OPTION PRICING USING THE TERM STRUCTURE OF INTEREST RATES TO HEDGE SYSTEMATIC DISCONTINUITIES IN ASSET RETURNS

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This paper demonstrates the use of term-structure-related securities in the design of dynamic portfolio management strategies that hedge certain systematic jump risks in asset returns. Option pricing formulas based on the absence of arbitrage opportunities in this context are also developed. The analysis is for the case where assets returns are driven by a finite number of Brownian motions and an m-variate point process. The inclusion of the additional traded assets in the term structure makes it possible to hedge systematic jumps imbedded in the m-variate point process.

KEY WORDS: unique martingale measures, market completeness, m-variance point process, arbitrage pricing

1. INTRODUCTION

Designing trading strategies to hedge price jumps is an issue of particular importance for the managers of fixed income and currency portfolios, as they face the risks of abrupt price movements caused by governmental intervention in financial markets. A successful hedging strategy must rely on positions in a wide collection of assets that have differential responses to the underlying disturbance in its various magnitudes. This paper sets out the theoretical structure that is necessary for pure discount bonds to serve as the instruments for effectively hedging such systematic jumps in asset prices. In particular, we investigate the hedging capabilities of term-structure-related securities in an economy in which are traded, risky assets (e.g., stocks, indexes, or currencies), default-free zero-coupon bonds of all future maturities, and a money market account. We identify the conditions under which the spectrum of zero-coupon bonds may be used to hedge particular types of jump risks faced by these financial markets. Examples of such jump risks that have been empirically investigated and identified in the literature include the response of asset returns to changes in federal discount rates (Waud 1970, Smirlock andYawitz 1985) and responses to changes in the prime rate (Slovin, Sushka, and Waller 1994).

The Black-Scholes methodology prices options on assets by arbitrage. Specifically, the price of an option on a stock is the cost of replicating the option payoff as the value at maturity of a dynamically rebalanced portfolio that takes a position in the asset and, in the absence of interest rate risk, in a single bond. In the presence of interest rate risk, other bonds may be used to hedge diffusion uncertainties affecting the movement of interest

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rates. These considerations are exploited in the hedging strategies of Heath, Jarrow, and Morton (1992) and Amin and Jarrow (1991). In this paper the role of term-structure-related securities is extended beyond that of hedging interest rate diffusion uncertainty to that of also hedging systematic jumps in asset prices.

The difficulty of pricing options by arbitrage, when the underlying asset price is subject to jumps, stems from the fact that in this case markets are seldom complete. Previous approaches to option pricing in this context have relied on equilibrium models employing specific utility functions (see Merton 1976, Naik and Lee 1990, and Madan and Milne 1991). A complete markets approach was employed by Jones (1984), who used derivatives to complete the market. The extension of this line of reasoning leads to static hedging strategies along the lines of Green and Jarrow (1987), Nachman (1988) and Madan and Milne (1994).

Realizing that the hedging of jumps, and the associated complete markets pricing of options, requires the use of a richer asset base in the replicating portfolio, attention is here focused on the term structure of pure discount bonds as a potential class of hedging instruments. We acknowledge that not all shocks to an underlying asset price will be reflected in pure discount bond price movements, and leave the hedging and pricing of such risks as an open problem for subsequent research. Nonetheless, extending the asset base to include the term structure of pure discount bonds is a first step in the solution to this problem. Such an extension, as proposed here, may in some cases prove insufficient but is unlikely to be unnecessary. This is because, in our view, the bond market serves as the alternative of first resort in combating investor uncertainty about capital market returns.

Our paper generalizes and extends existing models in option pricing theory. First, we generalize Merton (1976) to include stochastic interest rates, systematic jumps and random volatilities. Second, we extend the jump process in Cox and Ross (1976) to include a diffusion coefficient. Third, with respect to the literature on interest rate options, we generalize Heath et al. (1992) and Amin and Jarrow (1991, 1992) to include an m-variate point process. In this regard we also generalize the Poisson-Gaussian mixtures studied in Madan, Milne, and Shefrin (1989), He (1990) and Shirakawa (1991).

As argued earlier, the extension presented here may prove insufficient for practice and may therefore require a further generalization. Such a further generalization may be found in Jarrow and Madan (1994). This companion paper studies the equivalence between market completeness and the uniqueness of equivalent martingale measures when asset price processes are semimartingales, with price paths that are right continuous with left limits, and jump sizes are possibly unbounded. The special case examined here is a necessary but perhaps not a sufficient extension to the literature on option pricing.

An outline for the paper is as follows. Section 2 presents the model for the economy. Necessary and sufficient conditions for market completeness are presented in Section 3 along with option pricing formulas. The analysis is extended to nonhedgeable but diversifiable jump risks in Section 4. Section 5 presents an example with explicit option pricing formulas and hedging strategies. Section 6 concludes the paper.

2. THE ECONOMIC MODEL

Consider a continuous-time economy with frictionless trading over the time interval [0, T]. The underlying uncertainty is specified by a stochastic basis (Ω, F, Q), where (Ω, F, Q)

2A paper by Aase (1988) attempts to make this extension but the analysis is flawed, as the market studied is incomplete (see Naik and Lee 1990).
is a probability space and $\mathcal{F} = \{\mathcal{F}_t \mid t \in [0, T]\}, \mathcal{F}_t \subseteq \mathcal{F}$ is a complete, right-continuous, increasing, filtration. Adapted to this filtration is a $d$-dimensional standard Brownian motion $W = (W_1, \ldots, W_d)$, where $W_t = [W_t(t) \mid t \in [0, T]]$, and a nonexplosive $m$-variate point process $(T_n, Z_n; n \geq 1)$ as defined in Brémaud (1981, p. 20). The sequence $T_n$ is a sequence of jump times with $T_n$ interpreted as the time of occurrence of the $n$th jump. For each $n \geq 1$, at the time $T_n$, the random variable $Z_n$ takes one of $m$ possible values, $\{1, \ldots, m\}$. One may associate with the $m$-variate point process the counting processes $N = (N_1, \ldots, N_m)$, where $N_i = [N_i(t) \mid t \in [0, T]]$ is defined by $N_i(t) = \sum_{n \geq 1} 1_{(T_{n} \leq t)} 1_{(Z_n = i)}$ for $i = 1, \ldots, m$ and $1_A(\omega)$, the indicator function of the set $A$, equals 1 only if $\omega \in A$, and is zero otherwise. The coordinate point processes satisfy $\Delta N_i(t) \in \{0, 1\}$ and each process $N_i$ counts the occurrence of events that potentially cause jump discontinuities in the sample paths of asset prices or interest rates. Note further that at each stopping time $T_n$, exactly one of the $m$ processes $N_i$ has a jump and hence there are no simultaneous jumps possible. We suppose $\mathcal{F}$ is the canonical filtration generated by $(W, M)$.

**Assumption 2.1.** There exists an $m$-dimensional nonnegative process $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $\lambda_i = [\lambda_i(t) \mid t \in [0, T]]$ adapted to $\mathcal{F}$ such that

$$M_i(t) = N_i(t) - \int_0^t \lambda_i(s) \, ds$$

is a martingale under the measure $Q$.

The process $\lambda$ is called the intensity process for the point process $N$ and the martingales $M = (M_1, \ldots, M_m)$ are called compensated jump martingales. Relevant examples for $N$ and the associated process $M$ include counting the occurrence of events like (i) low or high, upward or downward changes in the federal discount rate, (ii) low or high, upward or downward movements in the prime rate, and (iii) low or high, upward or downward surprises in inflation and growth rate statistics. The intensity process for these events would in general be functions of macroeconomic conditions. For example, the likelihood of changes in the federal discount rate could be related to open market interest rates and member bank borrowing (Lombra and Torto 1977 and Froyen 1975).

Traded in this economy at any time are a set of risky assets, default-free zero-coupon bonds and a money market account.

Let $A(t, x)$ denote the time $t$ price of an asset $x \in X$, where $0 \leq t \leq T$ and $X$ is an arbitrary index set for the risky assets. The set $X$ could be a singleton, say a stock or market index. Alternatively, it could be a continuum, such as foreign-currency-denominated zero-coupon bonds of all maturities.

Let $P(t, T)$ denote the time $t$ price of a zero-coupon bond paying $1$ at time $T$, where $0 \leq t \leq T \leq T$. We assume that $P(t, T) > 0$ for all $0 \leq t \leq T \leq T$, $P(t, t) = 1$ a.e. $Q$ for all $t \in [0, T]$, and $\partial \log P(t, T)/\partial T$ exists for all $0 \leq t \leq T \leq T$. The time $t$ forward rate for date $T$, $f(t, T)$, is defined by $f(t, T) = -\partial \log P(t, T)/\partial T$. The spot rate $r(t)$ is defined by $r(t) = f(t, t)$. 

The money market account earns interest continuously compounded at rate \( r(t) \) at time \( t \) with a dollar invested at time 0 accumulating to \( B(t) \) by time \( t \), where

\[
B(t) = \exp \left\{ \int_0^t r(y) \, dy \right\}.
\]

The definition of asset price processes for our economy will employ the following regularity conditions on integrability for stochastic processes. We define a process \( u(t) \) to be \( L^1 \) in \( t \) if it is almost surely integrable in \( t \), in the \( L^1 \) sense with respect to Lebesgue measure, or that

\[
\int_0^T |u(t)| \, dt < \infty \quad \text{a.e. } Q.
\]

In addition we require some regularity conditions connected with the continuum of assets represented by the zero-coupon bonds for all maturities. For a continuum of assets indexed by \( T \), \( v(t, T) \) is said to be of \( L^{1,1} \) in \( (t, T) \), if \( v \) defined on \( \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \) is \( \mathcal{F} \times \mathcal{B} \times \mathcal{B} \) measurable, where \( \mathcal{B} \) is the \( \sigma \)-algebra of Lebesgue measurable sets on the positive half of the real line, \( v(t, T) \) is zero for \( t \) exceeding \( T \), \( v(\cdot, T) \) is predictable, and

\[
\int_0^T \int_0^y |v(s, y)| \, ds \, dy < \infty \quad \text{a.e. } Q.
\]

Finally, for a continuum of assets indexed by \( T \), \( w(t, T) \) is said to be \( L^{2,1} \), if \( w \) is \( \mathcal{F} \times \mathcal{B} \times \mathcal{B} \) measurable, \( w(t, T) \) is zero for \( t \) exceeding \( T \), \( w(\cdot, T) \) is predictable for each \( T \), and

(a)

\[
\int_0^T w(s, y)^2 \, ds < \infty \quad \text{a.e. } Q.
\]

(b)

\[
\int_0^T \int_0^y w(s, y) \, dW_i(s) \, dy < \infty \quad \text{a.e. } Q \quad \text{for all } i = 1, \ldots, d,
\]

(c)

\[
\int_0^T \left[ \int_s^T |w(s, y)| \, dy \right]^2 \, ds < \infty \quad \text{a.e. } Q.
\]

We generalize the Heath et al. (1992) model by incorporating the responses of asset prices to the compensated jump martingales \( M \). The process also generalizes Shirakawa (1991)

\(^3\)Enough structure is imposed in the subsequent assumptions to ensure the existence of \( B(t) \).
by allowing the jump intensities to be dependent upon past information. We assume that forward rates satisfy the following stochastic process.

**Assumption 2.2. (Forward Rate Stochastic Process).**  For $0 \leq t \leq T$, 

$$
(2.1) \quad f(t, T) = f(0, T) + \int_0^t \alpha(s, T) \, ds + \sum_{i=1}^m \int_0^t \theta_i(s, T) \, dM_i(s) \\
+ \sum_{i=1}^d \int_0^t \sigma_i(s, T) \, dW_i(s),
$$

where

(a) $\{f(0, T): T \in [0, T]\}$ is a fixed, nonrandom initial forward rate curve
(b) $\alpha(t, T), \theta_i(t, T), \lambda_i(t), \sigma_i^2(t, T), i = 1, \ldots, d$ are $L^1$ in $t$
(c) $\alpha(t, T), \theta_i(t, T), \lambda_i(t)$ are $L^{1,1}$ in $(t, T)$
(d) $\sigma_i(t, T)$ is $L^{2,1}$ for $i = 1, \ldots, d$.

This assumption states that the instantaneous change in forward rates is decomposable into three terms. The first, $\alpha(s, T) \, ds$, is a drift. The second, $\sum_{i=1}^m \theta_i(s, T) \, dM_i(s)$, is the sum of the effects of the compensated jump events. The forward rates' sensitivity to these jump events is measured by the volatility parameter $\theta_i(s, T)$. Finally, the third term, $\sum_{i=1}^d \sigma_i(s, T) \, dW_i(s)$, is a sum of $d$-Brownian motions with volatility coefficients $\sigma_i(s, T)$. The drift and volatility coefficients are assumed to satisfy the regularity conditions (b), (c), and (d) so that various integrals used in the subsequent development are well defined.

Nonnegativity of the forward rate process may be ensured by modeling the forward rate process as the Doléans-Dade exponential of a process whose jumps are restricted to exceed negative unity (Elliott 1982, p. 164). This requires that we model the rate of return on the forward rate itself, and, hence, we may define a continuum of processes $\rho(t, T)$ that are zero for $t$ exceeding $T$ with the interpretation that $\rho$ is the arithmetically cumulated return on the forward rate $f(t, T)$ so that

$$
(2.2a) \quad df(t, T) = f(t, T) \, d\rho(t, T),
$$

or equivalently that

$$
(2.2b) \quad f(t, T) = \mathcal{E}[\rho(t, T)]
$$

where $\mathcal{E}$ is the Doléans-Dade exponential operator. Hence to impose nonnegativity of forward rates we may make the following assumption.

**Assumption 2.2a. The coefficient functions of the forward rate process satisfy**

$$
\begin{align*}
\alpha(t, T) &= f(t_-, T)\alpha'(t, T), \\
\theta_i(t, T) &= f(t_-, T)\theta_i'(t, T) \quad \text{for } i = 1, \ldots, m, \\
\sigma_i(t, T) &= f(t_-, T)\sigma_i'(t, T) \quad \text{for } i = 1, \ldots, d,
\end{align*}
$$
where for \( i = 1, \ldots, m \) we have that \( \theta_i(t, T) \) exceeds negative unity. The cumulated forward rate of return process is then

\[
\rho(t, T) = \int_0^t \alpha'(s, T) \, ds + \sum_{i=1}^m \int_0^t \theta_i'(s, T) \, dM_i(s) + \sum_{i=1}^d \int_0^t \sigma_i(s, T) \, dW_i(s).
\]

Assumption 2.2a need not be imposed. Relaxing this assumption allows forward rates to become negative in principle, but the probabilities of this happening may be low enough to ignore in an estimated model using data with strictly positive forward rates. Furthermore, the relaxation of this assumption lends itself to greater analytical tractability by incorporating a wider class of models. We isolate Assumption 2.2a as an additional condition to be imposed for ensuring positive forward rates. The paper will proceed with the specification of only Assumption 2.2.

Following a similar analysis to that used by Heath et al. (1992), one may derive the bond price process and the process for the money market account implied by this forward rate specification. Note that any model for the bond price process itself implies, under suitable differentiability conditions, an implicit model for the forward rate process and, if these are positive, then also an implicit cumulated rate of return on the forward rate process or the process \( \rho(t, T) \). A calculation made in the Appendix shows that

\[
P(t, T) = P(0, T) \exp \left\{ \int_0^t (r(s) + b(s, T)) \, ds + \sum_{i=1}^m \int_0^t \Theta_i(s, T) \, dM_i(s) \\
+ \sum_{i=1}^d \int_0^t a_i(s, T) \, dW_i(s) \right\},
\]

where

\[
b(s, T) = -\int_s^T \alpha(s, y) \, dy,
\]

\[
\Theta_i(s, T) = -\int_s^T \theta_i(s, y) \, dy,
\]

\[
a_i(s, T) = -\int_s^T \sigma_i(s, y) \, dy.
\]

The bond return's drift component is the sum of two terms: the spot rate \( r(t) \) plus a maturity-specific "risk premium" \( b(s, T) \). The bond price is also seen to be discontinuous due to the response of forward rates to the jump events accounted for by the \( m \)-variate point process \( M \). The process for the money market account is also determined by the forward rate process and is

\[
B(t) = \frac{1}{P(0, t)} \exp \left\{ -\int_0^t b(s, t) \, ds - \sum_{i=1}^m \int_0^t \Theta_i(s, t) \, dM_i(s) \\
- \sum_{i=1}^d \int_0^t a_i(s, T) \, dW_i(s, T) \right\}.
\]
The money market account from time 0 to time \( t \) earns the yield on a \( t \)-maturity bond \((1/P(0, t))\) adjusted for the randomness of spot rates as reflected in the exponential term in (2.5). The dependence of the coefficients of \( dM_i \) on \( t \), with \( \Theta_1(s, t) \) approaching zero as \( s \) tends to \( t \), keeps \( B(t) \) continuous even though there are jumps in \( M_t \). This may be contrasted with (2.4) for \( P(t, T) \) where the relevant coefficient is \( \Theta_0(s, T) \) instead of \( \Theta_1(s, t) \).

Our next assumption concerns the stochastic process followed by the risky assets.

**Assumption 2.3. (Risky Asset Stochastic Process).** For \( x \in X \),

\[
A(t, x) = A(0, x) \exp \left\{ \int_0^t \beta(s, x) \, ds + \sum_{i=1}^m \int_0^t \eta_i(s, x) \, dM_i(s) \right. \\
+ \left. \sum_{i=1}^d \int_0^t \psi_i(s, x) \, dW_i(s) \right\},
\]

(2.6)

where \( \beta, \eta, \lambda \) for \( i = 1, \ldots, m \), \( \psi_i \) for \( i = 1, \ldots, d \) are \( L^1 \) in \( t \).

The risky asset prices follow an exponential process with drift component \( \beta(s, x) \, ds \), response to the jump events given by the \( m \)-variate point process \( \sum_{i=1}^m \eta_i(s, x) \, dM_i(s) \), and a \( d \)-dimensional Brownian shock of \( \sum_{i=1}^d \psi_i(s, x) \, dW_i(s) \).

Assumptions 2.1 to 2.3 complete the specification of the behavior of the asset prices for the assets traded in the economy. To price options we will utilize the risk-neutrality approach initially developed by Cox and Ross (1976) and formalized by Harrison and Pliska (1981). The first step in this analysis is to impose enough additional structure to ensure the existence of a unique equivalent martingale measure. The second step is to demonstrate that the economy is complete and to discuss the construction of synthetic options. To make the presentation less abstract, we provide the analysis for the case where \( X \) consists of a single asset. The more general analysis follows easily as in Amin and Jarrow [1991].

3. **Existence and Uniqueness of Equivalent Martingale Measures**

The absence of arbitrage opportunities in the economy described in Section 2 is guaranteed by ensuring the existence of an equivalent martingale measure (Harrison and Pliska 1981). This is a probability measure \( \tilde{Q} \) on \( F \) that is equivalent to \( Q \) and is such that for all the traded assets the discounted asset prices \( P(t, T)/B(t) \) and \( A(t, x)/B(t) \) are \( \tilde{Q} \) martingales. In general, there may be many equivalent martingale measures, but as shown in Harrison and Pliska (1981), when this measure is unique, markets are complete. In this case, all claims can be replicated by dynamic self-financing trading strategies in the primary traded assets with unique prices determined by the cost of the replication strategy.

To ensure the existence of a unique equivalent martingale measure, we introduce some further structural assumptions on the economy. These assumptions may be motivated from two distinct sets of considerations. The first focuses on the martingale conditions under the equivalent measure, and the assumptions are those necessary for unique solutions to the equations identifying the change of measure. The second focuses on the associated completeness requirement and obtains the assumptions as those necessary for replicating arbitrary claims. We present both of these considerations, introduce the formal assump-
tions, and then establish both the uniqueness of the equivalent martingale measure and the completeness of markets under these assumptions.

3.1. The Martingale Conditions

We begin by characterizing equivalent measures in the context of the economy described in Section 2. For the canonical filtration generated by the processes \((W, M)\) the set of equivalent measures are characterized by what may be interpreted as processes for the market prices of risk. Each component of \(W\) and each component of \(M\) may be viewed as a separate risk with its own time-dependent market price. We will denote by \(\mu_i(s)\), the market price of \(M_i\), and by \(\gamma_i(s)\) the market price of \(W_i\) at time \(s\).

**PROPOSITION 3.1.** (Characterization of Equivalent Measures). Let \(\hat{Q}\) be equivalent to \(Q\) and let \(Z = E[d \hat{Q} / d Q \mid \mathcal{F}_t]\) be the density process of \(\hat{Q}\) with respect to \(Q\). There exist \(\mathbf{F}\) adapted processes \((\mu_1(s), \ldots, \mu_m(s); \gamma_1(s), \ldots, \gamma_d(s))\) such that

(i) For all \(i = 1, \ldots, m\), \(\mu_i(s)\) is nonnegative, \(\mu_i(s)\lambda_i(s)\) is \(L^1\) in \(t\).

(ii) For all \(i = 1, \ldots, d\), \(\gamma_i^2(s)\) is \(L^1\) in \(t\).

(iii) The \(Q\) martingale \(Z\) is defined by

\[
Z(t) = \exp \left\{ \sum_{i=1}^{m} \int_0^t \log(\mu_i(s)) \, dN_i(s) - \sum_{i=1}^{m} \int_0^t (\mu_i(s) - 1) \lambda_i(s) \, ds \right. \\
+ \sum_{i=1}^{d} \int_0^t \gamma_i(s) \, dW_i(s) - \frac{1}{2} \sum_{i=1}^{d} \int_0^t \gamma_i^2(s) \, ds \right\}.
\]

(iv) Under the measure \(\hat{Q}\) the processes

\[
\tilde{W}_i(t) = W_i(t) - \int_0^t \gamma_i(s) \, ds
\]

\[
\tilde{M}_i(t) = N_i(t) - \int_0^t \mu_i(s) \lambda_i(s) \, ds
\]

are martingales.

**Proof.** This is a direct application of Girsanov's theorem to the context of Section 2. (For further details see Bremaud 1981, p. 187 and Protter 1990, p. 157.)

The martingale measure is identified by determining the market prices of risk given by the change of measure processes \((\mu, \gamma)\). We shall see why these processes are properly called market prices of risk after we have identified their defining equations. The defining equations follow from the martingale conditions that after the change of measure to \(\hat{Q}\), the \(B(t)\) discounted asset prices are martingales.
For the purpose of developing these martingale conditions we rewrite the bond and risky asset price processes from (2.4) and (2.6) in terms of the processes \((\tilde{W}, \tilde{M})\) as follows:

\[
P(t, T) = P(0, T)B(t)\exp\left\{\sum_{i=1}^{m} \int_{0}^{t} \Theta_i(s, T) dN_i(s) - \sum_{i=1}^{m} \int_{0}^{t} (\exp(\theta_i(s, T)) - 1)\lambda_i(s)\mu_i(s) ds\right\} \\
\cdot \exp\left\{\sum_{i=1}^{d} \int_{0}^{t} a_i(s, T) dW_i(s) - \sum_{i=1}^{d} \int_{0}^{t} a_i(s, T)\gamma_i(s) ds \right. \\
\left. - \frac{1}{2} \sum_{i=1}^{d} \int_{0}^{t} a_i(s, T)^2 ds \right\} \\
\cdot \exp\left\{\int_{0}^{t} b(s, T) ds - \sum_{i=1}^{m} \int_{0}^{t} \Theta_i(s, T)\lambda_i(s) ds + \frac{1}{2} \sum_{i=1}^{d} \int_{0}^{t} a_i(s, T)^2 ds \\
+ \sum_{i=1}^{m} \int_{0}^{t} (\exp(\theta_i(s, T)) - 1)\lambda_i(s)\mu_i(s) ds + \sum_{i=1}^{d} \int_{0}^{t} a_i(s, T)\gamma_i(s) ds \right\},
\]

whereby we have that

\[(3.3) \quad P(t, T) = P(0, T)B(t)\mathcal{E}\left(\int_{0}^{t} \sum_{i=1}^{m} (\exp(\theta_i(s, T)) - 1) d\tilde{M}_i(s)\right) \\
\cdot \mathcal{E}\left(\int_{0}^{t} \sum_{i=1}^{d} a_i(s, T) d\tilde{W}_i(s)\right) \\
\cdot \exp\left\{\int_{0}^{t} b(s, T) ds - \sum_{i=1}^{m} \int_{0}^{t} \Theta_i(s, T)\lambda_i(s) ds \\
+ \frac{1}{2} \sum_{i=1}^{d} \int_{0}^{t} a_i(s, T)^2 ds \\
+ \sum_{i=1}^{m} \int_{0}^{t} (\exp(\theta_i(s, T)) - 1)\lambda_i(s)\mu_i(s) ds + \sum_{i=1}^{d} \int_{0}^{t} a_i(s, T)\gamma_i(s) ds \right\}.
\]

By a similar argument one may rewrite \(A(t, x)\) as

\[(3.4) \quad A(t, x) = A(0, x)B(t)\mathcal{E}\left(\int_{0}^{t} \sum_{i=1}^{m} (\exp(\psi_i(s, x)) - 1) d\tilde{M}_i(s)\right) \\
\cdot \mathcal{E}\left(\int_{0}^{t} \sum_{i=1}^{d} \psi_i(s, x) d\tilde{W}_i(s)\right)\]
\[
\exp \left\{ \int_0^t \left( B(s, x) - r(s) \right) ds - \sum_{i=1}^m \int_0^t \eta_i(s, x) \lambda_i(s) ds \right. \\
+ \frac{1}{2} \sum_{i=1}^d \int_0^t \psi_i(s, x)^2 ds + \sum_{i=1}^m \int_0^t (e^{\eta_i(s, x)} - 1) \lambda_i(s) \mu_i(s) ds \\
+ \sum_{i=1}^d \int_0^t \psi_i(s, x) \gamma_i(s) ds \right\}.
\]

Since these asset prices discounted by \( B(t) \) are \( \bar{Q} \) martingales and since the Doléans-Dade exponential operators are applied to \( \bar{Q} \) martingales (Elliott 1982, p. 164). As the two Doléans-Dade exponentials are orthogonal local martingales \((\bar{M}_t, \bar{W}_t) = 0\), the product is also the Doléans-Dade exponential of the sum of local martingales and hence is a local martingale. Asset prices discounted by \( B(t) \) in (3.3) and (3.4) may then be written in the form \( L(t)H(t) \), where \( L \) is a local martingale and \( H \), the last exponential expression of (3.3) and (3.4) is a continuous, hence also predictable, process of bounded variation. An application of Ito's lemma to this product shows that the semimartingale decomposition of the product has the form

\[
L(t)H(t) = L(0)H(0) + \int_0^t L(s-) dH(s) + \int_0^t H(s-) dL(s)
\]

and the product is a martingale as required just if the predictable bounded variation component is zero, or \( dH(t) = 0 \). Setting the derivative of \( H(t) \) to zero, we obtain the martingale conditions that identify the martingale measure in relation to the market prices of risk \((\mu, \gamma)\) processes. They yield the following drift restrictions on the asset price paths. From the bond price martingale condition we get that

(3.5) \[ \sum_{i=1}^m \left[ e^{b_i(s, T)} - 1 \right] \lambda_i(s) \mu_i(s) + \sum_{i=1}^d a_i(s, T) \gamma_i(s) = -\delta(s, T), \]

where

\[
\delta(s, T) = \left[ b(s, T) - \sum_{i=1}^m \Theta_i(s, T) \lambda_i(s) + \frac{1}{2} \sum_{i=1}^d a_i^2(s, T) \right].
\]

From the martingale condition for the risky asset prices we obtain

(3.6) \[ \sum_{i=1}^m \left[ e^{Q_i(s, x)} - 1 \right] \lambda_i(s) \mu_i(s) + \sum_{i=1}^d \psi_i(s, x) \gamma_i(s) = -\Delta(s, x), \]
where

\[
\Delta(s, x) = \left[ \beta(s, x) - r(s) - \sum_{i=1}^{m} \eta_i(s, x) \lambda_i(s) \, ds + \frac{1}{2} \sum_{i=1}^{d} \psi_i(s, x)^2 \right]
\]

Equations (3.5) and (3.6) assert that the excess drift over the risk-free rate \( r(s) \) on the bond and risky asset returns is proportionate to the instantaneous volatility of martingale risks \((W, M)\) times their instantaneous market prices \((\gamma, \mu)\). Hence, the interpretation of the coefficients identifying the change of measure as the market prices of risk. The drift restrictions (3.5) and (3.6) are the relevant instantaneous asset pricing model for this economy. Unique equivalent martingale measures are linked via (3.5) and (3.6) to nonsingularity of matrices with rows having \( m + d \) columns of the form \( (e^{\delta_k(s, T_i)} - 1) \) for \( i = 1, \ldots, m; a_i(s, T_j), i = 1, \ldots, d \) or \( (e^{\eta_i(s)}) - 1, i = 1, \ldots, m \); \( \psi_i(s), i = 1, \ldots, d \), depending on whether the row refers to a bond price condition or a risky asset price condition.

3.2. The Replication Conditions

From the perspective of replication consider \( c \geq (m + d - 1) \) zero-coupon bonds with maturities \( 0 < T_1 < T_2 < \cdots < T_c \leq T \), and the risky asset \( A(t, x) \). The system of stochastic differential equations satisfied by these \( m + d \) assets is, by Ito’s lemma, applied to (2.4) and (2.6):

\[
(3.7) \quad dP(s, T_j) = \left[ r(s) + \delta(s, T_j) \right] P(s, T_j) \, ds \\
+ \sum_{i=1}^{m} \left[ e^{\delta_k(s, T_j)} - 1 \right] P(s, T_j) \, dN_i(s) \\
+ \sum_{i=1}^{d} a_i(s, T_j) P(s, T_j) \, dW_i(t) \quad \text{for } j = 1, \ldots, c.
\]

\[
dA(s, x) = \left[ r(s) + \Delta(s, x) \right] A(s, x) \, ds \\
+ \sum_{i=1}^{m} \left[ e^{\eta_i(s, x)} - 1 \right] A(s, x) \, dN_i(s) + \sum_{i=1}^{d} \psi_i(s, x) A(s, x) \, dW_i(t),
\]

where \( \delta \) and \( \Delta \) are as defined in (3.5) and (3.6).

Next, consider hedging an option on the risky asset \( A(t, x) \). The option’s price will fluctuate randomly due to these same \( m + d \) risks

\[
\left( dN_1(s), \ldots, dN_m(s); dW_1(s), \ldots, dW_d(s) \right).
\]

Given there are also \( c + 1 \geq m + d \) traded assets, a hedge is possible as long as the volatility matrix expression in (3.7) has full rank of \( m + d \). This is the same matrix as that obtained from the martingale conditions (3.5) and (3.6).
3.3. Structural Conditions for Completeness and Uniqueness of Martingale Measures

The considerations for identifying martingale measures and replicating arbitrary claims motivate our further structural assumptions. We consider the implementation of trading strategies involving the risky asset and some \( c \geq m + d - 1 \) bonds with maturities \( \tau^* < T_1 < T_2 < \cdots < T_c \leq T \).

Assumption 3.1. (Hedgeable Jump Risks). For all \( s \in [0, \tau^*] \) a.e. \( Q \) the \((c + 1) \times (m + d)\) matrix

\[
H(s) = \begin{bmatrix}
  e^{\theta_1(s, T_1)} - 1 & \cdots & e^{\theta_m(s, T_1)} - 1 & a_1(s, T_1) & \cdots & a_d(s, T_1) \\
  \vdots & & \vdots & \vdots & & \vdots \\
  e^{\theta_1(s, T_c)} - 1 & \cdots & e^{\theta_m(s, T_c)} - 1 & a_1(s, T_c) & \cdots & a_d(s, T_c) \\
  e^{\theta_1(s, x)} - 1 & \cdots & e^{\theta_m(s, x)} - 1 & \psi_1(s, x) & \cdots & \psi_d(s, x)
\end{bmatrix}
\]

has full rank \( m + d \). The unique solutions to the equation system

\[
H(s) \begin{bmatrix}
  \lambda_1(s) \mu_1(s) \\
  \vdots \\
  \lambda_m(s) \mu_m(s) \\
  \gamma_1(s) \\
  \vdots \\
  \gamma_d(s)
\end{bmatrix} = - \begin{bmatrix}
  \delta(s, T_1) \\
  \vdots \\
  \delta(s, T_c) \\
  \Delta(s, x)
\end{bmatrix}
\]

satisfy the conditions that (a) \( \mu_i(s) \) is nonnegative and \( \mu_i(s)\lambda_i(s) \) are \( \mathcal{L}^1 \) in \( s \) for \( i = 1, \ldots, m \); (b) \( \gamma_i^2(s) \) are \( \mathcal{L}^1 \) in \( s \) for \( i = 1, \ldots, d \); (c) for \( Z(t) \) defined by (3.1), \( E[Z(\tau^*)] = 1 \). Furthermore we require that (d) \( E[Z(\tau^*)P(\tau^*, T_i)/B(\tau^*)] = 1 \), (e) \( E[Z(\tau^*)A(\tau^*, x)/B(\tau^*)] = 1 \), and (f) \( E[Z(t)[P(i, T_i)/B(t)]^2] < \infty \), \( E[Z(t)[A(t, x)/B(t)]^2] < \infty \) for \( t \in [0, \tau^*] \). The solutions are assumed to be independent of the \( c \) bonds selected.

Under Assumptions 2.1 to 2.3 and 3.1 we can state and prove our main proposition.

Proposition 3.1. (Existence and Uniqueness of an Equivalent Martingale Measure).
Under Assumptions 2.1 to 2.3 and 3.1, there exists an equivalent probability \( \tilde{Q} \) on \((\Omega, \mathcal{F})\) such that under \( \tilde{Q} \):

(a) \( P(t, T)/B(t) \) for all \( T \in [0, T] \) and \( A(t, x)/B(t) \) are square-integrable martingales.

(b) \( \tilde{W}_i(t) = W_i(t) - \int_0^t \gamma_i(s) \, ds \) for \( i = 1, \ldots, d \)
is a \( d \)-dimensional Brownian motion.

\[
\tilde{M}_i(t) = N_i(t) - \int_0^t \lambda_i(s) \mu_i(s) \quad \text{for} \quad i = 1, \ldots, m
\]

is a compensated \( m \)-variate point process (a martingale).

Furthermore, there is only one equivalent probability \( \tilde{Q} \) such that, under \( \tilde{Q} \), \( p(t, T)/B(t) \) for all \( T \in [0, T] \) and \( A(t, x)/B(t) \) for all \( x \) are \( \tilde{Q} \)-martingales.

Proof. See Appendix.

This proposition guarantees the existence and uniqueness of a risk-neutral valuation operator for our economy. At this stage, however, it only applies to the traded assets (bonds, money market account, and the risky asset). To apply the valuation operator to contingent claims on these assets (like an option on \( A(t, x) \)), we first need to discuss the issue of market completeness.

3.4. Call Valuation and Market Completeness

For economies where prices follow a continuous sample path, the existence of a unique equivalent martingale measure as in Proposition 2.1, is known to be both necessary and sufficient for market completeness. The result was established for a finite asset economy when the stochastic process for asset prices is a vector-valued semimartingale (see Harrison and Pliska 1981, 1983). For a clearer statement on the implicit definition of two potentially different concepts of dynamic self-financing strategies involved, the reader is referred to Jarrow and Madan (1991) and Chatelain and Stricker (1994) for the equivalence of these definitions.

More generally, uniqueness of equivalent martingale measures implies market completeness, but the converse does not hold. Artzner and Heath (1993) provided a counterexample to the converse proposition, exhibiting an approximately complete economy under two distinct equivalent martingale measures. This counterexample must, however, necessarily involve an infinite asset economy with an infinity of discontinuous price processes (Artzner and Heath 1993). The Harrison and Pliska (1981, 1983) result, for finite asset economies, does not apply in this case, and though approximate completeness and martingale representation prevail, uniqueness of the equivalent martingale measure fails. Our economy is an infinite asset economy, with bonds of every maturity forming an infinity of, in our case, discontinuous price processes. However, there are only finitely many jump processes causing these discontinuities, and we restrict trading strategies to the use of \( c \) bonds as implicit in Assumption 3.1. This reduces the model of this paper to the context of Harrison and Pliska (1983), and completeness follows directly from uniqueness of the equivalent martingale measure. For a context employing infinitely many underlying jump processes and infinitely many discontinuous price processes, yet yielding under certain further conditions, the equivalence of unique equivalent martingale measures and completeness the reader is referred to Jarrow and Madan (1994).
We define a self-financing trading strategy (s.f.t.s) to be a dynamic portfolio consisting of

(a) $\Phi_0(\omega, t)$ shares of the money market account at time $t \in [0, \tau^*]$ under state $\omega \in \Omega$
(b) $\Phi_i(\omega, t)$ shares of the zero-coupon bond with maturity $T_i$ at time $t \in [0, \tau^*]$ under state $\omega \in \Omega$ for $i = 1, \ldots, c \geq m + d - 1$, where $\tau^* \leq T_1 < T_2 < \cdots < T_c \leq T$
(c) $\Phi_x(\omega, t)$ shares of the risky asset with price $A(t, x)$ at time $t \in [0, \tau^*]$ under state $\omega \in \Omega$

such that (i) the share selections depend only on the information available prior to and at time $t$ (i.e., are adapted to $F$); (ii) the share selections are predictable and integrable with respect to the semimartingales, $1, P(t, T_i)/B(t), i = 1, \ldots, c, A(t, x)/B(t)$ under the measure $\tilde{Q}$ (for further details see the appendix of Jarrow and Madan 1991); and (iii) defining

$$V(t) = \Phi_0(t)B(t) + \sum_{j=1}^c \Phi_j(t, T_j)P(t, T_j) + \Phi_x(t)A(t, x)$$

as the portfolio's value at time $t \in [0, \tau^*]$ under state $\omega \in \Omega$, we have that the share selections satisfy the self-financing condition

$$\frac{V(t)}{B(t)} = \frac{V(0)}{B(0)} + \sum_{j=1}^c \int_0^t \Phi_j(s) d\left(\frac{P(s, T_j)}{B(s)}\right) + \int_0^t \Phi_x(s) d\left(\frac{A(s, x)}{B(s)}\right).$$

The significance of these trading restrictions is twofold: first, the portfolio's value can be written as a stochastic integral (3.11); second, the portfolio's value is a $\tilde{Q}$ martingale (conditions (i) and (ii)). For any s.f.t.s, we therefore have that

$$\frac{V(t)}{B(t)} = \tilde{E}\left(\frac{V(\tau^*)}{B(\tau^*)} \mid \mathcal{F}_t\right) \quad \text{a.e.}$$

where $\tilde{E}(\cdot)$ is expectation with respect to $\tilde{Q}$. This extends the risk-neutral operator from the traded assets to all s.f.t.s. of the traded assets.

Our purpose, of course, is to value contingent claims. A contingent claim is defined to be any random payoff $y(\omega)$ at time $\tau^*$ which depends only on the information available at time $\tau^*$ with $\tilde{E}(y(\omega)/B(\tau^*)) < +\infty$. For example, a European call option on the risky asset with exercise price $K$ and exercise date $\tau^*$ would be such a random variable with payoff $\max[A(\tau^*, x) - K, 0]$ at time $\tau^*$.

The economy is said to be complete if, given any contingent claim $y$, there exists a s.f.t.s. with market value process $(V(t) : t \in [0, \tau^*))$ such that the payoff to the s.f.t.s. at time $\tau^*$ equals the payoff to the contingent claim a.e.; that is,

$$V(\tau^*) = y \quad \text{a.e.} \quad Q.$$
Under this circumstance, the price of the contingent claim \( y \) is defined to be the initial
value, \( V(0) \), of the s.f.t.s. which replicates it. Our next proposition is that markets are
complete.

**Proposition 3.2.** (Market Completeness). Under Assumptions 2.1 to 2.3 and 3.1, the
economy is complete.


Given the market is complete, Proposition 3.1 can then be combined with (3.12) to extend
our risk-neutral valuation operator to value any contingent claim \( y \). The time \( t \) arbitrage-free
price of the contingent claim \( y \) is, therefore,

\[
V(t) = B(t) \mathbb{E} \left( \frac{y}{B(t^*)} \mid \mathcal{F}_t \right).
\]

For example, a European option on the risky asset \( A(t, x) \) with exercise price \( K \) and exercise
date \( t^* \) has a time \( t \) value equal to \( \mathbb{E}(\max[A(t^*, x) - K, 0] \mid \mathcal{F}_t) B(t) \).

Proposition 3.2 ensures that there exists a s.f.t.s. in the risky asset, the \( c \) bonds, and the
money market account to create a synthetic option (call or put). The exact positions to
hold are not revealed by this proposition. In special cases, when the asset returns are strong
Markov, Ito’s lemma and (3.12) can be used to determine the proper “deltas.” This procedure
is illustrated below when we consider some examples. For the general theory, however,
the asset returns are not strong Markov. In this circumstance, a practical alternative is to
use a discrete approximation to the above economy (along the lines of Madan, Milne, and
Shefrin 1989) and calculate the deltas numerically from the approximating tree.

4. NONHEDGEABLE AND DIVERSIFIABLE RISKS

This section examines the pricing of an option on a risky asset whose jump component can
be decomposed into two parts, one that may be hedged using the term structure and another
that is unrelated to the term structure. By introducing a jump component unrelated to the
term structure, we explicitly recognize, as noted in the introduction, that not all shocks to
the risky asset price may be reflected in the term structure. In addition there may also be a
diffusion component to the risky asset price motion that is orthogonal to the term structure
motions.

For clarity, the set of risky assets \( K \) is again kept as a singleton, although the general
case is an easy extension and follows Amin and Jarrow (1991). To capture the idea of jump
and diffusion components unrelated to term structure securities but with effects on the risky
asset, we introduce one such component of each type. Formally we suppose

**Assumption 4.1.** (Non-Term-Structure-Hedgeable Risks). The jump risk \( N_n(t) \) and
the diffusion risk \( W_d(t) \) are said to be non-term-structure-hedgeable if for all \( 0 \leq t \leq T \)
\( T, \Theta_m(t, T) = 0 \) a.e. \( Q \) and \( a_d(t, T) = 0 \) a.e. \( Q \). Furthermore, for \( t^* \leq t_1 < t_2 < \cdots <
$T_c \leq T$ where $c \geq m + d - 1$, the $(c + 1) \times (m + d)$ matrix $H(s)$

$$
\begin{bmatrix}
    e^{\Theta_1(s, T_1)} - 1 & \cdots & e^{\Theta_{m-1}(s, T_{m-1})} - 1 & 0 & a_1(s, T_1) & \cdots & a_{d-1}(s, T_1) & 0 \\
    \vdots & & \vdots & & \vdots & & \vdots & \\
    e^{\Theta_1(s, T_c)} - 1 & \cdots & e^{\Theta_{m-1}(s, T_{m-1})} - 1 & 0 & a_1(s, T_c) & \cdots & a_{d-1}(s, T_c) & 0 \\
    e^{\Theta_1(s, T)} - 1 & \cdots & e^{\Theta_{m-1}(s, T_{m-1})} - 1 & e^{\Theta_1(s, x)} - 1 & \psi_1(s, x) & \cdots & \psi_{d-1}(s, x) & \psi_d(s, x)
\end{bmatrix}
$$

has the maximal rank of $m + d - 1$ for all $t \in [0, \tau^*]$ a.e. $Q$, with all entries for $\Theta_i(t, T)$, for $i = 1, \ldots, m-1$ and $a_i$, for $i = 1, \ldots, d-1$, adapted to the filtration generated by $(M_1, \ldots, M_{m-1}, W_1, \ldots, W_{d-1})$.

The volatility matrix $(H(s))$ of the system of assets has a rank deficiency of 1 under Assumption 4.1. Consequently, the term structure can be used to hedge all but one of the risks present in an option on $A(t, x)$. Thus, the market is incomplete. However, this rank condition along with the condition that the entries are adapted to the smaller filtration generated by $(N_1, \ldots, N_{m-1}, W_1, \ldots, W_{d-1})$ is still consistent with that subsection of the economy consisting of the term structure itself being (internally) complete, in that all interest rate options can still be priced according to the methodology given in the previous section. Note in this connection the exclusion of $W_d$ from the coefficients associated with the bond price motions. The inclusion of $W_d$ would in general require the inclusion of the risky asset in $X$ in the hedge portfolios and this would nontrivially and unnecessarily expose the portfolios to the risk of $M_m$ as well.

More generally however, the system of equations (3.9) determining the market prices of risk $(\mu_1(s), \ldots, \mu_m(s); \gamma_1(s), \ldots, \gamma_d(s))$ now can have no solution or a continuum of solutions parameterized by $\mu_m(s)$. Ruling out the absence of arbitrage opportunities, we suppose the existence of a solution. Specifically, we impose the following assumption.

**Assumption 4.2.** (Nonuniqueness of the Market Prices for Risk). (i) The system of equations (3.9) has a solution parameterized by an arbitrary nonnegative value for $\mu_m(s)$.

(ii) There exists a choice of $\mu_m(s)$, for which the solutions $(\mu_1(s), \ldots, \mu_m(s); \gamma_1(s), \ldots, \gamma_d(s))$ to (3.9) satisfy the integrability conditions noted in Assumption 3.1 and are independent of the collection of $c$ bonds selected.

Under Assumptions 2.1–2.3, 4.1, and 4.2, the identical proof used in Proposition 2.1 shows that there exists a continuum of equivalent martingale measures that may be indexed by the parameter process $\mu_m$. This nonuniqueness, in turn, implies that there is no unique arbitrage-free price for contingent claims on the risky asset $A(t, x)$. Valuing $A(t, x)$ contingent claims in this context generally requires an equilibrium model or some further assumptions regarding preferences and the nature of the economic model.

One such assumption is that of diversifiability. As shown in Milne (1988) a risk is diversifiable if in equilibrium all investors wish to avoid taking a position in this risk and the economy allows them to do so. Though such diversifiable risks may affect the firm’s cash flows and market values, they may have no effect on the structure of required returns reflected in the excess drifts on the right-hand side of (3.9). This is essentially because, though we may have asset returns sensitive to these risks, the risks are not priced in equilibrium.

For a formal statement of this diversifiability condition, we need some preliminary definitions. In an equilibrium, assuming continuous, convex, and strictly monotonic preferences
on cash flows for all investors, one may construct for each investor a personalized equivalent martingale measure that is based on his or her personal marginal rates of substitution for state contingent dollars (see Duffie 1988, 17D, p. 158). Associated with this personalized equivalent martingale measure is the \( P \)-martingale density process defined by the conditional expectation under \( P \) of the Radon-Nikodym derivative of the personalized measure with respect to \( P \). We define the class of relevant martingale measures to be the collection of all personalized martingale measures derived from an economic equilibrium. More specifically, one has to formulate a continuous-time general equilibrium model of the economy that accommodates incomplete markets. Within such a formulation, there may be more than one equilibrium, and one then has to select a particular equilibrium. For the selected equilibrium, each individual provides us with a single relevant "to this equilibrium" equivalent martingale measure. The class of relevant martingale measures is the set of measures, one for each individual, obtained from the particular selection of a general equilibrium for the incomplete markets model. Further, for \( G = \{ G_t | t \in [0, T], G_t \subseteq \mathcal{F}_t \} \) a subfiltration of \( \mathcal{F} \), we define an equivalent martingale measure to be \( G \)-adapted if the associated density process is \( G \)-adapted. One may then formally state the diversifiable assumption as follows.

**Assumption 4.3. (Diversifiable Jump Risk).** The \( m \)th jump risk \( M_m \) is said to be diversifiable (nonsystematic) if all relevant equivalent martingale measures are adapted to the filtration generated by the risks \( (M_1, \ldots, M_{m-1}, W_1, \ldots, W_d) \) excluding the risk \( M_m \).

Assumption 4.3 supposes that equivalent martingale measures derived from individual preferences in an equilibrium satisfy some property. Hence, one is making here an assumption about an endogenous entity of an equilibrium, and it is not clear what assumptions on the primitives of the economy will deliver such a result. The consequences of 4.3 for the forward rate processes are also of interest, and we would expect that they would imply part of Assumption 4.1 in that the coefficients \( \Theta_i(t, T) \), and \( a_i(t, T) \), of the bond price processes would be adapted to the filtration generated by \( (M_1, \ldots, M_{m-1}, W_1, \ldots, W_d) \). The demonstration of an incomplete markets general equilibrium consistent with a diversifiable jump risk in the sense of Assumption 4.3 is an open question for future research. Here we analyze the consequences of the validity of Assumption 4.3 for the pricing of claims contingent on \( A(T, x) \).

Assuming the validity of Assumption 4.3, a unique valuation for claims contingent on \( A(t, x) \) may be obtained if we restrict attention to the use of a relevant equivalent martingale measure for valuation. It follows from Assumption 4.3 that as the density \( Z(t) \) in (3.9) of the equivalent martingale measure \( \hat{Q} \) with respect to \( Q \) is the Doléans-Dade exponential of the process

\[
\zeta(t) = \sum_{i=1}^{m} \int_0^t (\mu_i(s) - 1) \, dM_i(s) + \sum_{i=1}^d \int_0^t \gamma_i(s) \, dW_i(s),
\]

that \( \mu_m(t) = 1 \) for all \( t \in [0, \tau^*] \) a.e. \( Q \) for any relevant equivalent martingale measure.

Given Assumptions 2.1–2.3 and 4.1–4.3, we can unambiguously price contingent claims according to (3.12). However, Proposition 3.2 does not hold, as the market is incomplete and one cannot hedge the risk implicit in the \( m \)th jump risk \( M_m \). The resulting valuation is relevant for investors in the selected equilibrium, in that in this equilibrium all investors value the asset at this price.
More generally one may expand the asset space to include other risky assets from the set \( X \). There may, however, also be other jump and diffusion risks introduced as we expand the asset space. Suppose we have in all \( m \) jump risks, \( d \) diffusions, \( c \) bonds, and \( n \) risky assets from the set \( X \). The volatility matrix is then \((c + n) \times (m + d)\), and there may be a rank deficiency. One might first decompose the underlying risks \((M, W)\) into two components, nondiversifiable (systematic) and diversifiable (nonsystematic) risks, so that \( M = (M^S, M^U) \) and \( W = (W^S, W^U) \), where \( M^S, W^S, M^U, W^U \) have dimensions, respectively, \( m^S, d^S, m^U, d^U \). Formally, the systematic risks are precisely those generating a filtration with respect to which the density \( Z \) of the equivalent martingale measure is adapted. The volatility matrix of (3.9) is accordingly partitioned into \( H(t) = (H^S(t), H^U(t)) \). From (4.1) and the fact that \( Z \) is adapted to the filtration generated by the systematic risks one observes that the market prices of the nonsystematic jump risks are unity and those for the nonsystematic diffusion risks are zero. Provided the systematic risk volatility matrix \( H^S \) has full rank \( m^S + d^S \), the methods of this paper may be implemented to obtain unambiguous equilibrium valuations for claims contingent on the asset values. By market incompleteness, though these risks cannot be hedged completely, dynamic hedging strategies can be constructed that invest only in the nonsystematic risks. In fact, an investment in accordance with the rows of \((H^S H^S)^{-1} H^S \) in the set of \( c \) bonds and \( n \) risky assets effectively invests in \((M^S, W^S)\) plus nonsystematic risks, and such portfolios may be used to remove the effects of systematic risks on portfolio values, leaving one exposed to just the nonsystematic risks.

5. AN EXPLICIT EXAMPLE

Suppose the uncertainty underlying the term structure has a one-dimensional diffusion component that is unrelated to the diffusion driving the risky asset, which is represented by the market index. The economy therefore has two underlying diffusions \((W_1, W_2)\). Consider in addition a jump shock that affects inversely the stock market and bond markets. This possibility may be represented by two jump risks \( N_1 \) and \( N_2 \), where \( N_1 \) is associated with a positive effect on the stock market and a negative effect on the bond market or positive on forward rates, while \( N_2 \) negatively affects the stock market and also negatively affects forward rates. Suppose that the arrival rates for the two jump processes are constant and equal to \( \lambda \). Hence the two independent underlying jump martingales are

\[
M_1(t) = N_1(t) - \lambda t, \quad M_2(t) = N_2(t) - \lambda t
\]

The filtration \( F \) is generated by \((M, W)\), where \( M = (M_1, M_2) \) and \( W = (W_1, W_2) \).

Let the forward rate stochastic process be

\[
f(t, T) = f(0, T) + \int_0^T \alpha(s, T) \, ds + \sigma W_1(t) + \theta \xi e^{-\xi(T-t)} M_1(t) - \theta \xi e^{-\xi(T-t)} M_2(t),
\]

(5.2)

where \( \theta \xi e^{-\xi(T-t)} \) represents a movement of \( \theta \xi e^{-\xi(T-t)} \) basis points in the forward rate for \( T \) caused by the jump shock at \( t \), upward or downward depending on the whether the shock

- These may include, for example, a term structure of corporate bonds.
is \( N_1 \) or \( N_2 \). The maximum movement in forward rates is \( \theta \xi \), which occurs at the short end of the term structure. This reflects the view that this end of the term structure primarily serves as the alternative to investment in the stock market.

If we integrate the forward rate shocks over all maturities, we get \( \theta \), and we suppose that \( \theta \) is below unity to ensure nonnegative bond prices. Although the drift specification is quite general in (5.2), a particular functional form will be implied by the absence of arbitrage opportunities as per (3.5).

Let the singleton risky asset be a market index and suppose it follows the process

\[
A(t) = A(0)e^{\theta t + \xi W(t) - \eta M(t) - \eta M_2(t)},
\]

The shock \( N_1 \) raises stock prices by a percentage \( \eta \) at the expense of raising forward rates by \( \theta \) basis points and correspondingly lowering bond prices. The shock \( N_2 \) does the opposite.

Given the spot rate process \( r(t) = f(t, t) \), the money market account value process is defined by \( \log(B(t)) = \int_0^t r(y) \, dy \), and the explicit zero-coupon bond price process relative to the money market account value is

\[
P(t, T) = P(0, T) e^{\int_0^t b(s, T) \, ds - \frac{1}{2} \int_0^t \sigma(T - s) \, dW(s) - \theta \int_0^t \Theta(T - s) \, dM(s) + \int_0^t \Theta(T - s) \, dM_2(s)},
\]

where \( b(s, T) = -\int_s^T \alpha(s, y) \, dy \) and \( \Theta(u) = \theta (1 - e^{-\theta u}) \).

In this specification, bond and market index returns are correlated through the presence of the common jump shocks, and this correlation could be small if the diffusion coefficients have a substantive volatility. Yet, the presence of these jumps can motivate a diversified term structure position designed to combat shocks in the stock market.

Completeness of markets requires the hedging matrix of Assumption 3.1 to have full rank. For \( c \) bonds with maturities \( \tau^* < T_1 < T_2 < \cdots < T_c \leq T \), this is the \((c + 1) \times 4\) matrix:

\[
H(s) = \begin{bmatrix}
    e^{-\theta(1 - e^{-\xi(T_1 - s)})} - 1 & e^\theta(1 - e^{-\xi(T_1 - s)}) - 1 & -\sigma(T_1 - s) & 0 \\
    \vdots & \vdots & \vdots & \vdots \\
    e^{-\theta(1 - e^{-\xi(T_c - s)})} - 1 & e^\theta(1 - e^{-\xi(T_c - s)}) - 1 & -\sigma(T_c - s) & 0 \\
e^{\eta} - 1 & e^{-\eta} - 1 & 0 & \psi
\end{bmatrix}
\]

which has full rank for many choices of \( \theta, \xi, \sigma, \eta, \psi \), and the \( T_i \)'s.

In order to value index options for maturities under a year, one may choose three bonds of maturities say one, two, and three years along with the index itself to complete markets. With market completeness one may value any contingent claim as its discounted expectation under the martingale measure \( \tilde{Q} \). We illustrate this approach by valuing a European call option on the index.

Consider a European call option on \( A(t) \) with strike price \( K > 0 \) and maturity date \( \tau^* < T \). By (3.13), the call's value \( C(t) \) is given by

\[
C(t) = \tilde{E} \left( \max(A(\tau^*) - K, 0) \right) \frac{B(t)}{B(\tau^*)} | \mathcal{F}_t \), B(t).
\]
To evaluate this expression in closed form, we add the assumption that \( \mu_i(s) = \mu > 0 \) for \( i = 1, 2 \). From (3.4) and the drift restrictions of (3.9) observe that, under \( \tilde{Q} \),

\[
B(t) \frac{A(\tau^*)}{B(\tau^*)} = A(t) e^{\psi(\tilde{W}_T(t) - \tilde{W}_t(t)) - \psi^2(\tau^* - t)/2} \cdot e^{\gamma n_1 \lambda \mu (\tau^* - t)} \cdot e^{-\gamma n_2 \lambda \mu (e^{\gamma - 1} - 1)(\tau^* - t)}
\]

where \( n_1 \) and \( n_2 \) are the number of jumps of \( N_1 \) and \( N_2 \) in the interval \([t, \tau^*]\), respectively. It follows from (4.7) that conditional on the realization of the jump processes \( \tilde{M}, B(t) A(\tau^*) / B(\tau^*) \) is a geometric Brownian motion with initial value at \( t \) of

\[
J_1(\tilde{M}(\tau^*)) = A(t) e^{\theta(n_1 - n_2) - \lambda \mu (\tau^* - t)[(e^{\gamma - 1}) + (e^{-\gamma} - 1)]}.
\]

Also observe from (3.3) applied to \( P(t, \tau^*) \) and the drift restrictions (3.9) that

\[
K \frac{B(t)}{B(\tau^*)} = K \frac{B(t) P(\tau^*, \tau^*)}{B(\tau^*)}
\]

\[
= P(t, \tau^*) J_2(\tilde{M}(\tau^*)) e^{\int_t^{\tau^*} -\sigma(\tau^* - s) d\tilde{W}(s) - \int_t^{\tau^*} \sigma^2(\tau^* - s)^2/2 ds},
\]

where

\[
J_2(\tilde{M}(\tau^*)) = KL_1(\tilde{M}(\tau^*)),
\]

\[
L_1(\tilde{M}(\tau^*)) = \exp \left\{ \sum_{i=1}^{n_1} -\theta(1 - e^{-\lambda(\tau^* - s)}) - \int_t^{\tau^*} (e^{-\theta(1 - e^{-\lambda(\tau^* - s)})} - 1) \lambda \mu \, ds \right\},
\]

\[
L_2(\tilde{M}(\tau^*)) = \exp \left\{ \sum_{i=1}^{n_2} \theta(1 - e^{-\lambda(\tau^* - s)}) - \int_t^{\tau^*} (e^{\theta(1 - e^{-\lambda(\tau^* - s)})} - 1) \lambda \mu \, ds \right\}
\]

and \( s_i, t_i \) are the times of occurrence of the \( n_1, n_2 \) jumps of \( N_1 \) and \( N_2 \), respectively.

It follows from (4.9) that conditional on \( \tilde{M}(\tau^*), K B(t) / B(\tau^*) \) is a discounted, stochastic exercise price with initial value \( J_2(\tilde{M}(\tau^*)) \) and whose evolution is given by the positive martingale process

\[
e^{\int_t^{\tau^*} -\sigma(\tau^* - s) d\tilde{W}(s) - \int_t^{\tau^*} \sigma^2(\tau^* - s)^2/2 ds},
\]

Hence, it follows from Amin and Jarrow (1991) that the call option's value conditional on \( \tilde{M}(\tau^*), C(t, \tilde{M}(\tau^*)) \) is given by the Black-Scholes formula. Specifically, let \( \Lambda(s, K, R, (\tau^* - t), \nu) \) be the Black-Scholes value of a European call with stock price \( S \), exercise price

\footnote{We could have used instead that both \( \lambda(s) \) and \( \mu_i(s) \) are deterministic functions of time. This restriction implies corresponding restrictions on the excess drift terms via (3.9) of Assumption 3.1.}
$K$, interest rate $R$, time to maturity $(\tau^* - t)$, and volatility rate $\nu^2$. Define

$$R = -[\log P(t, \tau^*)]/(\tau^* - t)$$

and

$$\nu^2 = \int_t^{\tau^*} [\psi^2 + \sigma^2(\tau^* - s)^2] \, ds.$$  

Then

$$(5.9) \quad C(t, \tilde{M}(\tau^*)) = \Lambda(J_1(\tilde{M}(\tau^*)), J_2(\tilde{M}(\tau^*)), R, (\tau^* - t), \nu).$$

The option’s value is obtained by taking the expectation of $C(t, \tilde{M}(\tau^*))$ with respect to the process $\tilde{M}$ under $\tilde{Q}$. $\tilde{M}$ is, under the equivalent martingale measure $\tilde{Q}$, a Poisson process with arrival rates $\lambda \mu$ for $N_1, N_2$, and conditional on $n_1, n_2$ arrivals the times of arrival are uniform in the interval $[t, \tau^*)$. This gives the call option value as

$$(5.10) \quad C(t) = E[C(t, n_1, s_i, i = 1, \ldots, n_1; n_2, t_i, i = 1, \ldots, n_2)].$$

The revised prices $J_1(\tilde{M}(\tau^*))$ and $J_2(\tilde{M}(\tau^*))$ employed in the Black-Scholes formula are the starting values $A(t)$, $K$ adjusted by the pure jumps compensated by a drift term. For example, with respect to $A(t)$, the compensating drift is $(e^n - 1)\lambda \mu (\tau^* - t)$, and the jump shock associated with $N_1$ is $e^{n_1}$. The expectation of $e^{n_1}$ is precisely the exponential of the compensating drift. The adjusting starting price is higher than $A(t)$ if $n_1$ exceeds $(e^n - 1)\lambda \mu (\tau^* - t)/\eta$, which is approximately the expected number of jumps $\lambda \mu (\tau^* - t)$ under the martingale measure.

The call’s value depends on the five parameters $(\nu, \eta, \theta, \xi, \lambda \mu)$ required to estimate the joint volatility structure of the index and the bonds. The parameter $\lambda \mu$ is needed to get the volatility of the jump component $d\tilde{M}$ which is $\mu \lambda \, dt$. Second, $\nu$ is the volatility of the continuous component of the forward price of the stock at time $\tau^*$. Third, the parameter $\eta$ is the percentage shock to the index. Fourth, the parameter $\theta$ is the absolute drop in the short end of the forward rate structure. Fifth, the parameter $\xi$ represents the decline in sensitivity to shocks of the more distant forward rates. These parameters can be estimated either historically or implicitly by inverting market call prices.

For purposes of hedging the European call option, we need to determine the self-financing strategy in the index and at least three pure discount bonds with maturities exceeding $\tau^*$, which duplicates the call’s payout. Four assets are needed to hedge the instantaneous movements in $(\tilde{M}_1, \tilde{M}_2, \tilde{W}_1, \tilde{W}_2)$. This strategy can be obtained from the call option formula (5.10) using a generalized form of Ito’s lemma appropriate for jump processes and used in (3.7). Note that the call option value in this case depends upon $A(t)$ and $P(t, \tau^*)$, and the influence of $\tilde{W}_1$ on the call option value occurs through $P(t, \tau^*)$, while the effect of $\tilde{W}_2$ occurs through $A(t)$. The jump processes $N_i, i = 1, 2$, affect call values through both $A(t)$ and $P(t, \tau^*)$.

Let $A'(t), P'(t)$, and $C'(t)$ be the discounted price processes $A(t)/B(t), P(t, \tau^*)/B(t)$, and $C(t)/B(t)$. Expression (3.10) may be rewritten as

$$(5.11) \quad C'(t) = C'(A'(t), P'(t)).$$
A calculation in the Appendix shows that

\[
C'(A'(t), P'(t)) = C'(A'(0), P'(0)) + \int_{0^+}^{t} C'_A(s_-) A'(s_-) \psi d\tilde{W}_2(s) - C'_p(s_-) P'(s_-) \sigma (\tau^* - s) d\tilde{W}_1(s) + \int_{0^+}^{t} \Delta C'_1(s_-) d\tilde{M}_1(s) + \Delta C'_2(s_-) d\tilde{M}_2(s),
\]

where \(C'_A, C'_p\) are partials of \(C'\) with respect to \(A'\) and \(P'\), respectively, and

\[
\Delta C'_1(s_-) = C'(A'(s_-) e^0, P'(s_-) e^{-\theta(1 - e^{-\epsilon(s_--s)\tau^*})}) - C'(A'(s_-), P'(s_-)),
\]

\[
\Delta C'_2(s_-) = C'(A'(s_-) e^{-\epsilon}, P'(s_-) e^{\theta(1 - e^{-\epsilon(s_--s)\tau^*})}) - C'(A'(s_-), P'(s_-)).
\]

A dynamic self-financing trading strategy in the index and three pure discount bonds with maturities \(T_i, i = 1, 2, 3\), exceeding \(\tau^*\) require holdings \(N^A(t)\) and \(N^p_i\) for \(i = 1, 2, 3\) such that

\[
(5.13) \quad [N^p_1(s_-), N^p_2(s_-), N^p_3(s_-), N^A(t)] H(s) + [\Delta C'_1(s_-), \Delta C'_2(s_-), -C'_p(s_-) P'(s_-) \sigma (\tau^* - s), C'_A(s_-) A'(s_-) \psi].
\]

Expression (5.13) hedges the Brownian risk as well as the jump risk. As the matrix \(H(s)\) is invertible for the case of three pure discount bonds, (5.13) has a unique solution that provides the self-financing strategies position in the three bonds and the index. The position in the money market account is the call value minus the value of this position in these four assets.

6. CONCLUSION

This paper demonstrates how the term structure of interest rates with its continuum of potentially responsive securities may be used to hedge certain systematic jump risks in asset returns, and how to price options in this context. The analysis is for the case where assets returns are driven by a finite number of Brownian motions and an \(m\)-variate point process. The inclusion of the additional traded assets in the term structure makes it possible to hedge systematic jumps. These underlying stochastic processes can be generalized considerably (see Jarrow and Madan 1994) and the qualitative results still apply.

REFERENCES


**APPENDIX**

**Remark.** Under Assumptions 2.1 and 2.2, it can be shown that $r(t)$ satisfies

$$
\int_0^t \left| r(y) \right| dy < +\infty \quad \text{a.e. } Q.
$$

Furthermore, under these assumptions, by Fubini's theorem applied pathwise

$$
\int_0^t \int_0^y \alpha(s, y) ds dy = \int_0^t \int_s^t \alpha(s, y) dy ds < +\infty \quad \text{a.e. } Q \quad \text{for all } t \in [0, T],
$$

$$
\int_t^T \int_0^y \alpha(s, y) ds dy = \int_0^t \int_t^y \alpha(s, y) dy ds < +\infty \quad \text{a.e. } Q
$$

for all $(t, T) \in [0 \leq t \leq T \leq T]$,

$$
\int_0^t \int_0^y \theta_i(s, y) dM_i(s) dy = \int_0^t \left( \int_s^y \theta_i(s, y) dy \right) dM_i(s) < +\infty \quad \text{a.e. } Q
$$

for all $i = 1, \ldots, m$ and $t \in [0, T],$

and

$$
\int_t^T \int_0^y \theta_i(s, y) dM_i(s) dy = \int_0^t \left( \int_t^y \theta_i(s, y) dy \right) dM_i(s) < +\infty \quad \text{a.e. } Q
$$

for all $i = 1, \ldots, m$ and $(t, T) \in [0 \leq t \leq T \leq T]$.

By Heath et al. (1992, appendix),

$$
\int_0^t \int_0^y \sigma_i(s, y) dW_i(s) dy = \int_0^t \left( \int_s^y \sigma_i(s, y) dy \right) dW_i(s) < +\infty \quad \text{a.e. } Q
$$

for all $i = 1, \ldots, d$ and all $t \in [0, T],$. 
\[
\int_0^T \int_0^t \sigma_i(s, y) \, dW_i(s) \, dy = \int_0^t \left( \int_0^T \sigma_i(s, y) \, dy \right) \, dW_i(s) < +\infty \quad \text{a.e. Q}
\]

for all \( i = 1, \ldots, d \) and all \((t, T) \in [0 \leq t \leq T \leq T]\).

These expressions allow the interchange of integrals required in the subsequent derivation of (2.4) and (2.5).

**Derivation of (2.4) and (2.5)**

We first derive (2.5). By substitution of (2.1) we obtain

\[
\log B(t) = \int_0^t r(y) \, dy
\]

\[
= \int_0^t f(0, y) \, dy + \int_0^t \int_0^y \alpha(s, y) \, ds \, dy + \sum_{i=1}^d \int_0^t \int_0^y \theta_i(s, y) \, dM_i(s) \, dy
\]

\[
+ \sum_{i=1}^d \int_0^t \int_0^y \sigma_i(s, y) \, dW_i(s) \, dy.
\]

An interchange of integrals the definition of \( P(0, t) = \exp(-\int_0^t f(0, y) \, dy) \) yields (2.5).

Next, we derive (2.4). Substitution of (2.1) into the definition of

\[
P(t, T) = \exp \left( -\int_t^T f(t, y) \, dy \right)
\]

yields

\[
-\log P(t, T) = \int_t^T f(0, y) \, dy + \int_t^T \int_0^y \alpha(s, y) \, ds \, dy
\]

\[
+ \sum_{i=1}^m \int_t^T \int_0^y \theta_i(s, y) \, dM_i(s) \, dy
\]

\[
+ \sum_{i=1}^d \int_t^T \int_0^y \sigma_i(s, y) \, dW_i(s) \, dy.
\]
An interchange of integrals yields

\[ -\log P(t, T) = \int_t^T f(0, y) \, dy + \int_0^t \int_t^T \alpha(s, y) \, dy \, ds \]
\[ + \sum_{i=1}^m \int_0^t \int_t^T \theta_i(s, y) \, dy \, dM_i(s) \]
\[ + \sum_{i=1}^d \int_0^t \int_t^T \sigma_i(s, y) \, dy \, dW_i(s). \]

Adding and subtracting \( \log B(t) \) from the above (using (2.5)) and combining like terms generates (2.4).

Proof of Proposition 3.1

Uniqueness. Take any \( \tilde{Q} \) on \((\Omega, \mathcal{F}, \mathbb{F})\) such that \( Q \sim \tilde{Q} \) makes \( P(t, T)/B(t) \) for all \( T \in [\tau^*, T] \) and \( A(t, x)/B(t) \) \( \tilde{Q} \)-martingales for all \( t \in [0, \tau^*] \).

By Proposition 2.1, the process \( Z(t) \equiv E[d\tilde{Q}/dQ \mid \mathcal{F}_t] \) is a strictly positive martingale with a representation of the form given by (3.1). Algebra yields (3.3) and (3.4) and (3.5) and (3.6) follow from the condition that the discounted asset prices \( P(t, T)/B(t) \) and \( A(t, x)/B(t) \) are \( \tilde{Q} \)-martingales (see the discussion preceding (3.5)). Assumption 3.1 implies that the equation systems (3.5) and (3.6) rewritten as (3.9) have a unique solution and, hence, \( Z(t) \) is unique.

Existence. Define a random variable \( Z(t) \) on \([0, \tau^*]\) by (3.1) where the coefficients \( \mu_i(s) \) for \( i = 1, \ldots, m \) and \( \gamma_i(s) \) for \( i = 1, \ldots, d \) are the unique solutions to the equation system (3.9) given by Assumption 3.1. The process \( Z \) is a supermartingale by construction. The integrability Assumptions 3.1(a)–(c) yield that \( E[Z(t)] = 1 \) for all \( t \in [0, \tau^*] \) and hence that \( Z(t) \) is a \( \tilde{Q} \)-martingale and therefore the measure \( \tilde{Q} \) on \((\Omega, \mathcal{F})\) defined by \( \tilde{Q}(A) \equiv E(1_A Z(\tau^*)) \) is a probability.

Following the same algebra as in the uniqueness proof, we derive equation systems (3.3) and (3.4). By Proposition 2.1, \((\tilde{W}, \tilde{M})\) are \( \tilde{Q} \)-martingales and the integrability conditions of Assumption 3.1(a)–(f) yield that the Doléans-Dade exponentials expressions in (3.3) and (3.4) are also \( \tilde{Q} \)-martingales. Equations (3.9) of Assumption 3.1 then imply that the discounted asset prices \( P(t, T)/B(t) \) and \( A(t, x)/B(t) \) are \( \tilde{Q} \)-martingales as the final exponential expressions in (3.3) and (3.4) are unity for all \( t \).