Diffusion processes in finance. Diffusion processes have found numerous applications in finance. Two of the most successful applications have been in the area of portfolio theory and option pricing theory (see Jarrow and Rudd 1983 and Jarrow 1988 for overviews of both areas). We will briefly explore each application to illustrate the usefulness of diffusion processes.

A diffusion process can be thought of as a stochastic process which is \textit{strong Markov} and has \textit{continuous} sample paths (Protter 1990: 187). Let us first clarify this definition. A stochastic process is a random variable indexed by time. In finance, the stochastic process under consideration most often applies to the equilibrium price process followed by a financial security, say a stock. The stock’s price process is said to be \textit{Markov} if the current value of the stock at any date is sufficient to determine its future (probabilistic) evolution. It is a \textit{strong Markov} process if the stock price process satisfies the condition that when trading the stock at any time based on its past history, the price at which the trade is executed is still sufficient to determine the future evolution of the stock price process. Finally, the stock’s price process has \textit{continuous} sample paths if its graph exhibits no jumps; that is, if observed continuously, plotted stock prices would be a continuous curve.

The canonical example of a diffusion process for finance is the geometric Brownian motion. Formally, let $s(t)$ be the stock price at time $t$. The stock price is said to follow a geometric Brownian motion if

$$s(t) = s_0 \exp\{\mu t + \sigma W(t)\},$$

where $\mu$ and $\sigma$ are positive constants, $s_0$ is the initial stock price and $W(t)$ is a standard Brownian motion. A \textit{standard Brownian motion} is a stochastic process which starts at zero, and whose changes over non-overlapping time intervals are independent and normally distributed random variables.

This is the canonical stock price process in finance since it implies that instantaneous stock returns follow a random walk with drift. This property is often held synonymous with the notion of ‘market efficiency’.

Diffusion processes are useful for finance theory because they provide a stochastic process where one can utilize the powerful stochastic calculus, and in particular, a simple form of the Fundamental Theorem of Stochastic Calculus or Itô’s Lemma. Itô’s Lemma is a tool for stochastically differentiating functions of diffusion processes. Intuitively speaking, this implies that knowledge of the stochastic process followed by the stock will be sufficient to characterize the stochastic process followed by any suitably differentiable function of the stock price and time. This transmission of knowledge from the stock price process to a function of the stock price is the crucial inference needed in both option pricing theory and portfolio theory. We next illustrate this procedure for each of these areas within finance.

The traditional problem in the theory of option valuation is to value a European type call option on a stock where the underlying stock price follows a diffusion process. To simplify the analysis, we often assume that interest rates are constant across time. A \textit{European-type call option} is a financial security which gives its owner the right (but not the obligation) to purchase the stock at a fixed price, called the \textit{exercise price}, at a fixed future date, called the \textit{maturity} date. These securities (or their near relatives – American options) trade on organized exchanges. For illustration purposes, let the current date be labelled time 0. Consider a European call option on the stock $s(t)$ with an exercise price of $K > 0$ and a maturity date of $T > 0$. Investor rationality requires that at the maturity date, the call’s value is:

$$C(T) = \max[s(T) - K, 0].$$

That is, if the stock price at maturity exceeds the exercise price, the investor should purchase the stock for $K$ dollars, with a cash flow of $s(T) - K$. Otherwise, he should let the option expire worthless. This gives a cash flow of zero. Of course, we are assuming that there are no transaction costs nor taxes. Hence the call is worth $\max[s(T) - K, 0]$ at maturity. This is called the \textit{boundary condition} for the call.

The call’s value at maturity is seen to be a simple function of the stock’s price at maturity. One can now postulate that the call’s value at any earlier date $t$ is also a function of the stock price at that date, $s(t)$, and time, $t$. Formally, we assume that the call’s value is some twice continuously differentiable function of these arguments, written as:

$$C(s(t)).$$

If $s(t)$ is a diffusion process, then Itô’s Lemma can now be applied to characterize the stochastic process followed by the European call option. In fact, Itô’s Lemma implies that changes in the call’s value over an ‘infinitesimal’ time interval $[t, t + \Delta t]$ are perfectly correlated (linearly related) to changes in the stock’s value over the same ‘infinitesimal’ interval. The linear relationship is described by the proportionality coefficient, $\delta = dC(t, s(t))/ds(t)$. This partial derivative ($\delta$) is called the ‘hedge ratio’. It describes how much the call’s value changes for a unit change in the stock price.

Consequently, selling $\delta$ shares of the stock at time $t$ and buying one European call option will create a riskless
position over the ‘infinitesimal’ time interval \([t, t + \Delta t]\). To avoid arbitrage, therefore, it must earn the riskless rate. This is known as the ‘law of one price’. From this no-arbitrage condition, algebra generates a deterministic partial differential equation satisfied by the European call’s value, whose solution yields the desired result. If the stock price follows a geometric Brownian motion, for example, then the resulting formula is known as the Black–Scholes model. Of course, different diffusion processes for the stock price yield different call values. Another notable model obtainable in this way is the constant elasticity of variance diffusion model (see Jarrow and Rudd 1983).

In portfolio theory, the canonical problem is to determine the optimal dynamic portfolio holdings of a budget-constrained price-taking investor. The investor usually selects his portfolio from among a finite set of non-redundant securities and his selection is constrained by his available wealth. The optimal holdings are determined by selecting those securities which generate the maximum expected utility of consumption through time.

More formally, the objective function can be written as

\[ E(\int_0^T u(C(t), t) \, dt), \]

where \(E(\cdot)\) is the expectations operator conditional upon the information available at time 0, \(C(t)\) is the investor’s consumption at date \(t\), \(T\) is the end of the trading horizon, and \(u(C(t), t)\) is a twice continuously differentiable utility function of consumption and time. The budget constraint is that the investor’s consumption and portfolio holdings at any date \(t\) must be no greater than his aggregate wealth at that date. The aggregate wealth at date \(t\) consists of the initial wealth at time 0 (the start of the model) plus any previously accumulated portfolio gains/losses and less any previous consumption.

This description of the budget constraint implies that the wealth of the investor at any date \(t\) is a function of the past history of the previous consumption and portfolio selections (prior to time \(t\)) and the past evolution of the security prices.

For general stochastic processes for security prices, the time-\(t\) conditional expected utility of consumption over any future time period \((t, T)\) will depend on the entire past history of the evolution of security prices. This representation of the objective function is very difficult to manipulate. But if security prices are assumed to follow diffusion processes, then the time-\(t\) conditional expected utility function simplifies considerably.

The strong Markov characteristic of the diffusion process implies that this conditional expected utility function can be written as some function of only the time-\(t\) or current values of the security prices. This simplification, along with the sample path continuity of security prices, implies that the simple form of Itô’s Lemma can now be applied. This application characterizes the stochastic process for the instantaneous change in the time-\(t\) conditional expected utility as a function of the instantaneous changes in the security prices at time \(t\) and the portfolio position held at time \(t\). In fact, this instantaneous change in expected utility can be shown to be at most a quadratic function of the portfolio holdings. This characterization, along with Bellman’s principle of optimality, now allows standard calculus to be utilized to determine the optimal time-\(t\) portfolio holdings (see Jarrow 1988).

The restriction to diffusion process for the stochastic process for security prices provides the simplification needed to solve the canonical portfolio selection problem. It also enables us explicitly to solve the option valuation problem. Recent research endeavours, however, have been directed toward relaxing this diffusion process assumption. The relaxations have been concerned with both properties of the diffusion process, (i) the strong Markov property, and (ii) the continuous sample path condition. Relaxation of the strong Markov property allows one to consider path dependences in the evolution of security prices. This is of considerable importance in option valuation, for example, when pricing interest-rate options (see Heath, Jarrow and Morton 1992). Relaxation of the continuous sample path property allows one to consider discrete information flows. This is important in portfolio theory since information often derives from events (such as catastrophes) that happen suddenly, without prior notice. Significant progress has to be made in generalizing the previous theories in this direction, using the newly developed martingale approach to stochastic calculus (see Protter 1990). In the derivations, mathematical theorems about martingales replace the mathematical theorems about strong Markov processes used in the previous analytic derivations. Good articles on these methods and the results obtainable, respectively, in option pricing theory and portfolio theory are given by Harrison and Pliska (1981) and Karatzas, Lehoczky and Shreve (1987).

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See also CONTINUOUS TIME STOCHASTIC MODELS; CONTINUOUS TIME STOCHASTIC PROCESSES; DYNAMIC PROGRAMMING; MARTINGALES; OPTION PRICING THEORY; OPTIONS; STOCHASTIC CALCULUS.

BIBLIOGRAPHY


dilution. See BOND COVENANTS; GOING PUBLIC; WARRANTS.

direct investment. See FOREIGN DIRECT INVESTMENT.