EASIER DONE THAN SAID

The Heath-Jarrow-Morton (HJM) interest rate models represent a new approach to interest rate option pricing—one that allows a trader to price and hedge an entire interest rate book with a single consistent model. As in the Black-Scholes equity option model, the only inputs to HJM are an underlying and a measure of its volatility. The “underlying” is the entire term structure, and “volatility” describes how this term structure fluctuates over time. Thus, the initial term structure is an input to HJM in the same way that the current stock price is an input to Black-Scholes—it is the underlying’s starting position.

Spot rate models (such as Black-Derman-Toy, Vasicek, Hull-White, Cox-Ingersoll-Ross) treat the spot interest rate as the underlying state variable. Besides the current spot rate, inputs to these models include various parameters used to describe possible future paths of the spot rate. Since the current term structure is not a direct input, these models try to fit the term structure by searching for parameter values which cause calculated zero coupon bond prices to match the market.

Since spot rate models manipulate a single rate, other rates and prices must be calculated when needed. Some spot rate implementations price a three-month option on a 20-year bond by modelling 20 years of spot rate evolution. Under HJM, three months of term structure evolution would price the option.

In addition, the term structure approach of HJM allows for a wider class of models. As we will show, many popular spot rate models can be expressed rather simply under HJM.

HJM models describe the motion of forward interest rates. The forward interest rate ft(u) is the rate at which one could contract at date t to borrow and lend for a short period of time at date u (u > t). The spot rate at date t is therefore ft(t). We find it more convenient to represent the term structure by forward rates rather than by yields or pure discount bond prices.1 The initial forward rate curve is an input to HJM models, and is taken from market data.2

Volatility is the critical ingredient in fixed-income option pricing. To model the evolution of the entire term structure, however, one must specify the volatility of each forward rate. Equation (1) shows the general form of a one-factor model. In a one-factor model, a single source of randomness, (dz), drives all interest rate changes. The equation describes how the forward rate for date u can fluctuate over the instant of time from t to t + dt:

\[ f(t + dt, u) = f(t, u) + \sigma(u)dz + drift(u)dt \]  (1)

For simplicity, we have omitted the arguments of the functions drift(·) and \( \sigma(·) \). The theory allows these functions to depend on virtually any quantity observable by time t but, as we discuss below, simpler models are normally preferable.

The most important part of equation (1) is the volatility function, \( \sigma(·) \). By determining the (absolute) volatility of each forward rate, \( \sigma(·) \) describes the form of the term structure’s motion through time. Aside from the initial term structure, \( \sigma(·) \) is the only input needed to price and hedge all derivatives.

The simplest model arises if \( \sigma(·) \) is a constant. In this case, all forward rates have the same volatility, so that the entire forward curve is continuously subjected to upward or downward parallel shocks. This model can be shown to be the natural continuous-time limit of the Ho-Lee model. The forward rate curve representation reveals one undesirable feature of this model: since volatility is constant even if rates drop to very low values, interest rates can and do become negative.

A more complicated model arises if we let \( \sigma(·) \) be a function of forward rate \( ft(u) \)'s maturity (u - t). Since the short end of the forward curve is typically more volatile than the long end, one normally lets \( \sigma(·) \) be a decreasing function of (u - t). For example, a model developed by Vasicek and studied later by Brenner and Hull and White arises from choosing \( \sigma(u - t) = \sigma_0 \exp(-(\alpha(u - t)) \) for positive constants \( \alpha \) and \( \sigma_0 \).

Aside from the initial term structure, \( \sigma(·) \) is the only input needed to price and hedge all derivatives.

In the HJM framework, the drift(·) term in each equation serves a technical role but does not affect derivative prices.

As usual in continuous time modelling, to obtain prices we must resort to a discrete time approximation of the continuous evolution. In discrete time, a random walk replaces the Brownian motion. This random walk takes us up and down steps at discrete points in time.

In general, the evolution of the term structure under the HJM approach is path-dependent. An upward movement followed by a downward movement of rates does not necessarily lead to the same term structure as a downward movement followed by an upward one. As a result, the discrete time valuation method must use trees, which grow exponentially fast, rather than lattices, which grow more slowly (see figure 1).

---

Table 1: Specific HJM volatility function(s)

<table>
<thead>
<tr>
<th>Model</th>
<th>HJM volatility function of ft(u)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous Ho-Lee</td>
<td>( \sigma_0 )</td>
</tr>
<tr>
<td>Vasicek (Hull-White)</td>
<td>( \sigma_0 \cdot \exp(-\alpha(u - t)) )</td>
</tr>
<tr>
<td>Cox-Ingersoll-Ross1</td>
<td>( \sigma_0 \cdot \exp(-\alpha(u - t)) )</td>
</tr>
<tr>
<td>Historical</td>
<td>Best fit to historical</td>
</tr>
<tr>
<td></td>
<td>term structure movements</td>
</tr>
<tr>
<td>Implied</td>
<td>Best fit to market options data</td>
</tr>
</tbody>
</table>

---

1 For an explanation, see Heath, Jarrow, Morton (1992)

2 While we assume throughout that an initial curve is available, estimating it is not necessarily a trivial matter.

---

1 g(t) is a deterministic function. See Heath, Jarrow, Morton (1992)
Table 1 presents some specific interest rate models and their corresponding HJM volatility functions. Some simple volatility functions, such as the first three entries in table 1, actually lead to path-independent models. However, insisting on a path-independent evolution of rates severely limits the choice of volatility functions. Path-independent models often introduce negative interest rates. Rather than accept these constraints, we have developed efficient methods for path-dependent pricing. As a result, one is free to choose virtually any volatility function, and can even let market data determine the choice.

Figure 1 reveals that, under a path-dependent model, pricing path-dependent derivatives (such as mortgage-backed securities, exotic options or amortising swaps) is no more difficult than pricing an ordinary option. Whereas the path-independent lattice has many paths leading to the terminal nodes, the path-dependent tree has a unique path to each final node.

Although the exponential growth of the path-dependent tree is disconcerting, the surprising fact is: path-dependent prices can be computed in real time on personal computers. Table 2 shows running times (on a 25MHz/50 MHz DX2 486 PC) required to price a five-year cap and a one x five-year European swaption under a path-dependent one-factor model.

Due to the path-dependence, running time roughly doubles for each additional time step. But figure 2 demonstrates that surprisingly few steps are needed to obtain accurate prices. The graph shows the relative error in the HJM prices for eight different European swaptions as a function of the number of time steps. All swaptions were one x five-year but had different strike prices and initial term structures. We determined the error by comparing the price at each step to the 12-step price. Notice that beyond five steps the error is always within 0.5%, corresponding to three-digit accuracy. Other instruments exhibit similar convergence patterns.

Having shown that HJM is clearly fast enough for real-time deal valuation, consider pricing an entire book. The simplest

<table>
<thead>
<tr>
<th>Branches</th>
<th>Running time (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cap</td>
</tr>
<tr>
<td>3</td>
<td>0.05</td>
</tr>
<tr>
<td>4</td>
<td>0.12</td>
</tr>
<tr>
<td>5</td>
<td>0.26</td>
</tr>
<tr>
<td>6</td>
<td>0.51</td>
</tr>
<tr>
<td>7</td>
<td>1.00</td>
</tr>
<tr>
<td>8</td>
<td>2.02</td>
</tr>
<tr>
<td>9</td>
<td>4.05</td>
</tr>
</tbody>
</table>
way to price the book is to price each deal sequentially. Although five steps seem sufficient, even with seven steps 200 caps can be priced in 200 seconds and 200 swaps in 68 seconds. If necessary, running times can be reduced by:
- Reducing the number of time steps used in valuing each deal. This results in a dramatic decrease in running time, with only a small loss of precision. Moreover, when aggregating hundreds of deals, the small pricing errors on each deal tend to cancel out.
- Distributing the work among several workstations. This parallelisation is well suited to book pricing since individual deals can be priced independently and aggregated in one final step.

Volatility functions are at the heart of the HJM approach. How should they be chosen? One might force HJM to match a particular model such as Ho-Lee or Cox-Ingersoll-Ross by letting \( \sigma(t) \) equal one of the first three entries in Table 1. But, without the path-independence constraint, HJM volatility functions can assume more complex, realistic shapes. In fact, market data can be used to infer volatility functions. We outline two market-driven approaches for choosing volatility functions, analogous to Black-Scholes historical and implied volatility estimation.

Volatility functions can be estimated statistically from historical term structure movements. For example, Ron Kahn finds the volatility function which best explains the observed fluctuations of US Treasury rates over the period 1980–90. This function is shown in Figure 3. The volatility function describes a nearly parallel shift in the forward curve, but with short rates rising more (in an upward move) or falling more (in a downward move) than long rates. In other words, short rates are more volatile than long rates. Figure 4 shows a set of possible term structures after one month’s evolution of this one-factor model.

An implied method for volatility estimation chooses volatility functions that best fit market derivative prices. In practice, the model would be calibrated to the prices of the most liquid instruments and then used to price less heavily traded instruments. Figure 5 shows an implied volatility function for an HJM model fit to Libor cap data in March 1992. In contrast to the historical volatility function, the hump in volatility before one year is more pronounced.

To contrast HJM’s approach with a more conventional one, consider how a trader using the Black model prices and hedges a cap. The constituent caplets are valued separately and independently, each with its own Black volatility. A two-year caplet might be priced with a 20% volatility while a 2½-year caplet is priced with 19%. It is unclear how to aggregate and hedge the risk associated with these two caplets. Moreover, since the Black model is driven solely by the variance of the short rate at caplet expiration dates, to price swaps the trader must use another, potentially inconsistent model. HJM, on the other hand, offers one consistent model which can be used to value and hedge all interest rate options.

If plotted, Black caplet volatilities look like the HJM volatility function in Figure 5, but have completely different meanings. The HJM volatility function describes the level and direction of the random movements of each part of the term structure.
Under HJM, the price of a three-year caplet, for example, will be influenced by all values of the volatility function between zero and three years.

Under a one-factor model, all interest rate changes are driven by a single source of uncertainty. For most reasonable volatility functions, all rates will move instantaneously in the same direction. Options which depend on yield curve twists, however, can be more accurately valued with a two-factor model. In such a model, two sources of randomness drive the evolution. Equation (2) replaces equation (1):

\[
\begin{align*}
\text{ft} + \text{d}t, \text{u} & = \text{ft}, \text{u} + \sigma(\cdot) \text{dZ} + \\
& + \sigma(\cdot) \text{dZ}_2 + \text{drift} (\cdot) \text{dt}
\end{align*}
\]

Under a two factor model, the term structure is subject to two different kinds of random shocks, each characterised by its own volatility function. Figure 6 shows the best historical second factor as found by Kahn. This function describes a twist in the forward rate curve: a positive shock in Z_2 causes short rates to fall and long rates to rise; a negative shock has the opposite effect.

Finally we turn to the critical issue of risk management. One HJM model, with one consistent set of assumptions, prices and hedges an entire book. To understand HJM’s risk management applications, we examine traditional bucket analysis from an HJM perspective.

Bucket analysis is a popular technique for tracking the exposure of interest rate books. The term-to-maturity axis is divided into several (say 10) segments (eg, zero to three months, three to six months, six months to one year, and so forth). Each instrument’s sensitivity to shifts in rates over a particular segment is calculated by simply repricing the instrument with the perturbed term structure. These sensitivities are then aggregated over all instruments to give the book’s sensitivity to movements of various parts of the term structure.

While bucket analysis can be used with any pricing model, it has a special interpretation in the HJM framework. A shift in the zero to three-month rate that leaves the rest of the curve unchanged is the movement described by an HJM volatility function which is flat and positive between zero and three months and zero everywhere else. Hedging against 10 such shifts amounts to hedging against 10 different HJM factors, each with a volatility function which is flat and positive over part of the time axis and zero everywhere else. The kind of rate curve shifts described by these volatility functions are clearly unusual. Once the volatility function has been chosen, one should put special emphasis on hedging against the movements that the volatility function describes.

This leads us to the notion of delta in an HJM context. Under a one-factor model with a particular volatility function, the delta of an instrument is simply the change in price of the instrument caused by a shift in the forward rate curve of the form described by the volatility function. Deltas are added across instruments in the usual way, and a position is hedged (with respect to this factor) if its overall delta is zero. In a two-factor model, an instrument has two deltas, each measuring sensitivity to rate movements of the form described by each respective volatility function. In general, hedging these two risks requires two hedging instruments.

Of course, no two-factor, or even three-factor model will perfectly describe the future behaviour of the term structure. Realistic hedging strategies guard against changes both inside and outside the model. We recommend first hedging a position with respect to the one or two factors being used for pricing (with the volatility functions chosen by one of the methods outlined earlier). Second, the book’s sensitivity to the kinds of changes not prescribed by the model should be closely monitored. These include a bucket analysis of sensitivity to rate changes, and measurements of the sensitivities to changes in the level and structure of volatility.

HJM models provide a coherent framework for interest rate pricing and risk management. As computing technology advances, more sophisticated extensions of HJM will come within range. These include models with more than two factors, currency models where two interest rate curves evolve (see Amin and Jarrow) and models allowing default risk (see Jarrow and Turnbull). HJM establishes not only a consistent benchmark for valuation and risk management of interest rate derivatives but also facilitates future, more complex, applications.

References


Heath, D.R. and A. Morton, 1992, Bond pricing and the term structure of interest rates, Econometrica 60(1), pages 77–105


Jarrow, R. and S. Turnbull, 1990, Pricing options on financial securities subject to credit risk, unpublished working paper, Cornell University


David Heath is professor of operations research, and Robert Jarrow professor of finance at Cornell University, Ithaca, New York. Andrew Morton is assistant professor of information and decision sciences at the University of Illinois, Chicago. Mark Spindel is HJM project co-ordinator at Barra