Chapter 8

Pricing Interest Rate Options

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1. Introduction

After lying dormant for many years, the theory for pricing interest rate options has recently experienced an explosive expansion.¹ This expansion has been fueled, at least in part, by an increased understanding of the martingale pricing technology.² The purpose of this paper is to review and to synthesize this expanding literature. A secondary purpose is to clarify the use of the martingale pricing methodology in the context of interest rate options.

For pedagogical reasons, this theory is presented in the context of a discrete trading, discrete state space economy. The continuous trading economies studied in the literature can be obtained (formally and intuitively) by taking limits. However, all the relevant concepts and insights are illustrated in this simpler setting, for use in the continuous time analogue. Furthermore, most continuous trading models must be implemented, in practice, through a discrete trading, discrete state space approximation. Consequently, little is lost by taking the discrete time perspective.

This paper is divided into two parts. Each part studies a different problem to which the interest rate option pricing theory is applied. The first problem is to value the entire zero coupon bond price curve, given the prices of only a few bonds (one, two or three) which lie upon it. This is the academic's version of 'arbitraging the yield curve'. This problem was analyzed first historically as well [see Vasicek, 1977; Dothan, 1978; Richard, 1978; Brennan & Schwartz, 1979; Langetieg, 1980; Rendleman & Bartter, 1980; Cournot, 1982; Ball & Torous, 1983; Cox, Ingersoll & Ross, 1985; Artzner & Delbaen, 1989; Longstaff, 1989].

The second problem is to price contingent claims (options) on the zero coupon bond price curve. Here, however, we are given the prices of all the zero coupon bonds. This literature is more current and has recently exploded [see Ho & Lee, 1986; Bliss & Ronn, 1989; Heath, Jarrow & Morton, 1990, 1991, 1992; Hull & White, 1990; Babbs, 1990; Turnbull & Milne, 1991; Shirakawa, 1991]. Of course,

¹ See the references to this paper.
² See Harrison & Kreps [1979], Harrison & Pliska [1981], Heath & Jarrow [1987].
the two problems are related, and this relationship is analyzed below. To simplify
the subsequent discussion, let us refer to the first problem as 'zero-curve arbitrage'
and the second problem as 'option pricing'.

One contribution of this review is the identification of a theoretical difference
between the model structures used to solve these two different problems. The
distinction relates to the manner in which the spot rate process' parameters are
specified within the model. Zero curve arbitrage models require an exogenous
specification of the spot rate process. In option pricing models, however, the spot
rate process is endogenously determined by an exogenous specification of the
evolution of the entire zero coupon bond price curve. This distinction will be
important in the subsequent analysis.

This paper is subdivided into six sections. Section 2 introduces the notation and
terminology. This is the most difficult section to master. Section 3 studies the zero-
curve arbitrage models. It is subdivided into the two standard approaches utilized
(spot rate process models and bond price process models). The equivalence
between these two approaches is studied using the martingale pricing technology.
Section 4 studies the option pricing models. It is also subdivided into the two
different approaches utilized (spot rate process models and bond price process
models). A comparative analysis of these two approaches is provided. Section 5
explores the continuous time empirical specifications of the various models, and a
summary section completes the paper.

2. The model: terminology and notation

The most difficult part of the interest rate options literature is the terminology
and notation. This section presents the general framework, from which all the
subsequent models will be derived. For simplicity, and simplicity alone, a one
factor economy (a binomial branching process) is provided. Extension to multiple
factors (multinomial branching processes) is straightforward and briefly discussed
where appropriate.

We consider a discrete time, discrete state space economy. The trading intervals
are \{0, 1, 2, \ldots, T\} where \(T < +\infty\). The state space for the economy is best
described by examining the tree diagram given in Figure 1. The model starts at
time 0. At time 1, one of two states occur: up denoted by 'u' and down denoted by
'd'. Up occurs with probability \(q_0 > 0\) and down occurs with probability \(1 - q_0 > 0\).
To avoid an overly complicated diagram, only the up probabilities are displayed.
We let \(s_1\) denote the state at time 1, i.e., \(s_1 \in \{u, d\}\).

Next, depending upon the state at time 1, the process can go up again or down.
There are four possible states at time 2: \{uu, ud, du, dd\}. These are distinct states
as the ordering is important. We let \(s_2\) denote an arbitrary state at time 2. We
allow the probability of jumping up at time 1, \(q_1(s_1)\), to depend upon the state at
time 1.

The state process continues in this fashion until time \(T\). At time \(T\) there are \(2^T\)
possible states, represented by all sequences of 'u's and 'd's where the ordering is
important. A generic element is denoted $s_T$. The probability of an upward jump at
time $T - 1$, $q_{T - 1}(s_{T - 1})$, depends upon the state (or history of the process) at time
$T - 1$.

In summary, at time $t$, the state of the process is denoted by $s_t$ with $s_t \in \{\text{all} \ t \ \text{sequences of } u\text{'s and } d\text{'s (where the ordering is important)}\}$. The probability of
jumping up at time $t$ is $q_t(s_t) > 0$. If the process jumps up, the new state is $s_t u$. Otherwise, it is $s_t d$. The generalization of this process from two branches to three
or more branches is straightforward and omitted.

Traded are all maturity zero coupon bonds and a money market account. We let
$P(t, \tau; s_t)$ denote the time $t$ price at state $s_t$ of a default free zero-coupon bond
paying 1 dollar at time $\tau$ for sure where $0 \leq t \leq \tau \leq T$. We require that bond
prices are strictly positive, i.e., $P(t, \tau; s_t) > 0$ for all $t, \tau$ and $s_t$; and that the bonds
are default free, i.e., $P(\tau, \tau; s_{\tau}) = 1$ for all $\tau$ and $s_{\tau}$.

The time $t$ forward rate at state $s_t$ for the period $[\tau, \tau + 1]$, denoted $f(t, \tau; s_t)$, is defined by

$$f(t, \tau; s_t) \equiv \frac{P(t, \tau; s_t)}{P(t, \tau + 1; s_t)} \text{ where } 0 \leq t \leq \tau < T. \quad (1)$$

This corresponds to the rate contractible at time $t$ for a riskless loan over the time
period $[\tau, \tau + 1]$. Note that this is a modification of the usual definition because
the magnitude of $f(t, \tau; s_t)$ is one plus a percent. Expression $(1)$ can be re-
written as:
\[ P(t, \tau; s_t) = \prod_{j=t}^{\tau-1} \left( \frac{1}{f(t, j; s_t)} \right) \quad \text{for } t \leq \tau - 1. \]  

(2)

The spot rate at time \( t \) under state \( s_t \), denoted \( r(t; s_t) \), is defined by

\[ r(t; s_t) = f(t, t; s_t). \]  

(3)

Lastly, the money market account's time \( t \) value under state \( s_t \) denoted \( B(t; s_{t-1}) \) given an initial dollar investment at time 0 is defined by

\[ B(t; s_{t-1}) = \prod_{j=0}^{t-1} r(j; s_j) \quad \text{for } t \geq 1 \text{ where } B(0) = 1 \]  

(4)

and where the \( s_j \)'s on the right side of expression (4) are the first \( j \) coordinates of \( s_{t-1} \) on the left side of expression (4). The time \( t \) value of this account is known at time \( t - 1 \) because it earns the spot rate over \([t-1, t]\), and this is known at time \( t - 1 \). This observation accounts for the fact that the state \( s_{t-1} \) and not \( s_t \) is within expression (4). We can rewrite expression (4) recursively as:

\[ B(t; s_{t-1}) = B(t - 1; s_{t-2}) r(t - 1; s_{t-1}) \quad \text{for } t \geq 1. \]  

(5)

Next, we turn to the description of the stochastic processes followed by the above quantities. For the moment, we concentrate on the evolution of the zero-coupon bond price curve and the spot rate of interest.

It is easiest to describe the evolution of the zero-coupon bond price curve as given in Figure 2. At time 0 we have an initial zero-coupon bond price curve which is represented as the \( T + 1 \) vector of prices: \( (P(0, T), P(0, T - 1), \ldots, P(0, 1), 1) \).

At time 1 it either moves 'up' to a different \( T \)-vector of prices or 'down' to a \( T \)-vector of prices. The shortest maturity bond paying off a dollar at time 0 is removed from this new vector at time 1. The last entry in this new vector is the

![Tree diagram representation of the zero-coupon bond price curve evolution.](image-url)
bond which had one period until maturity at time 0 \((P(0, 1))\), and it pays one dollar for sure (independent of the state which occurs) at time 1. Note that each new \(T\)-vector of bond prices depends upon the state which occurs \(u\) or \(d\). The probability of going up at time 1 is the probability that \(s_1 = u\), and it is given as \(q_0\). The probability of going down is the complement, \(1 - q_0\).

Next, at time 2, the \(T\)-vector of prices moves up or down to a new and reduced \((T - 1)\)-vector of prices. At each time step, the shortest maturity zero-coupon bond is removed from the price curve after it matures. This reduces the remaining price curve vector by one. Each price vector depends upon the state of the process. The probabilities of moving up or down at time 2 are also state dependent and are given by \(q_1(s_1)\) and \(1 - q_1(s_1)\), respectively.

Continuing to the second to last date in the tree, time \(T - 1\), only one zero-coupon bond remains and it matures at time \(T\). The probabilities for each state possible at time \(T\), \(s_{T-1}\), are given by \(q_{T-1}(s_{T-1})\) for up and \(1 - q_{T-1}(s_{T-1})\) for down. Finally, at time \(T\) the bond matures and pays one dollar independent of the state which occurs.

For analysis and without loss of generality, we decompose the bond price process in Figure 2 as follows:

\[
P(t, \tau; s_t) = \begin{cases} 
P(t - 1, \tau; s_t)u(t - 1, \tau; s_{t-1}) & \text{if } s_t = s_{t-1}u \\
P(t - 1, \tau; s_t)d(t - 1, \tau; s_{t-1}) & \text{if } s_t = s_{t-1}d \end{cases}
\]

for all \(\tau > t, s_{t-1}\) \(\quad (6)\)

where

\[u(t, t + 1; s_t) = d(t, t + 1; s_t) = \frac{1}{P(t, t + 1; s_t)} \quad \text{for all } t, s_t\]

and

\[u(t - 1, \tau; s_{t-1}) > d(t - 1, \tau; s_{t-1}) \quad \text{for all } t - 1, \tau, s_{t-1}.\]

The zero-coupon bond price moves up at time \(t\) by the proportion \(u(t - 1, \tau; s_{t-1})\) or down at time \(t\) by the proportion \(d(t - 1, \tau; s_{t-1})\). Note that the first restriction following expression (6) guarantees that each bond pays one dollar for sure at maturity, i.e., \(P(t + 1, t + 1; s_{t+1}) = 1\) for all \(s_{t+1}\). By definition, the time \(t\) spot rate is:

\[r(t; s_t) = u(t, t + 1; s_t) = d(t, t + 1; s_t) = \frac{1}{P(t, t + 1; s_t)} \quad (7)\]

The stochastic process for the spot rate is depicted in Figure 3. At time 0, the spot rate is \(r(0)\). At time 1, the spot rate is either \(r(1; u)\) or \(r(1; d)\) depending upon whether the state is \(u\) or \(d\). The probability of moving up is given by \(q_0\) and the probability of moving down by \(1 - q_0\). As before, this process continues until time \(T\). For analysis and without loss of generality, we decompose the spot rate process as in expression (8):
Fig. 3. Tree diagram description of the spot rate process.

\[ r(t; s_t) = \begin{cases} 
  r(t - 1; s_{t-1}) \alpha(t - 1; s_{t-1}) & \text{if } s_t = s_{t-1} u \\
  r(t - 1; s_{t-1}) \beta(t - 1; s_{t-1}) & \text{if } s_t = s_{t-1} d 
\end{cases} \]  

(8)

where \( \alpha(t; s_t) \neq \beta(t; s_t) \) for all \( t, s_t \).

The spot rate jumps at time \( t \) by the proportionality factor \( \alpha(t - 1; s_{t-1}) \) if \( u \) occurs, or by the proportionality factor \( \beta(t - 1; s_{t-1}) \) if \( d \) occurs. By expression (7), if the bond price \( P(t, t + 1; s_t) \) moves up at time \( t \), then \( r(t; s_t) \) moves down (and vice-versa). The probability of \( u \) occurring is \( q_{T-1}(s_{T-1}) \).

From expressions (6) and (8) we can derive the stochastic process followed by forward rates and the money market account, respectively. These processes are obtained by direct substitution of expression (6) into expression (1) for forward rates; and expression (8) into (4) for the money market account. This analysis is left for the reader. Given the above, it is now easy to see how to extend the processes to incorporate multiple branches. Simply, at each node, the tree expands into multiple directions and the notation must be expanded accordingly. This expansion can incorporate three branches, four branches, five branches, etc.

3. Zero curve arbitrage

This section studies the valuation models designed to price all bonds on the zero-coupon bond price curve using only the price of a few bonds and an exogenous specification of the spot rate process (or equivalently, a money market account). There are two forms taken by these models. The first model type focuses solely upon the spot rate process, using it as the focal point of the analysis. The second model type focuses upon the specification of a few bond price processes and the money market account. These two types of zero-curve arbitrage models, under various reparameterizations, are the discrete time analogues of Vasicek [1977], Dothan [1978], Richard [1978], Brennan & Schwartz [1979], Langetieg [1980], Rendleman & Bartter [1980], Couradon [1982], Ball
3.1. Spot rate process models

This class of models takes as its initial specification the spot rate process given in Figure 3 and the decomposition provided by expression (8), i.e., given are: \( (r(0), \alpha(t), \beta(t); 0 \leq t \leq T - 1) \). This initial value \( r(0) \) and the stochastic processes \( (\alpha(t), \beta(t); 0 \leq t \leq T - 1) \) completely specify the spot rate process given in Figure 3. The purpose of this model is to endogenously determine \( \left( P(0, \tau) \right) \) for \( \tau = 1, 2, \ldots, T \) and \( (u(t, \tau), d(t, \tau); 0 \leq t < \tau \leq T) \).

The model is completed with the following assumption.

**Assumption R** (Existence and uniqueness of equivalent martingale probabilities). There exist unique strictly positive (time and state) conditional probabilities\(^3\) \( (\pi_0, \pi_1(s_1), \pi_2(s_2), \ldots, \pi_{T-1}(s_{T-1})) = \Pi \) such that \( P(t, \tau; s_t)/B(t; s_t-1) \) are martingales with respect to \( \Pi \) for all \( \tau \in [1, \ldots, T] \).

This assumption states that the time \( t \) conditional expected value under the probabilities \( \Pi \) of a bond’s payout at maturity is its time \( t \) value, after normalization by the money market account:

\[
E^\Pi_t \frac{1}{B(t; s_{t-1})} = \frac{P(t, \tau; s_t)}{B(t; s_{t-1})} \quad \text{for all } 0 \leq t \leq \tau \leq T
\]  

(9)

where \( E^\Pi_t(\cdot) \) is the conditional expectation at time \( t \) with respect to the probabilities \( \Pi \).

Using expression (4), we can rewrite expression (9) as:

\[
E^\Pi_t \left( \frac{1}{\prod_{j=t+1}^{T-1} r(j; s_j)} \right) = P(t, \tau; s_t).
\]  

(10)

This shows that all the bond prices \( (P(0, \tau); 0 \leq \tau \leq T) \) and the stochastic processes determining their evolution through time \( (u(t, \tau), d(t, \tau); 0 \leq t < \tau \leq T) \) are specified by knowing: \( \{\Pi, r(0), (\alpha(t), \beta(t); 0 \leq t \leq T)\} \). Note that knowing the spot rate \( r(0) \) is equivalent [by expression (7)] to knowing the first bond price \( P(0, 1) \). Hence, one bond price on the curve and knowledge of the quantities \( (\Pi, (\alpha(t), \beta(t); 0 \leq t \leq T)) \) is sufficient to determine all the remaining zero-coupon bond prices \( (P(0, 2), \ldots, P(0, T)) \). If these calculated prices differ from the observed market prices, then the price discrepancy indicates the existence of ‘profitable’ trading opportunities. These ‘profitable trading’ opportunities will be discussed in a subsequent section.

\(^3\) Formally, the product of these conditional probabilities \( \pi_0 \pi_1(s_1) \pi_2(s_2) \ldots \pi_{T-1}(s_{T-1}) \) forms a probability measure \( \Pi \) on the discrete state space consisting of all possible \( s_T \).
Note that the bond valuation formula given in expression (10) is independent of the actual probabilities \( q_i(s_t) \) driving the spot rate process. This implies that two different traders who disagree about \( q_i(s_t) \) but agree on \( \{\Pi, r(0), (\alpha(t), \beta(t): 0 \leq t \leq T - 1)\} \), will agree on the bond prices. This is a trivial, but important observation.

The procedure for constructing a synthetic zero-coupon bond from another zero-coupon bond and the money market account, in the above model, will be discussed in Section 3.3 below. The papers for which the above spot rate process model can be interpreted as the discrete time analogue include Vasicek [1977], Dothan [1978], Richard [1978], Brennan & Schwartz [1979], Rendleman & Bartter [1980], Courtadon [1982], Cox, Ingersoll & Ross [1985] and Longstaff [1989].

3.2. Bond price process models

This class of models takes as its initial specification one maturity bond process \( (P(t, T)) \) from the vector stochastic process given in Figure 2 with the decomposition as in expression (6), plus the specification of the spot rate process given in Figure 3 with the decomposition as in expression (8). That is, given are:

\[
\begin{align*}
\{P(0, T), (u(t, T), d(t, T): 0 \leq t \leq T - 1)\} & \quad \text{and} \\
\{r(0), (\alpha(t), \beta(t): 0 \leq t \leq T - 1)\}
\end{align*}
\]

(11)

The purpose of this model is to endogenously determine \( (P(0, r) \) for \( \tau = 1, 2, \ldots, T - 1 \) and \( (u(t, \tau), d(t, \tau): 0 \leq t < \tau \leq T - 1) \).

Given these exogenous processes, we can prove the following lemma.

**Lemma 1** (Complete markets). Given \( \{P(0, T), (u(t, T), d(t, T): 0 \leq t \leq T - 1)\} \) and \( \{r(0), (\alpha(t), \beta(t): 0 \leq t \leq T - 1)\} \) as in expressions (6) and (8), respectively; the market is complete.

By complete, we mean that given any time \( T - 1 \) state contingent payout \( X(s_{T-1}) \in \{X^u(s_{T-2}), X^d(s_{T-2})\} \), there exists a dynamic self-financing trading strategy initiated at time 0 in the bond \( P(t, T) \) and the money market account \( B(t) \) with share holdings \( (n_P(t; s_t), n_B(t; s_t)) \) at time \( t \) under state \( s_t \) for all \( t \in \{0, 1, \ldots, T - 2\} \) such that the payoff to the portfolio at time \( T - 1 \) matches \( X(s_{T-1}) \), i.e.,

\[
\begin{align*}
n_P(T - 2; s_{T-2})P(T - 1, T; s_{T-1}) + \\
+ n_B(T - 2; s_{T-2})B(T - 1, T; s_{T-1}) = X(s_{T-1}) & \quad \text{for all} \ s_{T-1}
\end{align*}
\]

(12)

By self-financing we mean that there are no cash inflows or outflows from the
portfolio after its initiation, i.e.,

\[
np(t; s_t) P(t + 1; T; s_{t+1}) + n_B(t; s_t) B(t + 1; s_{t+1}) =
\]

\[
= np(t + 1; s_{t+1}) P(t + 1; T; s_{t+1}) + n_B(t + 1; s_{t+1}) B(t + 1; s_t)
\]

for all \( t, s_{t+1} \). \( (13) \)

Two observations about this definition are important. First, the contingent claim's payoff must terminate at time \( T - 1 \) as there is no security trading whose payouts differ across the states at time \( T \) (see Figure 2). Second, any contingent claim whose payoffs occur prior to time \( T - 1 \) can also be duplicated using this type of dynamic self-financing trading strategy. The trick is to deposit the contingent claim's payoff at the earlier date into a money market account which transfers it to time \( T - 1 \). The above self-financing trading strategy then applies.

The proof of the lemma contains some important expressions useful for hedging options (or equivalently, constructing synthetic options).

**Proof.** The proof is by backward induction. Consider forming a portfolio at time \( T - 2 \) given state \( s_{T-2} \) to duplicate \( X(s_{T-1}) \). The desire is to find \( np(T - 2; s_{T-2}), n_B(T - 2; s_{T-2}) \) such that at time \( T - 1 \):

\[
np(T - 2; s_{T-2}) P(T - 1; T; s_{T-2}u) +
\]

\[
+ n_B(T - 2; s_{T-2}) B(T - 1; s_{T-2}) = X^u(s_{T-2}) \quad (14a)
\]

and

\[
np(T - 2; s_{T-2}) P(T - 1; T; s_{T-2}d) +
\]

\[
n_B(T - 2; s_{T-2}) B(T - 1; s_{T-2}) = X^d(s_{T-2}) \quad (14b)
\]

Substitution of expression \( (6) \) into \( (14) \), algebra, and noting that \( u(T - 2, T; s_{T-2}) > d(T - 2, T; s_{T-2}) \) implies that equation \( (14) \) has a solution, and it is uniquely given by

\[
np(T - 2; s_{T-2}) = \frac{X^u(s_{T-2}) - X^d(s_{T-2})}{P(T - 2, T; s_{T-2})[u(T - 2, T; s_{T-2}) - d(T - 2, T; s_{T-2})]} \quad (15a)
\]

and

\[
n_B(T - 2; s_{T-2}) = \frac{X^d(s_{T-2})u(T - 2, T; s_{T-2}) - X^u(s_{T-2})d(T - 2, T; s_{T-2})}{B(T - 2, T; s_{T-2})r(T - 2, s_{T-2})[u(T - 2, T; s_{T-2}) - d(T - 2, T; s_{T-2})]} \quad (15b)
\]

The time \( T - 2 \) value of this position, denoted \( X(s_{T-2}) \), is

\[
X(s_{T-2}) = np(T - 2; s_{T-2}) P(T - 2, T; s_{T-2}) +
\]

\[
+ n_B(T - 2; s_{T-2}) B(T - 2; s_{T-2}). \quad (16)
\]
This can be written as:

\[ X^u(s_{T-3}) = n_P(T - 2; s_{T-3}u)P(T - 2, T; s_{T-3}u) + n_B(T - 2; s_{T-3}u)B(T - 2; s_{T-3}u) \]  
(17a)

\[ X^d(s_{T-3}) = n_P(T - 2; s_{T-3}d)P(T - 2, T; s_{T-3}d) + n_B(T - 2; s_{T-3}d)B(T - 2; s_{T-3}d). \]  
(17b)

The next goal is to find \( n_P(T - 3; s_{T-3}), n_B(T - 3; s_{T-3}) \) such that at time \( T - 2 \):

\[ n_P(T - 3; s_{T-3})P(T - 2, T; s_{T-3}u) + n_B(T - 3; s_{T-3})B(T - 2; s_{T-3}) = X^u(s_{T-3}), \]  
(18a)

and

\[ n_P(T - 3; s_{T-3})P(T - 2, T; s_{T-3}d) + n_B(T - 3; s_{T-3})B(T - 2; s_{T-3}) = X^d(s_{T-3}). \]  
(18b)

But, the system in (18) is identical to the problem given in expression (14), except for a time change. As we have already solved that problem, the solution is again given by (15) but with the appropriate time change ('\( T - 2 \)' replaced with '\( T - 3 \)').

By backward induction, we can finally get the shares at time 0, \((n_P(0), n_B(0))\) which yield a position at time 1, which in turn yield a position at time 2, \ldots, so forth, which in turn yields \( X(s_{T-1}) \) at time \( T - 1 \). By construction [expressions (17) and (18)] this portfolio is self-financing. This completes the proof. \( \square \)

We point out, in passing, that if Figure 1 was a trinomial process, then the completeness condition [expression (14)] would involve the nonsingularity of a system of three equations involving the 'u's and d's of two distinct maturity zero-coupon bonds (e.g., the bonds with maturities \( T - 1 \) and \( T \)) plus the money market account. The analysis and proof is a straightforward extension of the above. This can be further generalized to four branches, five branches, and so forth.

An **arbitrage opportunity** is defined to be any dynamic, self-financing trading strategy initiated at time 0 \((n_P(t; s_t), n_B(t; s_t); 0 \leq t \leq T - 1)\) such that its time \( T - 1 \) value, \( X(s_{T-1}) \), satisfies

\[ X(s_{T-1}) \geq 0 \quad \text{for all } s_{T-1}, \]  
(19a)

\[ X(s_{T-1}) > 0 \quad \text{for some } s_{T-1}; \]

and its time 0 value is nonpositive, i.e.,

\[ n_P(0)P(0, T) + n_B(0)B(0) \leq 0. \]  
(19b)

Such a portfolio has positive or zero initial cash flow at time 0, and positive or zero cash flow at time \( T - 1 \). It is a 'money pump'.
To finish the description of the model, we add:

**Assumption P** (No-arbitrage opportunities). There exist no-arbitrage opportunities in this economy.

We have as a direct consequence of Assumption P:

**Lemma 2** (The money market account return versus the $T$-maturity bond return). Given Assumption P,

$$u(t, T; s_t) > r(t; s_t) > d(t, T; s_t) \quad \text{for all } s_t \text{ and } t < T + 1 \quad (20)$$

**Proof.** Suppose not, say that $r(t; s_t) \geq u(t, T; s_t)$ for some $t, s_t$. Then, an arbitrage opportunity can be constructed. To see this, note that over $[t, t + 1]$, the money market account pays off more than the $T$-maturity bond given the state $s_t$ occurs at time $t$. So at time $t$, if state $s_t$ occurs, buy one unit of the money market account and finance it (zero investment) by selling the $T$-maturity bond. The resulting portfolio at time $t$ satisfies expression (19b) and at time $t + 1$, expression (19a). It is straightforward to transform this zero initial wealth position to time 0 (do nothing until time $t$). This arbitrage opportunity yields a contradiction of Assumption P, and completes the proof. □

Lemma 2 relates the returns on the $T$-maturity zero-coupon bond with those of the money market account. But, Assumption P is even stronger. It also relates the returns on the $T$-maturity bond and the money market account to *all* the other zero-coupon bonds. To see this, we need to expand the above definitions of a trading strategy and arbitrage opportunity. The previous definitions only involve the $T$-maturity bond and the money market account. The corresponding generalization of the definition of a trading strategy and the definition of an arbitrage opportunity to include the remaining zero-coupon bonds is straightforward.

**Lemma 3** (Zero-curve valuation). Given Assumption P, let $(n_p(t; s_t), n_B(t; s_t); 0 \leq t \leq \tau - 1)$ be the dynamic self-financing trading strategy in the bond with maturity $T$ and the money market account which pays one dollar at time $\tau$ across all states $s_\tau$ (this exists by Lemma 1), then

$$n_p(0) P(0, T) + n_B(0) B(0) = P(0, \tau). \quad (21)$$

**Proof.** The proof is simple. If expression (21) is violated, say the left side is strictly less than the right side, then form the portfolio $(n_p(t; s_t), n_B(t; s_t); 0 \leq t \leq \tau - 1)$ and short one bond maturing at time $\tau$. By construction, this is an arbitrage opportunity. This contradicts Assumption P. Thus, the left side of expression (21) must be greater than or equal to the right side. If the inequality is strict, repeat the above argument, but reverse the signs of the positions. This also yields a contradiction, and the result. □
Expression (21) shows that all the bond prices \( P(0, \tau): \ 0 \leq \tau \leq T \) and the stochastic processes for its evolution through time \( (u(t, \tau), d(t, \tau): \ 0 \leq t < \tau \leq T - 1) \) can be determined by knowing \( (P(0, T), (u(t, T), d(t, T): \ 0 \leq t \leq T - 1)) \) and \( (r(0), (\alpha(t), \beta(t): \ 0 \leq t \leq T - 1)) \). If the prices given by expression (21) differ from the observed market prices, then the price discrepancy indicates the existence of an arbitrage opportunity. We label such an arbitrage opportunity, 'zero-curve arbitrage'.

The above procedure, in expression (21) and by implication [expression (15)], gives the procedure for constructing a synthetic zero-coupon bond from another zero-coupon bond (maturity \( T \)) and the money market account. This synthetic construction of a zero-coupon bond is an important attribute of the bond price process models.

3.3. Equivalence between the spot rate process and bond price process models

This section shows the equivalence between the spot rate process and bond price process models using the martingale pricing technology. This equivalence follows through a sequence of two propositions.

**Proposition 1** (The spot rate model implies the bond price model). Given the spot rate process model, assumption R, implies that

(i) the market is complete, and

(ii) expressions (20) and (21) hold for the bond price process (6) constructable from expression (10).

**Proof.**  *Step (i).* Expression (10) together with \( \alpha(t; s_t) \neq \beta(t; s_t) \) implies \( u(t, T; s_t) > d(t, T; s_t) \). This is the crucial condition in Lemma 1, so the proof of Lemma 1 applies as written.

*Step (ii).* Expression (6), (9) and algebra yield

\[
\frac{u(t, T; s_t)}{r(t; s_t)} - \pi(t; s_t) + \frac{d(t, T; s_t)}{r(t; s_t)} (1 - \pi(t; s_t)) = 1.
\]

Together with \( u(t, T; s_t) > d(t, T; s_t) \), this implies (20).

From step (i), let \( (n_P(t; s_t), n_B(t; s_t): \ 0 \leq t \leq T - 1) \) be the self-financing trading strategy generating \( P(t, \tau) \). From the proof of Lemma 1 we have at time \( \tau \):

\[
n_P(\tau - 1; s_{\tau-1}) P(\tau, T; s_\tau) + n_B(\tau - 1; s_{\tau-1}) B(\tau; s_{\tau-1}) = 1 \quad \text{for all } s_\tau.
\]

Taking expected values of both sides under \( \Pi \) [using (6)] gives

\[
n_P(\tau - 1; s_{\tau-1}) P(\tau - 1, T; s_{\tau-1}) + n_B(\tau - 1; s_{\tau-1}) B(\tau - 1; s_{\tau-2}) = P(\tau - 1, \tau; s_{\tau-1}).
\]

This is expression (21) at time \( (\tau - 1) \).
Next, using the self-financing conditions [(17) and (18)], we get

\[ n_P(t - 2; s_{\tau - 2})P(t - 1, T; s_{\tau - 1}) + \\
+ n_B(t - 2; s_{\tau - 2})B(t - 1; s_{\tau - 2}) = P(t - 1, \tau; s_{\tau - 1}). \]

Taking expectations under \( \Pi \) [using (6)] gives (21) at time \( (\tau - 2) \). Continuing by backward induction yields (21). \( \square \)

Proposition 1 shows that the spot rate process model in conjunction with Assumption R, implies the bond price process model. The bond price model is defined to be expression (11) and the results given in expressions (20) and (21). These results are the implications of assumption P. In fact, expressions (20) and (21) are equivalent to assumption P, but this fact requires a more sophisticated argument and is therefore omitted. For practical applications, expressions (20) and (21) are the relevant conditions.

The order of argument in Proposition 1 is important, however. Given are \( \{r(0), (\alpha(t), \beta(t) t \in [0, T])\} \) and \( \Pi \). These, in turn, via expression (10) determine \( \{P(0, \tau), (u(t, \tau), d(t, \tau))\} \) for all \( \tau \). These derived quantities enable us to calculate the hedge ratios (or deltas) needed to construct a synthetic zero-coupon bond from another distinct zero-coupon bond and the money market account. The deltas are given in expression (15). These deltas are essential in order to execute any zero-curve arbitrage discovered via expression (10). Next, we state and prove the converse to Proposition 1.

**Proposition 2** (The bond price model implies the spot rate model). Expressions (20) and (21) imply Assumption R; in particular, \( \Pi \) is given by

\[ \pi^T(t; s_t) = \frac{r(t; s_t) - d(t, T; s_t)}{u(t, T; s_t) - d(t, T; s_t)} \quad \text{for all } 0 \leq t \leq T - 1 \text{ and } s_t. \] (22)

**Proof.** First, by expression (20), there exists a unique \( \pi^T(t; s_t) \in (0, 1) \) such that

\[ \pi^T(t; s_t)u(t, T; s_t) + (1 - \pi^T(t; s_t))d(t, T; s_t) = r(t; s_t). \]

In fact, the unique \( \pi^T(t; s_t) \) is given in expression (22). To show that \( P(t, \tau; s_t)/B(t; s_{\tau - 1}) \) is a martingale under the \( \pi^T \) given in expression (22) we use expression (21) [and (15)]. The cash flow at time \( \tau \) to be duplicated is \( X(s_\tau) = 1 \) if \( s_\tau = s_{\tau - 1}u \) and 1 if \( s_\tau = s_{\tau - 1}d \). Substitution of (15) in (21) evaluated at time \( \tau - 1 \) and algebra yields

\[ \frac{P(\tau - 1, \tau; s_{\tau - 1})}{B(\tau - 1; s_{\tau - 2})} = \\
= \frac{\pi^T(\tau - 1; s_{\tau - 1})P(\tau, \tau; s_{\tau - 1}u) + (1 - \pi^T(\tau - 1; s_{\tau - 1}))P(\tau, \tau; s_{\tau - 1}d)}{B(\tau; s_{\tau - 1})}. \]

This is the result for time \( \tau - 1 \). Continuing this procedure in a backward inductive fashion completes the proof. \( \square \)
Proposition 2 shows that the bond price process model, i.e., expressions (11), (20) and (21), imply the spot rate process model. The order of the argument is, again, important. Given are \([r(0), (\alpha(t), \beta(t): t \in [0, T])\] and \([P(0, T), (\mu(t, T), d(t, T): t \in [0, T - 1])\]. These, in turn, via expression (22) determine \(\Pi\). These \(\Pi\) enable us to use expression (10) to calculate the values of the remaining zero-coupon bonds as discounted expected values. This present value procedure facilitates numerical computations within the bond price process model.

3.4. Summary

The two models just analyzed provide a method for arbitraging price discrepancies across the zero-coupon bond price curve. In fact, this procedure can also be extended to price contingent claims (or options) written on the zero-coupon bond price curve evolution. A contingent claim is defined as a state contingent cash flow received at a particular date, say \(X(s_{T-1})\) received at time \(T - 1\). The identical argument used as in the proof of Proposition 2 implies (under either model) that the arbitrage free value of this contingent claim at time 0, denoted \(X(0)\), is given by:

\[
X(0) = E_0^{\Pi} \left( \frac{X(s_{T-1})}{B(T - 1; s_{T-2})} \right) B(0).
\]  

(23)

This argument can be extended to incorporate multiple cash flows and random stopping times, see Carr & Jarrow [1995].

Although this approach has been utilized in the literature [see Vasicek, 1977; Brennan & Schwartz, 1979; Courtadon, 1982; Cox, Ingersoll & Ross, 1985; Longstaff, 1989], it has some problems. The problems are related to the observation that the theoretical zero-coupon bond price curve generated by these models will (almost certainly) differ from the observed zero-coupon bond price curve. This is true because all models, including this one, are approximations to reality and contain errors. These are isolated as arbitrage opportunities in the above models. As such, for any nontrivial contingent claim dependent on the zero bond price curve (with multiple cash flows across multiple dates), the theoretical price in expression (23) will also differ from the observed price due to these zero coupon bond price differences. Again, this indicates an arbitrage opportunity.

With respect to the observed zero-coupon bond price curve, however, there may in fact be no-arbitrage opportunities using these contingent claims. The relative prices of the contingent claims (relative to the observed zero coupon bond price curve) may be arbitrage free. The above models do not incorporate this distinction. This distinction is best handled by using the class of models directly developed for pricing interest rate contingent claims. This is the topic of the next section.
4. Option pricing

This section shows how to extend the previous zero-curve arbitrage models to contingent claim valuation models where the entire initial zero-coupon bond price curve is given. The major difference between this approach and the previous one is that we now deduce the spot rate process endogenously from the bond price process given in Figure 2. Indeed, as each bond matures, it determines the spot rate for that period. The ability to construct the spot rate process from the bond price process in Figure 2 is the key insight to what follows.

4.1 Spot rate process models

This section shows how to extend the spot rate process model of the previous section to price contingent claims taking the entire initial zero-coupon bond price curve as a given. This class of models, under various reparameterizations, include Black, Derman & Toy [1990] and the discrete time analogue of Hull & White [1990]. The purpose of this approach is to endogenously determine \( \Pi \) and \((\alpha(t), \beta(t)): 0 \leq t \leq T - 1\) to match the initial bond price curve.

This extension reduces to one of solving a system of simultaneous equations in multiple unknowns. The equations are given by expression (10) for \( \tau = 0, 1, \ldots, T \) (\( T + 1 \) equations). Given now are \((P(0, 1), \ldots, P(0, T))\), the right side of expression (10). The unknowns are \( \Pi \) and \((\alpha(t), \beta(t)): 0 \leq t \leq T - 1\) from Figure 3 and expression (8). There are \( 2^T \) unknowns within \((\alpha(t), \beta(t)): 0 \leq t \leq T - 1\) and \([T - 1][T - 2]/2\) unknowns within \( \Pi \). The goal is to find solutions for the unknowns \((\Pi, (\alpha(t), \beta(t)): 0 \leq t \leq T - 1)\) such that the \((T + 1)\) equations implied by expression (10) with the observed bond prices are satisfied. This overabundance of unknowns implies that we can usually find a solution to this system.

It is usual practice [see Black, Derman & Toy, 1990] to arbitrarily set each element in \( \Pi \) equal to \((1/2)\). This reduces the number of unknowns considerably. The determination of \((\alpha(t), \beta(t)): 0 \leq t \leq T - 1\) from the system of equations described above must be done numerically (in general). Fortunately, a forward inductive procedure can be utilized. At step 1, \( \alpha(0; u) \) and \( \beta(0; d) \) are determined by expression (10) and \( P(0, 2) \). Given these, step 2 solves for \( \alpha(1; uu), \alpha(1; du), \beta(1; dd), \beta(1; ud) \) given expression (10) and \( P(0, 3) \). The process continues in this fashion until all bond prices \((P(0, 1), \ldots, P(0, T))\) are used. Counting equations and unknowns reveals (even given \( \Pi \)'s elements equal \( 1/2 \)) that additional constraints can be imposed (for example, the binomial tree for \( r(t) \) could be required to recombine). The above procedure guarantees that the spot rate process evolution yields bond prices [under expression (10)] that match the observed initial bond price curve.

To value contingent claims written on the zero-coupon bond price curve evolution, we use expression (23) with the spot rate process and \( \Pi \) as determined above. The quality of this class of models for pricing options is determined by
the goodness of fit of the spot rate process (just determined) to the empirically observed spot rate process. Empirical research is needed along these lines.

4.2. Bond price process models

This section shows how to extend the bond price process model of the previous section to price contingent claims taking the entire initial zero-coupon bond price curve as a given. This class of models includes under various reparameterizations those of Ho & Lee [1986], Bliss & Ronn [1989], Heath, Jarrow & Morton [1990, 1991, 1992], Babbs [1990], and Shirakawa [1991].

The first step in this extension is to expand the initial specification of the model to include the stochastic process for all the zero-coupon bonds (and not just the \( T \)-maturity bond). This implies that given exogenously is now the entire process in Figure 2 and its decomposition as in expression (6), i.e., \( ((P(0, 1), \ldots, P(0, T)), (u(t, \tau), d(t, \tau)): \text{for all } t, \tau) \). From this specification, the spot rate process is determined as given in expression (7). The next step in the procedure is to determine the additional restrictions that the implication of Assumption \( \mathcal{P} \), the absence of arbitrage, has on the remaining parameters \( (u(t, \tau), d(t, \tau)): \text{for } 0 \leq \tau \leq T - 1 \) in the bond price processes. This additional restriction is given in the next proposition.

**Proposition 3** (Arbitrage restrictions on the zero-coupon bond price processes). Given the specification of \( ((P(0, 1), \ldots, P(0, T)) \) and \( (u(t, \tau), d(t, \tau): \text{for all } t, \tau) \), expressions (20) and (21) hold if and only if

\[
1 > \frac{r(t; s_1) - d(t, T; s_t)}{u(t, T; s_t) - d(t, T; s_t)} = \frac{r(t; s_1) - d(t, \tau; s_t)}{u(t, \tau; s_t) - d(t, \tau; s_t)} > 0
\]

for all \( 0 \leq t \leq T - 1 \) and \( s_t \). (24)

**Proof. Step 1.** From Lemma 3 and Proposition 2 we have

\[
P(t, \tau; s_t) = \pi^T(t; s_t) P(t, \tau; s_t) u(t, \tau; s_t)
\]

\[
B(t; s_{t-1}) = \pi^T(t; s_t) B(t; s_{t-1}) r(t; s_t)
\]

\[
+ (1 - \pi^T(t; s_t)) \frac{P(t, \tau; s_t) d(t, \tau; s_t)}{B(t; s_{t-1}) r(t; s_t)}.
\]

Algebra gives (24).

**Step 2.** First, note that the strict inequalities in (24) imply (20). Second, reversing the algebra in step 1 gives that expression (9) holds. But this was all that was required for the proof of Proposition 1 [step (ii)] to obtain expression (21). □

Proposition 3 provides necessary and sufficient conditions upon the specification of the bond price process parameters \( (u(t, \tau), d(t, \tau): 0 \leq t \leq \tau \leq T) \) such that expression (23) can be used to price contingent claims on the evolution of the zero-coupon bond price curve, taking the initial curve \( (P(0, 1), \ldots, P(0, T)) \) as given. The probabilities \( \Pi \) are given by expression (22). Expression (24) implies
that these probabilities are independent of the particular maturity zero-coupon bond selected. These restrictions allow a great degree of flexibility in the selection of the bond price process.

4.3. A comparison of the spot rate process and bond price process models

By the equivalence propositions of section 3, both models will give the identical contingent claim values if each process generates the other. The models only differ in the initial parameterization and the manner in which the initial zero-coupon bond price curve is introduced into the model specification. The spot rate process approach takes the spot rate process and the martingale probabilities as the given, and then inverts the valuation procedure [expression (10)] to find those spot rate processes such that the theoretical bond prices match the observed initial zero-coupon bond price curve. In contrast, the bond price approach takes the entire bond price curve process as a given, but restricts the parameters of its stochastic process to guarantee that the evolution of the initial curve through time is consistent with the absence of arbitrage. The different models are just 'opposite sides of the same coin'.

Of the two approaches, for the exponentially expanding trees in Figures 2 and 3, the bond price process approach appears to be the simplest from a computational perspective. Indeed, it requires no inversion to determine the spot rate processes parameters. This inversion process usually entails numerical approximation procedures. However, this preference for the bond price process approach may no longer hold for special cases of the general model. In special cases, the spot rate process may have a recombining tree whereas the bond prices process may not. This could lead to computational efficiencies in constructing the tree which dominate the inversion difficulties. Additional research is needed to resolve these issues.

5. Empirical specifications

To implement these models, it is convenient to specify either the spot rate process's parameters (for the spot rate process models) or the bond price process's parameters (for the bond price process models) in terms of the continuous time limits. The motivation for this specification is that the discrete time, discrete state space model is, in fact, only a reasonable approximation to reality for small time steps between trading intervals. As such, the continuous time limit is the relevant process being approximated.

To illustrate this procedure, two examples are provided. One, for the spot rate process models, and two, for the bond price process models. We only consider the specifications for the option pricing application, although the same techniques can be applied for zero-curve arbitrage. For notational convenience, denote the unit of time between each trading interval \([t, t + 1]\) by \(\Delta\). The exact size of \(\Delta\) is usually determined by the particulars of the application itself. We will be interested in the limits of the various processes as \(\Delta \to 0\).
5.1. Spot rate process models

To illustrate this procedure, we give the specification contained in Black, Derman & Toy [1990]. Recall that in the spot rate process model, given are \( r(0), \Pi, (\alpha(t), \beta(t): 0 \leq t \leq T - 1) \). To simplify the parameterization, first set all the probabilities in \( \Pi \) equal to 1/2. Second, let \( \alpha(t), \beta(t) \) be dependent only on time, and independent of the state of the process. To facilitate estimation, let us reparameterize the model by defining \( \mu(t), \sigma(t) \) as:

\[
\sigma(t) = \frac{\log(\alpha(t)/\beta(t))}{2\sqrt{\Delta}}
\]

and

\[
\mu(t) = \frac{\log(\alpha(t)\beta(t))}{2\Delta}
\]

This implies

\[
\alpha(t) = e^{\mu(t)\Delta + \sigma(t)\sqrt{\Delta}}
\]

and

\[
\beta(t) = e^{\mu(t)\Delta - \sigma(t)\sqrt{\Delta}}
\]

Expressed in terms of the spot rate process given in expression (8), we get:

\[
r(t + \Delta; s_{t+\Delta}) = \begin{cases} 
  r(t; s_t) e^{\mu(t)\Delta + \sigma(t)\sqrt{\Delta}} & \text{if } s_{t+\Delta} = s_t u \\
  r(t; s_t) e^{\mu(t)\Delta - \sigma(t)\sqrt{\Delta}} & \text{if } s_{t+\Delta} = s_t d 
\end{cases}
\]

From (27) we see that \( \mu(t) = \mathbb{E}^{\Pi} \left[ \log r(t + \Delta; s_t + \Delta) - \log r(t; s_t) \right] / \Delta \) and \( \sigma(t)^2 = \text{var}^{\Pi} \left[ \log r(t + \Delta; s_t + \Delta) \right] / \Delta \) where \( \text{var}^{\Pi}(\cdot) \) is the \( \Pi \)-conditional variance given \( t \). Thus, these parameters can be interpreted as the instantaneous mean and volatilities of changes in the spot rate.

Using standard techniques, it can be shown that the limiting distribution of the spot rates as \( \Delta \to 0 \) is a lognormal. Black, Derman & Toy [1990] select \( \sigma(t): 0 \leq t \leq T - 1 \) to match the term structure of volatilities, and determine \( \mu(t): 0 \leq t \leq T - 1 \) such that expression (10) matches the initial bond price curve \( (P(0, 1), \ldots, P(0, T)) \). This completes the specification. Other specifications can be found in Hull & White [1990].

5.2. Bond price process models

The empirical specification for the bond price process models is much more subtle than for the spot rate process models. This is due to the facts that (i) each zero-coupon bond must equal a dollar for sure at maturity, and (ii) the no-arbitrage restriction (24) needs to be satisfied. Recall that in the bond price process model, given are \( \left( P(0, 1), \ldots, P(0, T), (u(t, \tau), d(t, \tau): \text{for all } t, \tau \right) \).

To illustrate this procedure, we give the continuous time specification of Ho & Lee [1986] contained in Heath, Jarrow & Morton [1990, 1991, 1992]. The easiest
method for guaranteeing that each bond equals a dollar for sure at maturity is not to specify \((u(t, \tau), d(t, \tau))\) for all \(t, \tau\) directly; but, to specify a stochastic process for forward rates instead. Expression (2) can be then used to guarantee that each bond's value at maturity is a dollar, and the \((u(t, \tau), d(t, \tau))\) for all \(t, \tau\) deduced from it.

Following this approach, let the state independent parameters \(\mu(t, \tau), \sigma\) be given and let the forward rate process satisfy the following expression:

\[
f(t + \Delta, \tau; s_{t+\Delta}) = \begin{cases} 
  f(t, \tau; s_t)e^{\mu(t, \tau)\Delta - \sigma \sqrt{\Delta}} & \text{if } s_{t+\Delta} = s_tu \\
  f(t, \tau; s_t)e^{\mu(t, \tau)\Delta + \sigma \sqrt{\Delta}} & \text{if } s_{t+\Delta} = s_td \\
\end{cases}
\]  
(28)

for \(t + \Delta \leq \tau\).

This implies, in the limit, that these discrete forward rates are lognormally distributed. Expression (2), rewritten in terms of \(\Delta\)'s is:

\[
P(t + \Delta, \tau; s_{t+\Delta}) = \prod_{j=t+\Delta}^{\tau-\Delta} \frac{1}{f(t + \Delta, j; s_{t+\Delta})} \quad \text{for } t + \Delta \leq \tau - \Delta
\]  
(29)

Substitution of (28) into (29) yields:

\[
P(t + \Delta, \tau; s_{t+\Delta}) = \begin{cases} 
  \prod_{j=t+\Delta}^{\tau-\Delta} \frac{1}{f(t, j; s_t)}e^{-\sum_{j=t+\Delta}^{\tau-\Delta} \mu(t, j)\Delta + \sigma (\tau - t - \Delta)\sqrt{\Delta}} & \text{if } s_{t+\Delta} = s_tu \\
  \prod_{j=t+\Delta}^{\tau-\Delta} \frac{1}{f(t, j; s_t)}e^{-\sum_{j=t+\Delta}^{\tau-\Delta} \mu(t, j)\Delta - \sigma (\tau - t - \Delta)\sqrt{\Delta}} & \text{if } s_{t+\Delta} = s_td \\
\end{cases}
\]  
(30)

Thus, we can identify:

\[
u(t, \tau; s_t) \equiv r(t; s_t)e^{-\sum_{j=t+\Delta}^{\tau-\Delta} \mu(t, j)\Delta + \sigma (\tau - t - \Delta)\sqrt{\Delta}} \quad \text{for } t \leq \tau - 2\Delta,
\]  
(31a)

\[
d(t, \tau; s_t) \equiv r(t; s_t)e^{-\sum_{j=t+\Delta}^{\tau-\Delta} \mu(t, j)\Delta - \sigma (\tau - t - \Delta)\sqrt{\Delta}} \quad \text{for } t \leq \tau - 2\Delta,
\]  
(31b)
and

\[ u(t, t + 1; s_t) = (t; s_t) = d(t, t + 1; s_t). \] (31c)

Next, we impose the no-arbitrage restriction (24). This implies that:

\[
\begin{align*}
    r(t; s_t) - r(t; s_t)e^{-\sum_{j=t+\Delta}^{t-\Delta} \mu(t, j) \Delta - \sigma(t - \Delta) \Delta} &= \frac{r(t; s_t)e^{-\sum_{j=t+\Delta}^{t-\Delta} \mu(t, j) \Delta + \sigma(t - \Delta) \Delta} - r(t; s_t)e^{-\sum_{j=t+\Delta}^{t-\Delta} \mu(t, j) \Delta - \sigma(t - \Delta) \Delta}}{1 - e^{-\sum_{j=t+\Delta}^{t-\Delta} \mu(t, j) \Delta - \sigma(t - \Delta) \Delta}} \\
    &= \pi(t; s_t) \\
    &\text{for all } t \leq \tau < 2\Delta \text{ and } \tau \leq T. \quad (32)
\end{align*}
\]

Simplification generates:

\[
\begin{align*}
    1 - e^{-\sum_{j=t+\Delta}^{t-\Delta} \mu(t, j) \Delta - \sigma(t - \Delta) \Delta} &= \pi(t; s_t) \\
    &\text{for all } t \leq \tau < 2\Delta \text{ and } \tau \leq T. \quad (33)
\end{align*}
\]

First, note that as \( \Delta \to 0 \), by a Taylor series expansion, the left side of expression (33) is approximately \( \sum_{j=t+\Delta}^{t-\Delta} \mu(t, j) \Delta + \sigma(t - \Delta) \Delta/(2\sigma(t - \Delta) \Delta) \). Thus, we see that:

\[
\pi(t; s_t) = \frac{1}{2} + O(\sqrt{\Delta}) \text{ where } \lim_{\Delta \to 0} 0(\sqrt{\Delta}) = 0 \text{ and } \lim_{\Delta \to 0} \frac{0(\sqrt{\Delta})}{\sqrt{\Delta}} < +\infty. \quad (34)
\]

For computational convenience, set \( \pi(t; s_t) = 1/2 \). Under this restriction, expression (33) uniquely determines \( \mu(t, \tau) \). It implies that

\[
\begin{align*}
    e^{\sum_{j=t+\Delta}^{t-\Delta} \mu(t, j) \Delta} &= \left( \frac{1}{2} e^{\sigma(t - \Delta) \Delta} + \frac{1}{2} e^{-\sigma(t - \Delta) \Delta} \right) \\
    &= \cosh(\sigma(t - \Delta) \Delta). \quad (35)
\end{align*}
\]

Expression (35) is the necessary and sufficient condition for no arbitrage.

In summary, an arbitrage-free empirical specification of the bond price process model is given by \( \{(P(0, 1), \ldots, P(0, T)), (u(t, \tau), d(t, \tau); \text{all } t, \tau)\} \) where

\[
\begin{align*}
    u(t, \tau; s_t) &= \left( \frac{1}{P(t, t + \Delta; s_t)} \right) \frac{e^{\sigma(t - \Delta) \Delta}}{\cosh(\sigma(t - \Delta) \Delta)} \\
    d(t, \tau; s_t) &= \left( \frac{1}{P(t, t + \Delta; s_t)} \right) \frac{e^{-\sigma(t - \Delta) \Delta}}{\cosh(\sigma(t - \Delta) \Delta)} \text{ for } t \leq \tau < 2\Delta. \quad (36a)
\end{align*}
\]
and
\[ u(t, t + \Delta; s_t) = d(t, t + \Delta; s_t) = \frac{1}{P(t, t + \Delta; s_t)} \quad (36c) \]
or equivalently,
\[ P(t + \Delta, \tau; s_{t+\Delta}) = \left\{ \begin{array}{ll}
\frac{P(t, \tau; s_t)}{P(t, t + \Delta; s_t)} \frac{e^{\sigma (\tau - t - \Delta) \sqrt{\Delta}}}{\cosh (\sigma (\tau - t - \Delta) \sqrt{\Delta})} & \text{if } s_{t+\Delta} = s_t u \\
\frac{P(t, \tau; s_t)}{P(t, t + \Delta; s_t)} \frac{e^{-\sigma (\tau - t - \Delta) \sqrt{\Delta}}}{\cosh (\sigma (\tau - t - \Delta) \sqrt{\Delta})} & \text{if } s_{t+\Delta} = s_t d
\end{array} \right. \]
and all \( 0 \leq t \leq \tau - 2\Delta \) and \( \tau \leq T \). \quad (37)

For calculating interest rate option prices with expression (23), this implies (through expression (24)) that \( \Pi \) is identically equal to (1/2). Furthermore, by expression (36), the entire evolution of the zero-coupon bond price curve is determined by \( (P(0, 1), \ldots, P(0, T)) \) and the \textit{single parameter} \( \sigma \), which corresponds to the volatility of the forward rate process given in (28). This completes the empirical specification. Other empirical specifications can be found in Heath, Jarrow & Morton [1992].

6. Conclusion

This paper has reviewed and synthesized the various approaches to pricing interest rate options. The review has concentrated on a discrete time, discrete state space model. Because of space limitations, the continuous time analogues were not discussed in great detail. This is an important area of research for which the reader must necessarily consult the literature. Before concluding, however, a comment is appropriate. As true in the field of option pricing in general, the empirical literature on interest rate options lags behind the available theory. Little is currently known about which subcase of the above models provides the "best" pricing method for interest rate options. This determination still awaits further research.

References


