Option Pricing with Random Volatilities in Complete Markets*

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Abstract. This article presents the theory of option pricing with random volatilities in complete markets. As such, it makes two contributions. First, the newly developed martingale measure technique is used to synthesize results dating from Merton (1973) through Eisenberg, (1985, 1987). This synthesis illustrates how Merton's formula, the CEV formula, and the Black-Scholes formula are special cases of the random volatility model derived herein. The impossibility of obtaining a self-financing trading strategy to duplicate an option in incomplete markets is demonstrated. This omission is important because option pricing models are often used for risk management, which requires the construction of synthetic options.

Second, we derive a new formula, which is easy to interpret and easy to program, for pricing options given a random volatility. This formula (for a European call option) is seen to be a weighted average of Black-Scholes values, and is consistent with recent empirical studies finding evidence of mean-reversion in volatilities.

Key words: option pricing, synthetic options, martingale measure, European call option

1. Introduction

Much of the existing literature on pricing options with random volatilities is for complete markets (see Merton, 1973; Cox and Ross, 1976; Eisenberg, 1985, 1987; Johnson and Shanno, 1987; and Scott, 1987). Notable exceptions are the analyses by Wiggins (1987), Hull and White, (1987), and Stein and Stein, (1991). There are two advantages to maintaining complete markets when pricing options: one, simplicity and two, the ability to construct a synthetic option. Simplicity is important for practical implementation, as it is very difficult to identify a usable, yet realistic general equilibrium model for pricing assets. In addition, the equilibrium based pricing models of Wiggins (1987), Hull and White (1987), and Stein and Stein (1991) do not provide a procedure for constructing synthetic options. The ability to construct a synthetic option position is essential for modern risk management techniques. Given these advantages, the purpose of this article is to extend the insights of Eisenberg (1985, 1987) and Johnson and Shanno (1987) by utilizing the newly developed martingale measure techniques to investigate the pricing of options with random volatilities in complete markets.

A synthesis of the existing techniques for random volatility option pricing, in the context of this new methodology, is developed below. To provide this synthesis, a general model for random volatility stock price dynamics is postulated. Using this process as a frame

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of reference, the restrictions necessary to obtain Merton's (1973) complete markets model are detailed. Merton's model includes, as special cases, the well-known Black-Scholes and CEV option pricing models. A discussion of why synthetic options cannot be constructed using the stock and bond alone, without these restrictions, is provided. We show how the introduction of an additional traded asset, imperfectly correlated to the stock, can complete the market. This asset could be a market index or a derivative security on the stock. General valuation formulae and the procedure for constructing synthetic options are discussed. This technique, in its most abstract form, can also be used to price options given any two sources of risk.

To illustrate the general theory, we provide an example that should prove useful in practical applications. This example is for a stock whose random volatility exhibits both mean reversion and is correlated to the instantaneous random returns on a market index like the S&P 500. This stochastic process is consistent with recent empirical evidence regarding the form of the volatility process, see Merville and Piepea (1989) for supporting evidence. An easily computed expression for a European option is provided. The option's value is seen to be a weighted average of Black-Scholes values.

An outline for this article is as follows. Section 2 presents a complete markets economy. Section 3 provides an example completed by a market index. A conclusion summarizes the article.

2. The Complete Markets Economy

We consider a continuous trading economy with the trading interval $[0, \tau]$ for $\tau < +\infty$. We are given a probability space $(\Omega, F, P)$ and two standard independent Brownian motions $\{W_1(t), W_2(t): t \in [0, \tau]\}$ initialized at zero. We denote the augmented\footnote{An easily computed expression for a European option is provided. The option's value is seen to be a weighted average of Black-Scholes values.} filtration generated by $\{W_1(t), W_2(t): t \in [0, \tau]\}$ as $\{F_t: t \in [0, \tau]\}$ where $F_t = F_{\tau}$. Let $E(\cdot)$ denote expectation with respect to the probability $P$.

Initially, two assets trade: a stock with a random volatility and a money market account. For convenience, we assume that the spot rate of interest is constant and equal to $r > 0$. This assumption is easily relaxed along the lines of Heath, Jarrow, Morton (1992) and it is not crucial to the subsequent analysis. The traded money market account, therefore, earns interest at this rate ($r$), i.e.,

$$B(t) = \exp\{rt\} \text{ for } t \in [0, \tau].$$

The stock price process has no cash dividends and is given by

$$S(t, \omega) = S(0)\exp\left\{\int_0^t \mu(y, \omega)dy - (1/2)\int_0^t \sigma^2(y, \omega)dy + \int_0^t \sigma(y, \omega)dW_1(y)\right\}$$

where

$S(0)$ is a strictly positive constant, the volatility and drift coefficients $(\sigma(y, \omega), \mu(y, \omega))$ are both predictable with respect to $\{F_t: t \in [0, \tau]\}$, and satisfy
\[
\int_0^T \frac{\mu(y, \omega)}{dy} < +\infty \text{ a.e. } P, \int_0^T \sigma^2(y, \omega)dy < +\infty \text{ a.e. } P, \text{ with } \\
P(\sigma(t, \omega) > 0 \text{ for all } t \in [0, \tau]) = 1.
\]

In its stochastic differential form, this can be written as (omitting the dependence on \( \omega \in \Omega \)):\(^3\)

\[
dS(t) = S(t)[\mu(t)dt + \sigma(t)dW_1(t)].
\] \( (3) \)

This stock price process has a random volatility because \( \sigma(t, \omega) \) is functionally dependent (through \( \omega \in \Omega \)) on the information set generated by both Brownian motions \( \{W_1(t), W_2(t): t \in [0, \tau]\} \). Special cases of expression (3) are of some interest. If \( \sigma(t, \omega) \) is set equal to a fixed positive constant independent of \( \omega \in \Omega \), then expression (3) yields the Black-Scholes economy. Alternatively, if \( \sigma(t, \omega) \) is a deterministic function of time \( t \), the current stock price \( S(t) \), and independent of the second Brownian motion \( (W_2(t)) \), then we get Merton's (1973) economy (of which the constant elasticity of variance (CEV) process is a special case (see Jarrow and Rudd, 1983).

From the perspective of option pricing theory, however, the interesting aspect of expression (3) is due to the fact that both the volatility and drift coefficients can depend on the path of the second Brownian motion \( \{W_2(t): t \in [0, \tau]\} \). Without this additional dependence, and given some mild integrability conditions, the above economy is easily shown to be complete.\(^4\) As this subcase has already been adequately investigated by Merton (1973), we restrict our attention to the more complicated, but realistic situation. For the remainder of the article, we assume that both \( \mu(t) \) and \( \sigma(t) \) depend nontrivially on the information set generated by \( \{W_2(t): t \in [0, \tau]\} \).

Under the above structure, we claim that the economy is incomplete. To see this intuitively, suppose that the volatility has the stochastic differential:\(^5\)

\[
d\sigma(t) = \pi(t)dt + \sum_{i=1}^2 \beta_i(t)dW_i(t).
\] \( (5) \)

Given this characterization of the volatility process and expression (2), two independent sources of randomness are seen to influence the stock's value at a future date \( \{W_1(t), W_2(t)\} \). Thus, to create a synthetic option, we need the ability to hedge these two risks. But, as there is only one traded asset, the stock, this is impossible.\(^6\)

To value contingent claims, we would like to utilize the equivalent martingale measure to construct the risk neutral value operator.\(^7\) In incomplete markets, not all contingent claims can be synthetically constructed, and these nonredundant claims cannot be uniquely priced by arbitrage arguments alone. To solve this nonuniqueness problem, equilibrium pricing arguments can be invoked, for example see Hull and White (1987) or Wiggins (1987). Instead, following Eisenberg (1985, 1987), to complete the market we allow trading in an additional asset whose time \( t \) price \( A(t) \) satisfies the following stochastic process:
\[ A(t, \omega) = A(0) \exp \left\{ \int_0^t \alpha(y, \omega) dy - \sum_{i=1}^{2} \int_0^t (1/2) \eta_i(y, \omega)^2 dy + \sum_{i=1}^{2} \int_0^t \eta_i(y, \omega) dW_i(y) \right\} \]

where

\[ A(0) \] is a strictly positive constant, the drift and volatility coefficients \( \alpha(y, \omega) \), \( \eta_i(y, \omega) \) for \( i = 1, 2 \) are predictable \( \{F_t; t \in [0, \tau]\} \), and satisfy

\[ \int_0^t |\alpha(y, \omega)| dy < +\infty \text{ a.e. } P, \int_0^t \eta_i(y, \omega)^2 dy < +\infty \text{ a.e. } P \]

for \( i = 1, 2 \) with \( P(\eta_2(t, \omega) \neq 0 \text{ for all } t \in [0, \tau]) = 1 \).

In a differential form,

\[ dA(t) = A(t) \left[ \alpha(t) dt + \sum_{i=1}^{2} \eta_i(t) dW_i(t) \right]. \]  

The asset's price process is similar to that given for the stock in expression (3). Note however, that the second Brownian motion influences this asset's price dynamics through a nonzero volatility coefficient \( (\eta_2) \). The first volatility coefficient \( (\eta_1) \) could be zero.

The asset \( A(t) \) could be a market index, \(^8\) like the S&P 500, or a contingent claim issued against the stock, like a call option. The first example is emphasized below. The second example is similar to that used in Jones (1984) for jump-diffusion processes. The subsequent mathematics is identical for either of these two cases. Indeed, (6) allows the asset to have random volatilities, which is implied by either of these two examples. These two examples are by no means exhaustive. In fact, any asset correlated with both the stock \( (S(t)) \) and the volatility \( (\sigma(t)) \) is acceptable.

Intuitively, since there are now two random shocks \( \{W_1(t), W_2(t); t \in [0, \tau]\} \) and two imperfectly correlated traded risky assets \( (S(t), A(t)) \) to hedge these risks, the markets should be complete. This is, in fact, the case as the next proposition shows.

**Proposition (Complete Markets)**

Define \( \phi_1(t) = -(\mu(t) - r)/\sigma(t) \) and

\[ \phi_2(t) = -[\alpha(t) - r/\eta_2(t)] - [\eta_1(t)\phi_1(t)/\eta_2(t)] \]

If \( \int_0^T \phi_1(t)^2 dt < +\infty \text{ a.e. } P \) and \( \int_0^T \phi_2(t)^2 dt < +\infty \text{ a.e. } P \),

\[ E \left[ \exp \left\{ \sum_{i=1}^{2} \int_0^T \phi_i(t) dW_i(t) - (1/2) \sum_{i=1}^{2} \int_0^T \phi_i(t)^2 dt \right\} \right] = 1, \]
E \left\{ \exp \left\{ \int_0^t (\phi(t) + \sigma(t))dW_1(t) + \int_0^t \phi_2(t)dW_2(t) - \frac{1}{2} \int_0^t \phi_1(t) \right\} \right\} = 1,
\left(\frac{1}{2} \int_0^t \phi_2(t)^2 dt \right) = 1;

E \left\{ \exp \left\{ \sum_{i=1}^2 \int_0^t (\phi_i(t) + \eta_i(t))dW_i(t) - \frac{1}{2} \sum_{i=1}^2 \int_0^t \phi_i(t)^2 dt \right\} \right\} = 1;

\text{then there exists a unique probability } \bar{P}\text{ defined by}
\begin{align*}
\frac{d\bar{P}}{dP} &= \exp \left\{ \sum_{i=1}^2 \int_0^t \phi_i(t)dW_i(t) - \frac{1}{2} \sum_{i=1}^2 \int_0^t \phi_i(t)^2 dt \right\} 
\tag{10}
\end{align*}

\text{such that both } \{S(t)/B(t): t \in [0, \tau] \}\text{ and } \{A(t)/B(t): t \in [0, \tau] \}\text{ are } \bar{P}\text{ martingales with respect to } \{F_t: t \in [0, \tau] \}.

\text{Proof: In the Appendix.} \quad \text{Q.E.D.}

This proposition states that under the stated hypotheses, there exists a unique equivalent martingale measure for this economy. The two stochastic processes \(\phi_1(t)\) and \(\phi_2(t)\) defined in expressions (8) and (9) correspond to the market prices of risk associated with the two Brownian motions \(W_1(t)\) and \(W_2(t)\), respectively. These quantities are the unique solutions to the well-known arbitrage-free restrictions:

\begin{align*}
\mu(t) - r &= -\phi_1(t) \sigma(t) \\
\alpha(t) - r &= -\phi_1(t) \eta_1(t) - \phi_2(t) \eta_2(t). \tag{11}
\end{align*}

These arbitrage-free restrictions guarantee that the excess return on the stock and the asset must be linearly related to the market prices of risk times the sensitivity of the stock and the asset to each of these risks. These restrictions are needed in a crucial step in the proof of this proposition (see the Appendix).

The first four conditions of this proposition guarantee that the market price of risk processes are well-behaved. By a result from Harrison and Pliska (1981), the uniqueness of the equivalent martingale measure implies that markets are complete. Hence, any contingent claim can be synthetically constructed using a self-financing trading strategy in the stock \(S(t)\) and index \(A(t)\).

A useful corollary to this proposition is obtained by invoking Girsanov's Theorem (see Karatzas and Shreve, 1988), which asserts that
\[ \tilde{W}_i(t) = W_i(t) - \int_0^t \phi_i(y)dy \quad \text{for } i = 1, 2 \]  
(12)

are independent, standard Brownian motions on the stochastic basis \( \{\Omega, F, \tilde{P}, (F_t; t \in [0, \tau])\} \). The corollary is that there exists a risk neutral transformation of the original economy where both the stock and asset earn the risk free rate. This transformation is obtained by substituting (12) into expressions (3), (7) along with expression (11), i.e.,

\[ dS(t) = S(t)[rdt + \sigma(t)d\tilde{W}_1(t)] \quad \text{and} \]
\[ dA(t) = A(t) \left[ rdt + \sum_{i=1}^{2} \eta_i(t)d\tilde{W}_i(t) \right]. \]
(13)

Since the markets are complete, given any contingent claim \( \{C(t); t \in [0, \tau]\} \) with a random payoff \( X \) at time \( \tau \), which is \( F \) measurable and satisfying \( \tilde{E}(X^2) < +\infty \), its time \( t \) price is:

\[ C(t) = \tilde{E}(X/B(\tau))B(t). \]
(14)

For example, if the contingent claim is a European call option on the stock \( S(t) \) with exercise price \( K > 0 \) and maturity date \( \tau \), then \( X = \max[S(\tau) - K, 0] \) and

\[ C(t) = \tilde{E}(\max[S(\tau) - K, 0]/B(\tau)|F_t)B(t). \]
(15)

Substitution of expression (13) into expression (15) yields:

\[ C(t) = \tilde{E} \left[ \max \left( S(t)e^{-(1/2)\int_t^\tau \sigma(y)dy + \int_t^\tau \sigma(y)d\tilde{W}_1(y)} - Ke^{-(r\tau - t)} \right) \bigg| F_t \right]. \]
(16)

The difficulty in evaluating expression (16) is in the fact that both \( \int_t^\tau \sigma(y)d\tilde{W}_1(y) \) and \( \int_t^\tau \sigma^2(y)dy \) are random with unknown distributions. The distribution for \( \int_t^\tau \sigma(y)d\tilde{W}_1(y) \) and \( \int_t^\tau \sigma^2(y)dy \) could be obtained via monte-carlo simulation, given fixed specifications for the volatility coefficients \( \sigma(y) \). Alternatively, for practical applications we can compute expression (16) for specified \( \sigma(y) \) by using a multinomial tree along the lines of Madan, Milne, and Shefrin (1989). With a specification of \( \sigma(y) \), the multinomial approximation of \( \int_t^\tau \sigma(y)d\tilde{W}_1(y) \) and \( \int_t^\tau \sigma^2(y)dy \) can be explicitly computed. We illustrate this procedure with an example in the next section.

3. Example: Creating a Synthetic Option Using a Market Index

This section illustrates the general analysis of the preceding section using an example. The example itself, however, is of considerable independent interest as it is selected based on recent empirical evidence relating to the form of the volatility process.
Merville and Piepea (1989) investigated the volatilities of 25 optioned stocks and the futures index over the ten year period (1975–1985). Using implicit volatilities from Black-Scholes models, they find evidence consistent with the belief that stock volatilities follow a mean-reverting diffusion process with noise. Stein and Stein (1991) argue that the noise component could be due to a misspecification from using the Black-Scholes model to approximate a random volatility option model (see Jarrow and Wiggins, 1989 for related discussion). Consequently, Stein and Stein (1991) ignore the noise term and concentrate on only the diffusion component. We will do likewise. Last, Merville and Piepea also show that changes in volatilities are correlated across stocks. As we choose our volatility process to be correlated to a market index, different stock volatilities will be correlated with each other and this condition is satisfied as well (see footnote 1). The process we specify reflects these observations.

Let the stochastic process for the stock’s volatility be:

\[ d\sigma(t) = (\tilde{\sigma} - \gamma \sigma(t))dt + \sigma(t)\beta_2 dW_2(t) \]  \hspace{1cm} (17)

where

\[ \sigma(0), \tilde{\sigma}, \gamma, \text{ and } \beta_2 \text{ are strictly positive constants, and} \]

\[ \int_0^T (\tilde{\sigma} - \gamma \sigma(y))^2 dy < +\infty \text{ a.e. P.} \]

Expression (17) has four notable features. The first is that the volatility follows a mean reverting process with a long-run value of \( \tilde{\sigma} \). This is consistent with Merville and Piepea (1989). The second is that the volatility coefficient of the volatility is proportional to (\( \beta_2 \)), a constant. Given the restrictions on the other parameters of the volatility process, non-negative \( \beta_2 \) implies non-negative volatilities. This is an improvement over the process employed in Stein and Stein (1991) who allow volatilities to be negative with positive probability. Third, the randomness in the stock’s volatility is correlated with the index (\( A(t) \)) through the second Brownian motion \( \{W_2(t) : t \in [0, \tau]\} \). If other volatilities follow similar processes, then they will be correlated to each other through the \( dW_2(t) \) term. This is consistent with the evidence in Merville and Piepea (1989). Interpreting \( A(t) \) as a market index, expression (17) then states that the stock’s volatility is correlated with it. As both expressions (3) and (7) hold, the stock’s return is also correlated with the market index (through the first Brownian motion \( W_1(t) \)). Fourth, the first Brownian motion does not influence the volatility’s dynamics. The consequence of which is that \( \{\sigma(s) : s \leq t\} \) is independent of the information set generated by the first Brownian motion \( \{W_1(s) : s \leq t\} \). If we assume that the market prices of risk, \( \phi_1(t) \) and \( \phi_2(t) \), are constants, then \( \{\sigma(s) : s \leq t\} \) is also independent of the information set generated by the risk adjusted Brownian motion \( \{W_1(s) : s \leq t\} \). This simplifies the valuation formula (16) considerably. We impose this restriction upon the model.

Consider a European type call option on the stock \( (S(t)) \) with exercise price \( K \) and maturity date \( \tau \). By expression (16), its time 0 value is:
\[ C(0) = \mathbb{E} \left( \mathbb{E} \left[ \max \left( S(0)e^{-(1/2) \int_0^{\tau} \sigma^2(y) \, dy + \int_0^{\tau} \sigma(y) \, d\tilde{W}_2(y) - Ke^{-r\tau}, 0 \right) \right] \mid \{ \sigma(t) : t \in [0, \tau] \} \right) \right]. \]  

(18)

Conditional on \{ \sigma(t) : t \in [0, \tau] \} (or equivalently \{ \tilde{W}_2(s) : t \in [0, \tau] \}), the inner conditional expectation generates the well-known Black-Scholes formula. A simplification yields:

\[ C(0) = \int_0^\infty (S(0)e^{-r\tau} \Phi(h - \nu) - Ke^{-r\tau} \Phi(h - \nu)) \, dF(\nu) \text{ where} \]

\[ h = (\log(S(0)/Ke^{-r\tau}) + (1/2)\nu^2)/\nu, \]

\[ \nu^2 = \int_0^\tau \sigma^2(u) \, du, \text{ and} \]

\[ F(\nu) \text{ is the distribution function for } \nu. \]

The call’s value is thus a weighted average of Black-Scholes values, each with a differing “modified” volatility \( \nu \). The weights of the differing “modified” volatilities \( \nu \) correspond to the likelihood of each occurring \( (dF(\nu)) \). This call value differs from that obtained in Stein and Stein (1991) only to the extent that the distribution \( dF(\nu) \) differs. The distribution for \( F(\nu) \) is unknown, yet it can be shown to depend upon the parameters \( (\sigma(0), \sigma, \gamma, \beta_2; \phi_2) \). To see this, expression (17) is used to rewrite \( \nu^2 \) as:

\[ \nu^2 = \sigma(0)^2 \int_0^\tau \exp \left\{ \int_0^y (2(\bar{\sigma}/\sigma(u) - \gamma) - \beta_2^2 + 2\beta_2 \phi_2) \, du + 2\beta_2 \tilde{W}_2(y) \right\} \, dy. \]

(20)

Using expression (20), it is easy to see that as \(-\phi_2 \) increases, everything else constant, call values decline. Thus, as investors become more risk averse, call values decline. As the underlying volatility increases \( (\sigma(0) \text{ increases}) \) or if its long-run value is expected to increase \( (\bar{\sigma} \text{ increases}) \), call values increase.

To obtain the distribution for the “modified” volatility \( \nu \), one can perform a monte-carlo simulation on expression (20), and use the resulting distribution to compute expression (19). More importantly, because markets are complete, there is a dynamic trading strategy in the stock and market index \( \{S(t), A(t)\} \) that creates the synthetic call. This trading strategy can be obtained via a multinomial lattice approximation to the stochastic system involving \( \{S(t), A(t), \sigma(t)\} \). The option’s “deltas” can then be obtained using the standard procedures (see Madan, Milne, Shefrin, 1989).

Stein and Stein (1991) observe that at-the-money Black-Scholes values are “nearly linear” in volatility. For out-of-the-money options, however, an increase in the volatility of the

volatility increases the value of the option. Note that this is a different effect than increasing the current volatility, and it generates the well-known "smile" in implied volatilities. Thus, Stein and Stein argue that their random volatility model is consistent with the observed "smile" in implied volatilities. These arguments also apply to our pricing formula as it is similarly an average of Black-Scholes prices.

4. Conclusion

This article presents the theory of option pricing with random volatilities in complete markets. The contribution of this article is two-fold. First, it extends and synthesizes the existing literature by using the newly developed martingale measure technique. Second, it provides an example where a computable solution for a European call option is obtainable. This example may prove useful in practical applications.

Notes

1. Among the findings reported by Merville and Piepea is standard deviations of both individual stocks and the market, as measured by implied standard deviations using non-random variance option pricing models, exhibit mean-reversion with arithmetic rather than geometric Brownian motion. They also report that these implied standard deviations have a noise component and that changes in individual stock-return standard deviations are correlated with changes in the standard deviation of the returns to the market.

2. The filtration is augmented to be $P$-complete and right continuous. For subsequent footnotes, we denote the augmented filtration generated by $\{W_t(t) : t \in [0, \tau]\}$ as $\{F^1_t : t \in [0, \tau]\}$. The restriction of $P$ to $F^1_\tau$ is denoted $P^1$ by as well.

3. This stock price process could be generalized to

$$dS(t) = S(t)[\mu(t)dt + \sigma_1(t)dW_1(t) + \sigma_2(t)dW_2(t)]$$

where $\sigma_1(t)$ and $\sigma_2(t)$ are random and satisfy the same measurability and integrability conditions imposed on $\sigma(t)$. The subsequent mathematics follows in an identical fashion. In the text, we concentrate on the simpler expression (3) for expositional clarity.

4. See the Appendix, proposition (A.1).

5. Necessary and sufficient conditions for this are that

(i) $\sigma(t)$ is a semi-martingale, and

(ii) its bounded variation component is absolutely continuous with respect to Lebesgue measure.

The proof of this statement follows from the martingale representation theorem.

6. Formally, to prove this assertion, we use a well-known result from Harrison and Pliska (1981) or Jarrow and Madan (1991). The result is: Given the existence of a probability measure $\bar{P}$ equivalent to $P$, making $\{S(t)/B(t) : t \in [0, \tau]\}$ a $\bar{P}$-martingale with respect to $\{F_t : t \in [0, \tau]\}$, then the economy is complete if and only if $\bar{P}$ is unique. We show that there exist a continuum of non-unique equivalent martingale measures in proposition A.2 of the Appendix.

7. Letting $X^t$ be $F_t$ measurable, the risk neutral present value operator is defined to be $\tilde{E}(X/B(\tau))$ where $\tilde{E}(\cdot)$ is expectation with respect to the equivalent martingale probability measure $\tilde{P}$.

8. To see that $A(t)$ could be a market index, let there be $m$ traded stocks with prices $s_i(t)$ for $i = 1, 2, \ldots, m$. Let $S(t) = s_1(t)$ and

$$A(t) = \sum_{i=1}^{m} N_i(t)s_i(t) \text{ with } N_i(t) > 0 \text{ and } \sum_{i=1}^{m} dN_i(t)s_i(t) = 0$$
Let $Z_i(t)$ for $i = 1, \ldots, m$ be independent Brownian motions, and let

$$dZ_i(t) = \mu_i(t) dt + \sigma_i(t) dZ_i(t)$$

for all $i$.

Then,

$$dA(t) = \sum_{i=1}^{m} N_i(t) \mu_i(t) dt + \sum_{i=1}^{m} N_i(t) \sigma_i(t) dZ_i(t).$$

Define

$$dW_1(t) = dZ_1(t)$$

and

$$dW_2(t) = \sum_{i=2}^{m} N_i(t) \sigma_i(t) dZ_i(t) / \left[ \sum_{i=2}^{m} N_i(t)^2 \sigma_i(t)^2(t) \right]$$

then

$$dS(t) = S(t)[\mu(t) dt + \sigma(t) dW_1(t)]$$

and

$$dA(t) = \sum_{i=1}^{m} N_i(t) \mu_i(t) dt + N_i(t) S(t) \sigma(t) dW_1(t) + \left[ \sum_{i=2}^{m} N_i(t)^2 \sigma_i(t)^2(t) \right] dW_2(t)$$

which satisfy (3) and (7).

This example could be extended to allow correlation across the various stocks. Indeed, just let $Z_i(t)$ be correlated for $i = 1, \ldots, m$. This, in turn, will induce a correlation across $dW_1(t)$ and $dW_2(t)$. An orthogonalization at this point returns us to the process in footnote 3 and expression (7).

9. The volatility process in expression (17) could be modified to that used in Stein and Stein (1991), i.e.,

$$d\sigma(t) = [\sigma - \gamma \sigma(t)] dt + \beta \sigma(t) dW_2(t).$$

Their process allows negative volatilities with positive probability.

In this case expression (19) still applies, but the distribution $dF(\nu)$ differs, and is given in Stein and Stein (1991); expression (12).

10. The form of this expression is not new, see Hull and White (1987) or Stein and Stein (1991).


12. Even though there are complete markets, the option price still depends on the second market price of risk $\phi_2$. This result is due to the transformation of the volatility process (17) to that process which holds under the risk adjusted Brownian motion $d\tilde{W}_2(t) = d\tilde{W}_2(t) + \phi_2 dt$.

Appendix

Proposition A.1. If both $\mu(\gamma, \omega)$, $\sigma(\gamma, \omega)$ are predictable with respect to $\{F_t^1: t \in [0, \tau]\}$, and suitably integrable (to be made precise in the proof) then the market is complete.

Proof. Using the result stated in the text from Harrison and Pliska (1981) or Jarrow and Madan (1991), we show that there exists a unique equivalent probability measure $\tilde{P}^1$ on $(\Omega, F_\tau^1)$ making $S(t)/B(t)$ a $\tilde{P}^1$-martingale. The definition of $\tilde{P}^1$ and $F_t^1$ are given in footnote 2.
(Existence)

Define \( \phi_1(t) = \frac{\mu(t) - r}{\sigma(t)} \) and \( Z_1(t) = \exp \left\{ \int_0^t \phi_1(u) dW_1(u) - \frac{1}{2} \int_0^t \phi_1^2(u) du \right\} \).

Assume \( E(Z_1(\tau)) = 1 \) and \( E(Z_1(\tau)S(\tau)/S(0)B(\tau)) = 1 \). These are the integrability conditions.

Define \( \tilde{P}^1(A) = \int_A Z_1(\tau) dP^1 \). The first two integrability conditions make \( \tilde{P}^1 \) a probability on \( F_1^\tau \). The third makes \( S(t)/B(t) \) a \( \tilde{P}^1 \) martingale with respect to \( \{F_t^\tau : t \in [0, \tau]\} \).

(Uniqueness)

The volatility matrix (a scalar) \( \sigma(y) \) is nonsingular for all \( t \in [0, \tau] \) a.e. P. By Jarrow and Madan (1991), \( \tilde{P}^1 \) is unique.

Q.E.D.

Proposition A.2. (Incomplete Markets)

Define \( \phi_1(t) = - (\mu(t) - r)/\sigma(t) \)

Let

\[
\int_0^\tau \phi_1^2(t) dt < +\infty \text{ a.e.}
\]

For any predictable process \( \{\phi_2(t) : t \in [0, \tau]\} \) such that

\[
\int_0^\tau \phi_2^2(t) dt < +\infty \text{ a.e.,}
\]

\[
E \left\{ \exp \left( \sum_{i=1}^2 \int_0^\tau \phi_i(t) dW_i(t) - \frac{1}{2} \sum_{i=1}^2 \int_0^\tau \phi_i^2(t) dt \right) \right\} = 1,
\]

and

\[
E \left\{ \exp \left( \int_0^\tau (\phi_1(t) + \sigma(t)) dW_1(t) + \int_0^\tau \phi_2(t) dW_2(t) - \frac{1}{2} \int_0^\tau (\phi_1(t) + \sigma(t))^2 dt - \frac{1}{2} \int_0^\tau \phi_2(t)^2 dt \right) \right\} = 1;
\]

then \( \tilde{P}(\phi_2) \) defined by \( d\tilde{P}(\phi_2)/dP = \exp \left\{ \sum_{i=1}^2 \int_0^\tau \phi_i(t) dW_i(t) - \frac{1}{2} \sum_{i=1}^2 \int_0^\tau \phi_i^2(t) dt \right\} \) is an equivalent probability measure making \( \{S(t)/B(t) : t \in [0, \tau]\} \) a martingale with respect to \( \{F_t^\tau : t \in [0, \tau]\} \).
Proof. The statement of proposition A.2 identifies the candidates for equivalent martingale probability measures. We show that each of these make $S(t)/B(t)$ a $\tilde{P}(\phi_2)$ martingale with respect to \( \{F_t: t \in [0, \tau]\} \).

Note that the first two conditions in the hypothesis guarantee that $\tilde{P}(\phi_2)(\Omega) = 1$.

Next, define $L(t) \equiv (d\tilde{P}(\phi_2)/dP) \mid F_t$. It is well known that $S(t)/B(t)$ is a $\tilde{P}$ martingale if and only if $S(t)L(t)/B(t)$ is a $P$ martingale. We show this later condition.

Writing out $S(t)L(t)/B(t)$ yields

\[
S(t)L(t)/B(t)S(0) = \exp \left\{ \int_0^t (\mu(y) - r)dy - (1/2) \int_0^t \sigma^2(y)dy + \int_0^t \sigma(y)dW_1(y) \right. \\
+ \left. \sum_{i=1}^2 \int_0^t \phi_i(y)dW_i(y) - (1/2) \sum_{i=1}^2 \int_0^t \sigma_i^2(y)dy \right\}
\]

\[
= \exp \left\{ \int_0^t \phi_1(y) + \sigma(y)dyW_1(y) + \int_0^t \phi_2(y)dyW_2(y) \right. \\
- \left. (1/2) \int_0^t [\phi_1(y) + \sigma(y)]^2dy - (1/2) \int_0^t \phi_2(y)^2dy \right\}
\]

\[
\cdot \exp \left\{ \int_0^t (\mu(y) - r)dy + \int_0^t \phi_1(y)\sigma(y)dy \right\}.
\]

By the definition of $\phi_1(t)$, the second term is equal to 1. The remaining expression is a non-negative supermartingale. The third condition in the hypothesis guarantees that it is a martingale.

Q.E.D.

Proof of the Proposition in Section 2 of the Text. The first two conditions following expressions (8) and (9) in the text yield that $\tilde{P}(\Omega) = 1$. Define $L(t) \equiv (d\tilde{P}/dP) \mid F_t$.

(Existence)

We show that $S(t)L(t)/B(t)$ and $A(t)L(t)/B(t)$ are $P$ martingales.

The identical argument as in proposition A.2 shows $S(t)L(t)/B(t)$ is a $P$ martingale. Next, we write (after algebra)

\[
A(t)L(t)/B(t) = \exp \left\{ -(1/2) \sum_{i=1}^2 \int_0^t [\phi_i(y) + \eta_i(y)]^2dy + \sum_{i=1}^2 \int_0^t [\phi_i(y) + \eta_i(y)]dW_i(y) \right\}
\]

\[
\exp \left\{ \int_0^t (\sigma(y) - r)dy + \sum_{i=1}^2 \int_0^t \phi_i(y)\eta_i(y)dy \right\}.
\]
By the definition of $\phi_2$, the second term in this product is 1. The remaining expression is a nonnegative supermartingale. The fourth condition in the hypothesis guarantees that this is a martingale.

(Uniqueness)

The volatility matrix,

$$
\begin{pmatrix}
\sigma(t) & 0 \\
\eta_1(t) & \eta_2(t)
\end{pmatrix}
$$

is non-singular for all $t \in [0, \tau]$ a.e. $P$, by the assumptions that both $\sigma(t)$ and $\eta_2(t)$ are nonzero. By Jarrow and Madan (1991), $\bar{P}$ is unique.

References


