t), \quad \text{and} \\
\beta(t) \cdot C(t) \cdot \beta(t) &= \sum_{j=1}^d \sum_{j'=1}^d \beta_j(t) C_{jj'}(t) \beta_{j'}(t).
\end{align*}

In the definition of a component s.f.t.s., condition (3.1a) implicitly defines the stochastic integral of \{\beta(t): t \in \tau\} with respect to \{S(t): t \in \tau\} as \sum_{j=1}^d \int_0^T \beta_j(s) \, dS_j(s). This is the coordinate-wise definition for a stochastic integral of a \(d\)-dimensional process. It is not the most general definition of the stochastic integral of \{\beta(t): t \in \tau\} with respect to \{S(t): t \in \tau\} possible. The stochastic integral is definable for a larger class of s.f.t.s. We say that the \((d + 1)\)-dimensional adapted stochastic process \{\beta_0(t), \ldots, \beta_d(t): t \in \tau\} is a vector s.f.t.s. if it satisfies conditions (3.1b), (3.1c), (3.1d), and

\begin{align}
(3.1a*) \quad & \int_0^T \beta(s) \cdot C(s) \cdot \beta(s) \, ds < +\infty \quad \text{a.e. } P, \\
(3.1e*) \quad & \nu_p(t) = \nu_p(0) + \int_0^t \beta(s) \cdot dS(s) \quad \text{for all } t \in \tau \quad \text{a.e. } P
\end{align}

where \(\int_0^T \beta(s) \cdot dS(s)\) is the stochastic integral as defined in Jacod and Shiryaev (1987, p. 166).
A component s.f.t.s. or a vector s.f.t.s. \( \{ \beta_0(t), \ldots, \beta_d(t): t \in \tau \} \) is said to be admissible if the value process \( \{ v_\phi(t): t \in \tau \} \) defined in condition (3.1e) or (3.1e*) is a \( \hat{P} \)-martingale with respect to \( \{ F_t: t \in \tau \} \).

Given any \( F \)-measurable random variable \( x: \Omega \to R \) which is \( \hat{P} \)-integrable (i.e., \( \hat{E}(|x|) < +\infty \)), the market is defined to be component complete if there exists an admissible component s.f.t.s. \( \{ \beta_0(t), \ldots, \beta_d(t): t \in \tau \} \) such that \( v_\phi(T) = x \) a.e. \( P \). This is Harrison Pliska's (1981) definition of completeness. Finally, we say the market is vector complete if there exists an admissible vector s.f.t.s. \( \{ \beta_0(t), \ldots, \beta_d(t): t \in \tau \} \) such that \( v_\phi(T) = x \) a.e. \( P \). This is the definition of completeness implicit in Harrison and Pliska (1983).\(^8\)

We now state and prove two theorems characterizing market completeness.

**Theorem 3.1 (Characterization of Component Complete Markets).** Suppose there exists a unique equivalent probability measure \( \hat{P} \) on \( (\Omega, F) \) making \( \{ S_t(t): t \in \tau \} \) \( \hat{P} \)-martingales with respect to \( \{ F_t: t \in \tau \} \) for \( i = 1, \ldots, d \). Then the market is component complete if and only if

\[
\begin{align*}
(3.2) \quad & \int_0^T \sum_{i=1}^d a_{ki}^2(t, \omega) \left( \sum_{j=1}^d \sigma_{ij}^2(t, \omega) \right) dt < +\infty \quad \text{a.e. } P \\
& \text{ for } k = 1, \ldots, d
\end{align*}
\]

where \( a_{ki}(t, \omega) \) is the \((k, i)\)th element of \( \Sigma^{-1}(t, \omega) \).

**Proof.**

Step 1. Assume the market is component complete. Then by definition there exist admissible component s.f.t.s. \( \{ \beta_{0k}(t), \ldots, \beta_{dk}(t): t \in \tau \} \) satisfying conditions (3.1a) and (3.1b) such that, for \( k = 1, \ldots, d \),

\[
(3.3) \quad \tilde{W}_k(T) = \beta_{0k}(T) + \sum_{i=1}^d \beta_{ik}(t) dS_i(t) \quad \text{a.e. } P
\]

with

\[
(3.4) \quad \sum_{i=1}^d \sum_{j=1}^d \int_0^T \beta_{ki}^2(t) S_i^2(t) \sigma_{ij}^2(t) dt < +\infty \quad \text{a.e. } P.
\]

Using condition (2.6) in (3.3), admissibility, and taking conditional expectations with respect to \( F_t \) yields

\[
(3.5) \quad \tilde{W}_k(t) = \beta_{0k}(t) + \sum_{j=1}^d \int_0^t \left[ \sum_{i=1}^d \beta_{ik}(s) S_i(s) \sigma_{ij}(s) \right] d\tilde{W}_j(s)
\]

for all \( t \in \tau \) a.e. \( P \).

By uniqueness of the integrands in the martingale representation theorem (see Karatzas and Shreve 1988, Ex. 4.22(ii), p. 189),

\[
P(\beta_{0k}(t) = 0 \text{ a.e. } t) = 1
\]

\(^8\) A justification for this statement can be found in the appendix.
and

\[
\begin{pmatrix}
\sum_{j=1}^{d} \beta_{jk}(t)S_j(t)\sigma_{ij}(t) = 0 & \text{for } j \neq k \\
\sum_{j=1}^{d} \beta_{jk}(t)S_j(t)\sigma_{ij}(t) = 1 & \text{for } j = k
\end{pmatrix}
\quad \text{a.e. } t
\]

= 1.

Solving the system in (3.6) yields \( \beta_{ik}(t) = a_{ki}(t)/S_i(t) \) for \( i = 1, \ldots, d \). Substitution into (3.4) yields (3.2).

Step 2. Assume (3.2). First, define \( \beta_{ki}(t) = a_{ki}(t)/S_i(t) \) for \( i = 1, \ldots, d \) and \( k = 1, \ldots, d \) and

\[
v_k(t) = \sum_{i=1}^{d} \int_{0}^{t} \beta_{ki}(t) \, dS_i(t).
\]

This integral is well defined since

\[
\int_{0}^{T} \beta_{ki}^2 \, d(S_i) = \int_{0}^{T} \sigma_{ki}^2 \left( \sum_{j=1}^{d} \sigma_{ij}^2 \right) \, dt < +\infty \quad \text{a.e. } P
\]

by (3.2). Note that from (2.6),

\[
v_k(t) = \sum_{i=1}^{d} \int_{0}^{t} \beta_{ki}(t) \left( \sum_{j=1}^{d} \sigma_{ij} S_i \, d\tilde{W}_j \right) = \tilde{W}_k(t) \quad \text{a.e. } \tilde{P}.
\]

Since \( \tilde{P} \) is equivalent to \( P \), a.e. \( P \) is identical to a.e. \( \tilde{P} \).

Second, take \( x: \Omega \to R \), \( F \)-measurable with \( \mathbb{E}(|x|) < +\infty \). By the martingale representation theorem (Karatzas and Shreve 1988, Ex. 4.22 (ii), p. 189), there exists an adapted stochastic process \( \{\eta_i(t), \ldots, \eta_d(t): t \in \mathbb{T}\} \) such that

\[
x = \tilde{E}(x) + \sum_{j=1}^{d} \int_{0}^{T} \eta_j \, d\tilde{W}_j \text{ a.e. } \tilde{P} \quad \text{and} \quad \sum_{j=1}^{d} \int_{0}^{T} \eta_j^2(s) \, ds < +\infty \text{ a.e. } \tilde{P}.
\]

Consider \( \int_{0}^{T} \eta_k \, d\tilde{W}_k \). By our first comment (use Ikeda and Watanabe 1981, Proposition 2.5, p. 58),

\[
\int_{0}^{T} \eta_k \, d\tilde{W}_k = \int_{0}^{T} \eta_k \, dv_k = \sum_{i=1}^{d} \int_{0}^{T} \eta_k \beta_{ki} \, dS_i
\]

holds with

\[
\int_{0}^{T} \eta_k^2 \beta_{ki}^2 \, d(S_i) < +\infty \quad \text{a.e. } \tilde{P}.
\]
Hence
\[
\sum_{k=1}^{d} \int_{0}^{T} \eta_k \, d\hat{W}_k = \sum_{i=1}^{d} \sum_{k=1}^{d} \eta_k \beta_{ki} \, dS_i.
\]

Define
\[
\delta_i(t) = \sum_{k=1}^{d} \eta_k \beta_{ki} \quad \text{for } i = 1, \ldots, d,
\]
\[
\nu_{6}(t) = \tilde{E}(x) + \sum_{i=1}^{d} \int_{0}^{t} \delta_i(s) \, dS_i(s),
\]
and
\[
\delta_0(t) = \nu_{6}(t) - \sum_{i=1}^{d} \delta_i(t)S_i(t).
\]

We claim that \(\{\delta_0(t), \ldots, \delta_d(t) : t \in \tau\}\) is an admissible component s.f.t.s. such that
\[
x = \delta_0(T) + \sum_{i=1}^{d} \int_{0}^{T} \delta_i(t) \, dS_i(t) \quad \text{a.e. } P.
\]

First, \(\{\delta_0(t), \ldots, \delta_d(t) : t \in \tau\}\) is obviously jointly measurable and adapted.
Second, (3.10) and (3.11) yield that this strategy is self-financing with \(\nu_{6}(0) = \tilde{E}(x)\).
Third, \(\sum_{i=1}^{d} \int_{0}^{t} \delta_i^2 \, dS_i(t) < +\infty \text{ a.e. } \hat{P}\) by (3.7). This implies \(\int_{0}^{T} \delta_i \, dS_i(t)\) is well defined.
Fourth, \(\sum_{i=1}^{d} \int_{0}^{T} |\delta_i(t)|S_i(t) \, dt < +\infty \text{ a.e. } \hat{P}\). Indeed, suppose not. Then
\[
P\left( \int_{0}^{T} \delta_i(t, \omega) \, dS_i(t, \omega) = +\infty \right) > 0.
\]

But, then
\[
\hat{P}\left( \int_{0}^{T} \delta_i(t, \omega) \, dS_i(t, \omega) = +\infty \right) > 0.
\]

This contradicts \(\int_{0}^{T} \delta_i(t) \, dS_i(t)\) being well defined.
Lastly, by (3.8)
\[
\nu_{6}(T) = \delta_0(T) + \sum_{i=1}^{d} \int_{0}^{T} \delta_i(t) \, dS_i(t)
\]
\[
= \tilde{E}(x) + \sum_{k=1}^{d} \int_{0}^{T} \eta_k \, d\hat{W}_k = x \quad \text{a.e. } P.
\]

Note that this statement also shows \(\nu_{6}(t)\) is a \(\hat{P}\)-martingale. \(\square\)
The necessary and sufficient condition (3.2) of Theorem 3.1 can be usefully rewritten in terms of instantaneous covariance matrices. Define, for this purpose, the instantaneous return covariance matrix by

\[ Q(t, \omega) = \Sigma(t, \omega) \Sigma(t, \omega)', \]

where the prime denotes transpose. Let \( \gamma(t, \omega) \) be the ratio of the largest to the smallest eigenvalue of \( Q(t, \omega) \), or the condition number of \( Q(t, \omega) \). The necessary and sufficient condition for component completeness (3.2) is restated in terms of \( Q \), and a useful sufficient condition is provided by the integrability of \( \gamma \) in the following corollary.\(^9\)

**Corollary 3.1.** Suppose there exists a unique probability measure \( \tilde{P} \) on \((\Omega, F)\) making \((S_t(t); t \in \tau) \) \( \tilde{P} \)-martingales with respect to \((F_t; t \in \tau)\) for \( i = 1, \ldots, d \). Then the market is component complete if and only if

\[
\int_0^T \sum_{i=1}^d Q_{ii}(t, \omega) [Q^{-1}]_{ii}(t, \omega) \, dt < +\infty \text{ a.e. } P.
\]

In particular, it is component complete if

\[
\int_0^T \gamma(t, \omega) \, dt < +\infty \text{ a.e. } P.
\]

**Proof.** The necessary and sufficient condition (3.2) for component completeness is equivalent to

\[
\int_0^T \sum_{i=1}^d \left[ \sum_{k=1}^d a_{ki}^2(t, \omega) \right] \left[ \sum_{j=1}^d \sigma_{ij}^2(t, \omega) \right] \, dt < +\infty \text{ a.e. } P,
\]

where the summation over \( k \) is justified by there being only finitely many terms in the sum. It follows from the definition of \( Q \) and \( a_{ki} \) that

\[
Q_{ii}(t, \omega) = \sum_{j=1}^d \sigma_{ij}^2(t, \omega)
\]

and

\[
[Q^{-1}]_{ii}(t, \omega) = \sum_{k=1}^d a_{ki}^2(t, \omega).
\]

Condition (3.12) follows on substitution of (3.15) and (3.16) into (3.14).

By a diagonalization of \( Q(t, \omega) \) as \( U(t, \omega)D(t, \omega)U(t, \omega)' \) for \( U(t, \omega) \) orthonormal and \( D(t, \omega) \) a positive diagonal matrix, we note that

\[
Q_{ii}(t, \omega) = \sum_{k=1}^d D_{kk}(t, \omega)U_{ki}^2(t, \omega) \leq \max_{k=1,\ldots,d} (D_{kk}(t, \omega)),
\]

\(^9\)We thank Freddy Delbaen for this corollary.
while

\[ [Q^{-1}]_{ii}(t, \omega) = \sum_{k=1}^{d} \frac{1}{D_{kk}(t, \omega)} U_{kk}^{i}(t, \omega) \leq \left[ \min_{k=1, \ldots, d} (D_{kk}(t, \omega)) \right]^{-1} \]

The summation over \( i \) in (3.12) is therefore bounded by \( dy(t, \omega) \) and the sufficiency of (3.13) follows. \( \Box \)

Theorem 3.1 shows that for the definition of a component complete market, Harrison and Pliska's conjecture is false. The additional integrability condition (3.2) or (3.12) is seen to be a necessary and sufficient condition for a component complete market. This condition is easily verified in practical applications and essentially rules out ill-conditioned instantaneous covariance matrices for the assets' returns. From (3.12) we observe that it is trivially satisfied for diagonal instantaneous covariance matrices, or when the Müller (1989) orthogonality condition holds. More generally, (3.12) holds whenever the eigenvalues of \( Q \) are uniformly bounded above and below. Müller's counterexample (1989, p. 40), however, violates condition (3.12).

**Theorem 3.2 (Characterization of Vector Complete Markets).** Suppose there exists an equivalent probability measure \( \bar{P} \) on \((\Omega, F)\) making \( \{S_i(t); t \in \tau\} \) for \( i = 1, \ldots, d \). Then the market is vector complete if and only if \( \bar{P} \) is unique.

**Proof:**

Step 1. Assume the market is vector complete but \( \bar{P} \) is not unique. Then there exists \( A \in F \) and \( \bar{P}; F \to [0, 1] \) such that \( \bar{P}(A) \neq \bar{P}(A), \bar{P} \sim P \), and \( \{S_i(t); t \in \tau\} \) for \( i = 1, \ldots, d \) are \( \bar{P} \) martingales with respect to \( (F_t; t \in \tau) \).

Given \( L_A \), there exists a \( \bar{P} \)-admissible vector s.f.t.s. \( \{\beta_0(t), \beta(t); t \in \tau\} \) with value function \( v_A(t) \) such that \( v_A(T) = 1_A \) a.e. \( P \). By the self-financing condition, \( 1_A = v_A(0) + \int_0^T \beta(s) \cdot dS(s) \) a.e. \( P \). Similarly, there exists a \( \bar{P} \)-admissible vector s.f.t.s. \( \{\alpha_0(t), \alpha(t); t \in \tau\} \) with value function \( v_A(t) \) such that \( v_A(T) = 1_A \) a.e. \( P \) and

\[ 1_A = v_A(0) + \int_0^T \alpha(s) \cdot dS(s) \text{ a.e. } P. \]

Thus

\[ v_A(0) - v_A(0) + \int_0^T \alpha(s) \cdot dS(s) = \int_0^T \beta(s) \cdot dS(s). \]

Because \( \{S_i(t); t \in \tau\} \) is a \( \bar{P} \) and \( P \) martingale and is \( \bar{P} \)-admissible, we have \( \beta \) is \( \bar{P} \)-admissible. Thus \( v_A(0) = v_A(0) \) and \( \bar{P}(A) = \bar{E}(1_A) = v_A(0) = \bar{E}(1_A) = \bar{F}(A) \). But this is a contradiction, so \( \bar{P} \) must be unique.

Step 2. Assume \( \bar{P} \) is unique. By Theorem 2.2, \( \Sigma^{-1}(t) = (a_{ij}), i, j = 1, \ldots, d \), exists for all \( t \in \tau \) a.e. \( P \). Define \( \beta^k(t) = (\beta_1^k(t), \ldots, \beta_d^k(t)) \) by \( \beta^k(t) = a_{ik}(t)/S_i(t) \) for \( i, k = 1, \ldots, d \). Define \( v_k(t) = \int_0^T \beta^k(s) \cdot dS(s) \). This integral is well defined since
\[
\int_0^T \beta^k(t) \cdot C(t) \cdot \beta^k(t) \, dt = \int_0^T \sum_{i=1}^d \sum_{j=1}^d \left[ a_{kj}(t) C_{ij}(t) a_{kj}(t) \right] \, dt
\]
\[
= \int_0^T \sum_{i=1}^d \sum_{j=1}^d \left[ a_{kj}(t) \left( \sum_{n=1}^d \sigma_{in}(t) \sigma_{jn}(t) \right) a_{kj}(t) \right] \, dt
\]
\[
= \int_0^T \sum_{n=1}^d \left( \sum_{i=1}^d \sigma_{in}(t) a_{ki}(t) \right) \cdot \left( \sum_{j=1}^d a_{kj}(t) \sigma_{jn}(t) \right) \, dt
\]
\[
= T < +\infty.
\]
Note \( v_k(t) = \int_0^t e^{k(t)} \cdot dS(t) = \tilde{W}_k(t) \) a.e. \( \tilde{P} \). Take \( x: \Omega \rightarrow R \) \( F \)-measurable with \( \mathbb{E}(|x|) < +\infty \). By the martingale representation theorem, there exists an adapted stochastic process \( \eta(t) = (\eta_1(t), \ldots, \eta_d(t)) \) such that
\[
x = \mathbb{E}(x) + \int_0^T \eta_1(t) \cdot d\tilde{W}(t) \quad \text{a.e. } \tilde{P} \quad \text{and} \quad \sum_{j=1}^d \int_0^T \eta_j^2(s) \, ds < +\infty \quad \text{a.e. } \tilde{P}.
\]
Define
\[
\delta_i(t) = \sum_{k=1}^d \eta_k(t) \beta^k_i(t) \quad \text{for } i = 1, \ldots, d,
\]
\[
v(t) = \mathbb{E}(x) + \int_0^t \delta_i(t) \cdot dS(t),
\]
\[
\delta_0(t) = v(t) - \sum_{i=1}^d \delta_i(t) S_i(t).
\]
Now \( \int_0^T \delta_i(t) \cdot dS(t) \) is well defined since
\[
\int_0^T \delta_i(t) \cdot C(t) \cdot \delta_i(t) \, dt = \int_0^T \left[ \sum_{k=1}^d \eta_k(t) \beta^k_i(t) \right] \cdot C(t) \cdot \left[ \sum_{k=1}^d \eta_k(t) \beta^k_i(t) \right] \, dt
\]
\[
= \int_0^T \sum_{k=1}^d \eta_k^2(t) \, dt < +\infty \quad \text{a.e. } P.
\]
Consider \( \{\delta_0(t), \delta_1(t), \ldots, \delta_d(t): t \in \tau\} \). We claim it is an admissible vector s.f.t.s. such that \( v_\tau(T) = x \) a.e. \( P \). The verification of the claim is the same as in Theorem 3.1 with one exception: the last step becomes
\[
v(T) = \delta_0(T) + \int_0^T \delta_i(t) \cdot dS(t) = \mathbb{E}(x) + \int_0^T \eta_i(t) \cdot d\tilde{W}(t) = x \quad \text{a.e. } P. \quad \Box
\]

Theorem 3.2 shows the conditions under which Harrison and Pliska's conjecture is correct. It is correct if the class of admissible s.f.t.s is enlarged to be the vector admissible s.f.t.s. To our knowledge, there is no economic reason why traders should be restricted
to the smaller class of component s.f.t.s. Indeed, Theorem 3.2 gives a compelling reason why the opposite is the more logical assumption. As shown by Harrison and Pliska (1983), Theorem 3.2 is generalizable to the class of models where security prices follow semimartingales. We comment briefly on this theorem in the appendix.

CONCLUSION

This paper provides a characterization of a complete securities market in a multidimensional diffusion model. As such, it clarifies a confusion in the literature, and it should also prove useful in practical applications.

APPENDIX

This appendix provides some comments on Harrison and Pliska (1983) to relate it to the above analysis. Harrison and Pliska (1983) refer to Jacod (1979) in their definition of a self-financing trading strategy, for their property (i) requires $\Phi \in L(Z)$, which is defined in Jacod (1979). Jacod, on p. 52, discusses only the widest class of predictable processes with respect to which one may define stochastic integrals over a single semimartingale $X$. Harrison and Pliska (1983), however, do state that $L(Z)$ is the set of all predictable processes that are integrable with respect to $Z$. If we look at the proof of their theorem, a critical step is the result $P^* \in M_0(Z)$ if the representation property holds, and this is the equivalence (i) $\Leftrightarrow$ (ii) of Theorem 11.2 of Jacod (p. 338).

Property (ii) asserts $H^i(P) = \mathcal{L}^1(\mathcal{X} \cup \{1\})$, where $H^i(P)$ is defined by Jacod on p. 27, so every local $P$-martingale $M$, for which $E[\sup_p |M_p|] < \infty$ holds, belongs to the class $\mathcal{L}^1(\mathcal{X} \cup \{1\})$.

For the definition of $\mathcal{L}^1(\mathcal{X} \cup \{1\})$, see Definition 4.4 on p. 114 of Jacod (1979). This is the smallest stable subspace of $H^1$ that contains all stochastic integrals $H \cdot M$ for $M \in (\mathcal{X} \cup \{1\})$ and $H \in L^1(M)$.

Jacod defines $L^1(M)$ on p. 42. It requires $H$ to be predictable with

\[ (\int_0^\infty H^2 \, d|M, M|) < \infty \text{ a.e.} \] 

The definition of a stable subspace is on p. 113 and is, in particular, a closed subspace of $H^1$ in the appropriate norm.

Hence, $\mathcal{L}^1(\mathcal{X} \cup \{1\})$ is, in general, bigger (by norm closure) than the space of stochastic integrals $H \cdot M$ for $H$ satisfying the integrability condition (**). Finally, this stable subspace is characterized by Jacod on p. 143 for finite families of martingales as Theorem 4.60, which asserts that for a vector of martingales $\mathbf{M}$,

\[ \mathcal{L}^1(\mathbf{M}) = \left\{ \sum_i \int_0^t H^i \, dM^i \mid H \in L^1(\mathbf{M}) \right\}. \]

Jacod, in (4.59), defines $\mathcal{L}^1(\mathbf{M})$ precisely by our condition (3.1a*) for the Brownian filtration case, or, more generally, that

\[ \int_0^\infty \sum_y H^i H^j \, d|M^i, M^j| < +\infty \text{ a.e.} \]
REFERENCES


