A SIMPLE FORMULA FOR OPTIONS ON DISCOUNT BONDS

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I. INTRODUCTION

This paper has two purposes. First, it provides the detailed derivation of the closed-form solution for a European call option on a default-free discount bond contained in Heath, Jarrow and Morton (1992). Second, this paper relates this solution to an open question relating to Merton's (1973) stochastic interest rate call valuation model.

In his paper, Merton obtained a closed-form solution for a European-type call option on a risky asset under stochastic interest rates. To obtain this formula, the risky asset's return dynamics were assumed to follow a diffusion process with a

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deterministic volatility, an example of which is the lognormal process. Furthermore, Merton postulated the existence of a default-free, discount bond maturing at the option's expiration date whose price process satisfied four conditions: (1) it is a diffusion process, (2) the bond's price at maturity is unity, (3) the instantaneous volatility of the bond's return is deterministic, and (4) the volatility is zero at the bond's maturity. Implicitly, a fifth condition, (5) that the bond price process be consistent (i.e., arbitrage free) with respect to some spot rate process was also imposed. The existence of a bond price process satisfying these five conditions and independent of the expectations hypothesis remains an open question.¹

Under this hypothesis, Merton obtained a closed-form formula for a European-type call option on the traded risky asset that does not explicitly depend upon investor preferences. The formula resembled the Black–Scholes model but with two modifications. First, the discount factor in the Black–Scholes model involving the riskless interest rate was replaced by the price of the zero-coupon bond maturing at the option's expiration date. Second, the volatility in the Black–Scholes model was replaced by an expression involving the volatilities of the risky asset, the discount bond, and their instantaneous correlations.²

Since the traded risky asset can itself be a default-free discount bond, this model generates a bond option formula. Recognizing this fact, Ball and Torous (1983) brought us a step closer to answering the open question by explicitly constructing a discount bond price process which satisfies conditions (1)–(3) of Merton's postulate. Unfortunately, Cheng (1987) showed that the Ball and Torous process was inconsistent with condition (5). This observation cast additional doubt upon the existence of a bond price process satisfying Merton's hypotheses and invalidated Ball and Torous's bond option formula.

Using the recent insights of Heath et al. (1992), this paper resolves this existence question by constructing a bond price process which simultaneously satisfies conditions (1)–(5) and is independent of the expectations hypothesis. This process, in turn, is then used to price a European-type call option on a default-free discount bond. This formula does not explicitly depend upon investor preferences, and it is easy to calculate since it resembles the Black–Scholes model but with different input parameters. The input parameters can be estimated from time series data on bond prices.

The next section of this paper provides the bond price process and the bond option formula and proves the above assertions. Detailed calculations are contained in an appendix. A brief summary section completes the paper.

II. THE MODEL

Consider an economy with frictionless markets and continuous trading. Let \( P(t,T) \) denote the time \( t \) price of a default-free discount bond which pays \$1 at date \( T = t \). We assume that \( P(t,T) > 0 \) for all \( t \in [0,T] \), \( P(T,T) = 1 \), and that \( \partial \log \)}
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\[ P(t,T) = \exp \left[ - \int_t^T f(t,v) \, dv \right] \]  

(1)

We assume that forward rates satisfy the following diffusion process. For fixed but arbitrary \( T > 0 \),

\[ df(t,T) = \alpha(t,T) dt + \sigma_1 \, dW_1(t) + \sigma_2 e^{-(\lambda/2)(T-t)} \, dW_2(t), \]  

(2)

where \( \alpha(t,T) \) is a random process, \( \sigma_1, \sigma_2, \lambda \geq 0 \) are nonnegative constants, and \( \{ W_1(t), W_2(t) \} \) are independent Brownian motions. Two independent Brownian motions are seen to influence changes in forward rates. The first, \( W_1(t) \), can be identified with a constant shift in all maturity forward rates. The second, \( W_2(t) \), influences short maturity forward rates more than it does long maturity forward rates due to the exponential term \( (e^{-(\lambda/2)(T-t)}) \). As such, it can be interpreted as a factor affecting the spreads between long and short maturity forward rates. Two Brownian motions, rather than one, are included to allow for a richer class of forward rate curve evolutions. Furthermore, it also allows different maturity discount bond prices to be imperfectly correlated. In contrast, a single Brownian motion would necessarily imply instantaneous perfect correlation between different maturity discount bonds.

Heath et al. (1992) show that this process is consistent with an arbitrage-free economy if and only if the drift term satisfies

\[ \alpha(t,T) = -\sigma_1 \phi_1(t) - \sigma_2 e^{-(\lambda/2)(T-t)} \phi_2(t) + \sigma_2^2 (T-t) - 2(\sigma_2^2/\lambda)(e^{-(\lambda/2)(T-t)} - 1), \]  

(3)

where \( \phi_1(t), \phi_2(t) \) are random functions, independent of \( T \), which represent arbitrary market prices for risk. We impose this restriction for the remainder of the paper.

From Itô’s lemma, the bond price process implied by condition (2) and (3) is:

\[ \frac{dP(t,T)}{P(t,T)} = \left[ -\int_t^T \alpha(t,v) \, dv + \frac{1}{2} \sigma_1^2 (T-t)^2 + \frac{1}{2} \sigma_2^2 \left( \int_t^T e^{-(\lambda/2)(v-t)} \, dv \right)^2 \right] dt \]

\[ -\sigma_1 (T-t) dW_1(t) + [2\sigma_2(e^{-(\lambda/2)(T-t)} - 1)/\lambda] dW_2(t). \]  

(4)

This satisfies the five conditions of Merton’s hypothesis. Conditions (1)–(4) are obvious by inspection (using Equation [1]). Condition (5) is guaranteed by the theorems of Heath et al. (1992). Hence, expression (4) gives a nontrivial bond price process, consistent with arbitrage-free pricing and consistent with arbitrary market prices for risk.

The bond process is seen to have two factors and, hence, bonds of different maturities are imperfectly correlated. Unfortunately, from expression (2), we see...
that this model allows negative forward rates with positive probability. This will be true, however, for any process satisfying Merton’s hypothesis as condition (3) implies a Gaussian process for the logarithm of the bond’s price.

Next, consider a European call option on the bond $P(t,T)$ with an exercise price of $K$ and maturity date $t^*_n$, where $0 \leq t \leq t^*_n \leq T$. Let $C(t)$ denote the value of this call option at time $t$. By definition, the value of the call option at maturity is

$$C(t^*_n) = \max[P(t^*_n, T) - K, 0].$$

To apply Merton’s model, the traded asset is the long-term bond, $P(t, T)$, with dynamics given by expression (4). The bond in Merton’s model is the short-term bond, $P(t, t^*_n)$, with maturity $t^*_n$. This bond also follows the price dynamics given by expression (4) with $T$ replaced by $t^*_n$. Direct substitution into Merton’s model gives the following bond option formula:

$$C(t) = P(t, T)\Phi(h) - KP(t, t^*_n)\Phi(h-q),$$

where $h = [\log(P(t, T)/KP(t, t^*_n)) + (\lambda/2)q^2]/q$, $q^2 = \sigma^2 (T-t^*_n)^2 + (4\sigma^2 \lambda^3)(e^{\lambda t^*_n} - e^{\lambda t}) - \lambda (e^{(\lambda/2)t} - e^{(\lambda/2)t^*_n})$, and $\Phi(*)$ is the cumulative normal distribution function. This formula resembles the traditional Black–Scholes model with the simple volatility parameter replaced by

$$q^2 = \text{var}(d[P(t, T)/P(t, t^*_n)])/[P(t, T)/P(t, t^*_n)].$$

the instantaneous variance of the ratio of the bond prices, that is, the variance of the forward price of a bond $P(t^*_n, T)$.

This model does not explicitly depend upon investor preferences. It depends only on the two bond prices $\{P(t, t^*_n), P(t, T)\}$, the contract’s provisions $\{K, t^*_n\}$, and the forward rate’s process parameters $\{\sigma_1, \sigma_2, \lambda\}$. These parameters can be estimated from past observations of the forward rates.

There are two one-factor special cases of this model, one obtained by setting $\sigma_2 = \lambda = 0$ and the other obtained by setting $\sigma_1 = 0$. The first model is a continuous-time analogue of the Ho and Lee path-independent model (Heath et al. 1992), where the volatility parameter is $q^2 = \sigma^2 (T-t^*_n)^2 (t^*_n-t)$. In this case, only one parameter from the forward rate’s process needs to be estimated. The second model, an analogue to the Vasicek model (Brenner 1989), has two parameters where short-term rates are more volatile than longer-term rates. Unfortunately, these one-factor models have all maturity bonds being instantaneously perfectly correlated.

III. CONCLUSION

This paper provides a closed-form solution for a European-type call option on a default-free discount bond which does not explicitly depend upon investor preferences. This valuation formula corresponds to a specific example of Merton’s (1973)
model, where bond prices follow a specific diffusion process. The bond price process of this paper can be generalized to the case where \( \sigma_1 \) is replaced by the deterministic function \( \sigma_1(t,T) \), and \( \sigma_2 e^{-\lambda/2(T-t)} \) is replaced by the deterministic function \( \sigma_2(t,T) \).

**APPENDIX**

This appendix briefly outlines an alternative derivation of expression (5) based on Heath et al. (1992).

Define

\[
\tilde{W}_i(t) = W_i(t) - \int_0^t \Phi(v) dv \quad \text{for } i = 1,2.
\]

Under expressions (2) and (3), there exists a martingale measure \( \tilde{Q} \) such that \( \tilde{W}_i \) are independent Brownian motions and

\[
C(t) = \tilde{E}_t \{ \max \{ P(t^*,T) - K, 0 \} \exp \{-\int f(y,y) dy\} \},
\]

where \( \tilde{E}_t(\cdot) \) is the time \( t \) conditional expectation using \( \tilde{Q} \). Upon substitution,

\[
C(t) = \tilde{E}_t \{ \exp \{-\int f(t,y) dy - \int A(y,y) dy - \int \int \sigma_1(y-s) ds dy\} \}.
\]

\[
= \tilde{E}_t \{ \exp \{-\int \int \sigma_2 \exp \{-(\lambda/2)(y-s)\} d\tilde{W}_2(s) dy - \int \int \sigma_1 d\tilde{W}_1(s) dy\} \}.
\]

\[
= \max \{ \exp \{-\int f(t,y) dy - \int \int \sigma_2 \exp \{-(\lambda/2)(y-s)\} d\tilde{W}_2(s) dy \}
- \int \int \sigma_1 d\tilde{W}_1(s) dy - \int A(t^*,y) dy - \int \int \sigma_1(y-s) ds dy\} - K, 0 \}
\]

where

\[
A(t^*,T) = \sigma_2^2 \int_t^T \exp \{-\lambda \int_0^T dW\} \int \exp \{- \lambda \int_0^T (T-s) dy \} dy ds.
\]
This last expression is the restricted drift $\bar{G}(\tau^*, T)$ in Heath et al. (1992). To evaluate this expression, the trick is to define

$$Z = \int_0^T \int_0^T \sigma_1 d\tilde{W}_1(s) + \int_0^T \sigma_2 \exp \left(-\lambda/2(y - s)\right) dy d\tilde{W}_2(s),$$

which is normally distributed, and

$$X = \int_0^y \int_0^T \sigma_1 d\tilde{W}_1(s) dy + \int_0^y \sigma_2 \exp \left(-\lambda/2(y - s)\right) d\tilde{W}_2(s) dy,$$

which is also normally distributed. Next, calculate

$$E(X \mid Z = u) = \exp \left(\mu_1 + \frac{\mu_2^2}{2}\right)$$

where

$$\mu_1 = \mu \text{ cov}(X, Z)/\text{var}(Z)$$

$$\mu_2 = \text{var}(X) \left\{ 1 - \left[\text{cov}(X, Z)^2/\text{var}(X)\text{var}(Z)\right]\right\}.$$

These can be obtained by taking expectations over the stochastic integrals. With these definitions,

$$C(t) = \exp \left(-\int f(t, y) dy - \int A(y, y) dy - \int \sigma_t^2 (y - s) ds dy\right).$$

$$E^{\mu} [E(X \mid Z = u) \max \left\{ \exp \left(-\int f(y, y) dy - \int A(y, y) dy\right), \tau \right\} d\tau - \int \sigma_t^2 (y - s) ds dy - u - K, 0),$$

where $E^{\mu}(\cdot)$ is expectation over $Z = u$, and $E(X \mid Z = u)$ is expectation over $X$ given $Z = u$. Substitution and algebra give the solution in the text.

NOTES

1. Merton (1973, note 43) gives an explicit example of a price process satisfying these conditions, however, he assumes that the expectations hypothesis holds. Independent of this paper, Jamshidian (1989) also provides an example.
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2. An example of this formula is contained in expression (5) below, where \( P(t,T) \) is the risky asset price, \( P(t,t') \) is the discount bond's price maturing at the option's expiration date, and \( q \) is the expression for the modified volatility.

3. For the technical conditions concerning measurability of the random process \( \alpha(t,T) \), see Heath, Jarrow, and Morton (1992).

4. It is important to note that the presence of negative interest rates does not admit arbitrage opportunities. This follows since our model does not include cash. All trading is in terms of a money market account.

5. An alternate derivation of expression (5), based on Heath et al. (1992) is provided in the Appendix.

6. This case corresponds to a preference-free version of Merton's (1973, note 43) expectations hypothesis example. It also corresponds to a formula obtained independently by Jamshidian (1989) obtained under a one-factor model spot rate process but without the expectations hypothesis.

REFERENCES


