8
Credit Risk

ROBERT JARROW AND STUART TURNBULL

8.1 INTRODUCTION
We consider two facets of credit risk. First, the pricing of derivatives written on assets subject to default risk. An example is the pricing of derivatives written on corporate bonds, where there is a positive probability that default may occur on the part of the issuer of the bonds. Second is the pricing of derivatives where the writer of the derivative might default. Consider an over-the-counter option written on a Treasury bond. There is no default risk arising from the underlying asset—the Treasury bond. However, there is default risk arising from the fact that the writer of the option may not be able to honour the obligation if the option is exercised. This form of risk is referred to as counterparty risk. In the over-the-counter market counterparty risk is a major concern to financial institutions and regulatory bodies. We will describe a simple approach to the pricing and hedging of both forms of credit risk. We will give a number of examples: the pricing of options on credit risky bonds, the pricing of over-the-counter caps, and the pricing of credit default swaps.

8.2 PRICING CREDIT RISKY BONDS
Firms are allocated to particular risk classes AAA, AA, etc. on the basis of their current creditworthiness. A typical set of term structures is shown in Figure 8.1. A firm in credit class AAA is assumed to have the least credit risk among corporate firms. Firms of lower credit than AAA, such as those in credit class AA, trade at a lower price and are thus higher yield.

Suppose we want to price a derivative written on a zero coupon bond issued by a firm with credit rating ABC. We must price this derivative in such a way that it is (i) consistent with the absence of arbitrage; (ii) consistent with the relevant initial term structures of interest rates; and (iii) consistent with a positive probability of default. We do this by first constructing a lattice of one-period interest rates to model the term structure of default-free Treasury bills. This is described in Jarrow and Turnbull (1995). Next we consider zero coupon bonds for the firm belonging to the particular risk class, ABC.
8.2.1 Lattice of Default-free Interest Rates

The prices of default-free zero coupon bonds are given in Table 8.1. Following Black et al. (1991), it is assumed that spot interest rates are log-normally distributed. The lattice is shown in Figure 8.1. The value of the one-year default-free bond, face value 100, is

\[ B_F(0, 1) = 100 \exp(-0.047175) = 95.3921 \]

The value of the two-year default-free zero coupon bond, face value 100, at year one is

\[ B_F(1, 2) = 100 \exp(-0.053810) = 94.7612 \]

if the spot rate is 5.3810 per cent, and

\[ B_F(1, 2) = 100 \exp(-0.048689) = 95.2477 \]

if the spot rate is 4.8689 per cent.

We know from Black et al. (1991) that normalized prices are a martingale under the martingale probabilities. It is assumed that the martingale probability of the spot interest in an up-state is 0.5. Therefore,

\[ B_F(0, 2) = \exp(-0.047175)(0.5 \times 94.7612 + 0.5 \times 95.2477) \]

\[ = 90.6267 \]

<table>
<thead>
<tr>
<th>Maturity (Years)</th>
<th>Default-Free</th>
<th>Credit Class ABC</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( B_F(0, T) )</td>
<td>( v(0, t; D) )</td>
</tr>
<tr>
<td>1</td>
<td>95.3921</td>
<td>95.0486</td>
</tr>
<tr>
<td>2</td>
<td>90.6264</td>
<td>89.7056</td>
</tr>
<tr>
<td>3</td>
<td>85.7820</td>
<td>84.1008</td>
</tr>
</tbody>
</table>
\[ r(2)_1 = 6.0583\% \\
= B_F(2, 3) = 94.1216 \]

\[ r(1)_1 = 5.3810\% \]

\[ r(0) = 4.7175\% \]

\[ r(2)_2 = 5.4817\% \\
= B_F(2, 3) = 94.6658 \]

\[ r(1)_2 = 4.8689\% \]

\[ r(2)_3 = 4.9601\% \\
= B_F(2, 3) = 95.1609 \]

\[ \pi \] is the martingale probability of an up-state (\( \pi = 0.5 \)).

\[ 1-\pi \] is the martingale probability of a down-state (\( 1-\pi = 0.5 \)).

Volatility is 5 per cent.

**Figure 8.2** Default-free spot interest rates

which equals the number in Table 8.1, ignoring a small round-off error. Extending this analysis gives \( B_F(0, 3) = 85.7820 \).

### 8.2.2 Risky Debt

We want to value a zero coupon bond for a firm belonging to the credit class ABC. Let \( v(t, T; DS_t) \) denote the value at date \( t \) of a zero coupon bond issued by the firm. The debt matures at time \( T \) and the bondholders are promised the face value of the bond at maturity. Let the face value be USD 100. There is a positive probability that the firm might default over the life of the bond. If default occurs, the bondholders will receive less than the promised amount. The symbol \( DS_t \) is used to denote the default status of the bond at date \( t \):

\[
DS_t = \begin{cases} \\
D; \text{default has not occurred at date } t \\
\bar{D}; \text{default has occurred at or before date } t 
\end{cases}
\]

As the symbol \( DS_t \) indicates, there are two possibilities. One, default does not occur before or at date \( t \), denoted \( \bar{D} \); and two, default does occur before or at date \( t \), denoted \( D \).

We can always view the pricing of credit risky bonds in terms of a foreign currency analogy. Imagine a hypothetical currency, called ABCs. In terms of this currency, we can view the debt issued by the firm as default-free. Indeed, at maturity, the bondholder is issued the face value of debt in ABCs. But, this currency is useless to the bondholder, so we need to define an exchange rate which converts this hypothetical currency to dollars. After all, the bondholders are interested in the dollar value of their ABCs. If default has not occurred before or at date \( t \), then the exchange rate is unity. If default did occur, it is
assumed that we get some fraction, \( \delta \), of a dollar for each ABC. This is the same as being paid the fraction \( \delta \) of the face amount of the debt. The fraction \( \delta \) is also called the pay-off ratio or recovery rate. Defining \( e(t) \) as the date \( t \) exchange rate per ABCs, we have:

\[
e(t) = \begin{cases} 
1; & \text{with probability } 1 - \mu(t)h \text{ if } DS_t = \overline{D} \\
\delta; & \text{with probability } \mu(t)h \text{ if } DS_t = D
\end{cases}
\]  

(1)

where \( 0 \leq \delta < 1; \ h \) denotes the time interval and \( \mu(t)h \) is the martingale probability of default occurring, conditional upon no default at or before date \( t - h \). We are interested in the martingale probabilities of default because we want to develop pricing formulae which are arbitrage-free.\(^1\) If default has occurred at or before date \( t - h \), then it is assumed that the bond remains in default and the pay-off ratio constant at \( \delta \) dollars,

\[
e(t) \equiv \delta
\]  

(2)

The conditional martingale probabilities of default can be estimated using the observed term structures of interest rates. We will discuss how to do this below.

To simplify the analysis, we are going to assume that the default process is independent of the level of the default-free rate of interest. This implies that if interest rates are "high" or "low" this has no effect on the probability of default. It is a useful first approximation, and its relaxation is discussed in Jarrow and Turnbull (1995b).

### 8.2.3 Credit Risky Debt

In Table 8.1 we are given two sets of prices for zero coupon bonds. The first is for default-free bonds and the second is for bonds belonging to credit class ABC. The default-free bonds at each maturity are seen to be more valuable than the equivalent maturity bond issued by the firm in credit class ABC. This difference reflects the likelihood of default. We want to estimate these implicit martingale probabilities of default.

Before we can do this, however, we must first specify the pay-off ratio \( \delta \) in the event of default. This value comes from our credit risk analysts, who estimate that given the nature of the debt, we expect to receive USD 0.40 on the dollar in the event of default.\(^2\)

Consider first the one-year bond. For simplicity, we take the interval in the lattice to be one year. At maturity, the credit risky bond's value is:

\[
v(1, 1, DS) = 100 \begin{cases} 
1; & \text{probability } 1 - \mu(0)h \text{ if } DS_1 = \overline{D} \ (\text{no default}) \\
\delta; & \text{probability } \mu(0)h \text{ if } DS_1 = D \ (\text{default})
\end{cases}
\]  

(3)

where \( h = 1 \) and \( \delta = 0.40 \). The face value of the bond is 100.

The default process is shown in Figure 8.3. Given that default has not occurred at date \( t = 0 \), the conditional (martingale) probability that default occurs at \( t = 1 \) is denoted by \( \mu(0) \times h \), where \( h \) is the time interval. In this example \( h = 1 \). The conditional (martingale) probability that default does not occur is \( 1 - \mu(0)h \). We can use the term structures of interest rates for default-free bonds and for credit class ABC bonds to infer the value of \( \mu(0) \).

The expected value of the pay-off is

\[
v(0, 1, \overline{D}) = 0.9539 \times 100\{1 \times [1 - \mu(0)] + \delta \times \mu(0)\}
\]  

and discounting at the risk-free rate gives, using Table 8.1

\[
v(0, 1, \overline{D}) = 0.9539 \times 100\{1 \times [1 - \mu(0)] + \delta \times \mu(0)\}
\]  

(4)
From Table 8.1, we know that \( v(0, 1, D) = 95.0401 \). Therefore
\[
95.0486 = 0.9539 \times 100([1 - \mu(0)] + 0.40\mu(0))
\]
Equation (5)

Solving for martingale probability of default gives
\[
(1 - 0.40) \times \mu(0) = 1 - (95.0486/0.9539)/100
\]
or
\[
\mu(0) = 0.006
\]

The pricing of the two-period zero coupon bond is slightly more complicated because at the end of the first period both interest rates and the default status of the firm are uncertain. The default process is shown in Figure 8.4. If default has occurred at date \( t = 1 \), then the bond is assumed to remain in default. If default has not occurred at date \( t = 1 \), then one period later at date \( t = 2 \) either default occurs or it does not. The martingale probability of default occurring at date \( t = 2 \) conditional upon the fact that default has not occurred at date \( t = 1 \) is \( \mu(1)h \). The conditional (martingale) probability that default does not occur at date \( t = 2 \) is \( 1 - \mu(1)h \). Figure 8.4 is combined with Figure 8.2 and the possible states are shown in Figure 8.5. The same argument is used to determine the conditional martingale probability of default \( \mu(1) \).

Let us start at State A, at date \( t = 1 \). The value of a default-free bond, face value of 1, that matures at \( t = 2 \) is
\[
B(1, 2)_d = \exp(-0.0487)
\]
The subscript “d” refers to the “down” state for the default-free spot rate of interest. Default has occurred at date \( t = 1 \), so the pay-off to the bond at date \( t = 2 \) is
\[
v(2, 2; D) = 100\delta
\]
The value in State A at date \( t = 1 \) is
\[
v_A(1, 2; D) = \exp(-0.0487)(100\delta) = B(1, 2)_d(100\delta)
\]
A similar argument applies if State B occurs:

\[ v_B(1, 2, D) = \exp(-0.0538)(100\delta) \]

\[ = B(1, 2)_u(100\delta) \]

where \( B(1, 2)_u \) is the value at date \( t = 1 \) of a one-period default-free bond and is equal to \( B(1, 2)_u = \exp(-0.0538) \)

The subscript "u" refers to the "up" state for the default-free spot rate of interest.

If State C occurs, then the argument is more interesting. In State C default has not occurred, so that one period later, at maturity, one of two possible states can occur:

\[ v(2, 2, DS) = 100 \begin{cases} 1: \text{probability } 1 - \mu(1) \text{ if } DS_2 = \overline{D} \text{ (no default)} \\ \delta: \text{probability } \mu(1) \text{ if } DS_2 = D \text{ (default)} \end{cases} \]

Therefore, in State C the value of the bond is

\[ v_C(1, 2, \overline{D}) = B(1, 2)_u 100[(1 - \mu(1)) + \delta \mu(1)] \]

In State D a similar argument applies:

\[ v_D(1, 2, D) = B(1, 2)_u 100[(1 - \mu(1)) + \delta \mu(1)] \]

The value of the credit risky bond today, \( v(0, 2, D) \), is determined by calculating the expected value of the bond at date \( t = 1 \) using the martingale probabilities and discounting at the risk-free rate of interest. Referring to Figure 8.5, there are four possible states.
\[ r(0) = 4.72\% \quad e(0) = 1 \]

\[ r(1) = 5.38\% \quad e(1) = 1 \]

\[ r(2) = \delta \]

\[ e(2) = 1 \]

\[ e(2) = \delta \]

\[ r(1) = 5.38\% \quad e(1) = \delta \]

\[ r(2) = \delta \]

\[ r(1) = 4.87\% \quad e(1) = \delta \]

\[ r(2) = \delta \]

\[ r(0) = 4.72\% \quad e(0) = 1 \]

\[ r(1) = 5.38\% \quad e(1) = 1 \]

\[ r(2) = \delta \]

\[ e(2) = 1 \]

\[ e(2) = \delta \]

<table>
<thead>
<tr>
<th>State</th>
<th>Martingale Probability of Occurrence</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(1-(\pi)) (\mu(0))</td>
</tr>
<tr>
<td>B</td>
<td>(\pi \mu(0))</td>
</tr>
<tr>
<td>C</td>
<td>(1-(\pi)) ([1-\mu(0)])</td>
</tr>
<tr>
<td>D</td>
<td>(\pi \left[1-\mu(0)\right])</td>
</tr>
</tbody>
</table>

**Figure 8.5** Two-period credit risky debt: determining the implied probabilities

Therefore

\[ v(0, 2; \bar{D}) = 0.9539 \times [(1 - \pi)\mu(0)v_A(1, 2; D) + \pi\mu(0)v_B(1, 2, D) + \pi(1 - \mu(0))v_C(1, 2, \bar{D}) + \pi(1 - \mu(0))v_D(1, 2, \bar{D})] \]

Substituting the values of the bond in the four different states gives

\[ v(0, 2; \bar{D}) = 0.9539 \times [(1 - \pi)\exp(-0.0487) + \pi\exp(-0.0538)\mu(0)(100\delta) + 0.9539 \times [(1 - \pi)\exp(-0.0487) + \pi\exp(-0.0538)] \times [1 - \mu(0)]100\left[(1 - \mu(1)) + \mu(1)\delta\right] \]

The above calculation can be simplified by considering the pricing of a two-period default-free zero coupon bond. Consider the value of the default-free bond at \(t = 1\):

\[ B(1, 2) = \begin{cases} \exp(-0.0538); & \text{martingale probability } \pi \\ \exp(-0.0487); & \text{martingale probability } 1 - \pi \end{cases} \]
and the value of the default-free bond today at \( t = 0 \) is

\[
B(0, 2) = 0.9539 \times [(1 - \pi) \exp(-0.0487) + \pi \exp(-0.0538)]
\]

Therefore

\[
v(0, 2; \bar{D}) = B(0, 2)[\mu(0)(100\delta) + [1 - \mu(0)]100[1 - \mu(1)] + \mu(1)\delta]]
\] (6)

From Table 8.1 we have \( B_F(0, 2) = 90.6264 \), \( v(0, 2, \bar{D}) = 89.7056 \) and \( \delta = 0.40 \). We have estimated \( \mu(0) = 0.006 \). Therefore, substituting these values into equation (6) and solving for \( \mu(1) \) gives

\[\mu(1) = 0.011\]

The martingale probability of default at time 1, as implied by the bond prices, is almost twice that of default at time 0. Given \( \mu(0) = 0.006 \) and \( \mu(1) = 0.011 \), the pricing of the two-period credit risky debt is summarized in Figure 8.6.

Repeating this argument for the three-year zero coupon bond gives

\[\mu(2) = 0.016\]

Equation (5) can be written in the form

\[v(0, 1, \bar{D}) = B_F(0, 1)E_0^\bar{D}[e(1)]\] (7)

\[
\begin{align*}
t = 0 & \\
State D
& \\
\quad v(0,2,\bar{D}) = 89.7056 \\
\quad \mu(0) = 4.72\%
\end{align*}
\]

\[
\begin{align*}
t = 1 & \\
State D
& \\
\quad v(1,2,\bar{D}) = 94.14 \\
\quad r(1) = 5.38\% \\
\quad v(2,2,\bar{D}) = 40
\end{align*}
\]

\[
\begin{align*}
t = 2 & \\
State D
& \\
\quad v(2,2,\bar{D}) = 100
\end{align*}
\]

\[
\begin{align*}
t = 0 & \\
State C
& \\
\quad v(0,2,\bar{D}) = 89.7056 \\
\quad \mu(0) = 4.72\%
\end{align*}
\]

\[
\begin{align*}
t = 1 & \\
State C
& \\
\quad v(1,2,\bar{D}) = 94.62 \\
\quad r(1) = 4.87\% \\
\quad v(2,2,\bar{D}) = 40
\end{align*}
\]

\[
\begin{align*}
t = 2 & \\
State C
& \\
\quad v(2,2,\bar{D}) = 100
\end{align*}
\]

\[
\begin{align*}
t = 0 & \\
State B
& \\
\quad v(0,2,\bar{D}) = 37.60 \\
\quad \mu(0) = 5.38\%
\end{align*}
\]

\[
\begin{align*}
t = 1 & \\
State B
& \\
\quad v(1,2,\bar{D}) = 38.10 \\
\quad r(1) = 4.87\% \\
\quad v(2,2,\bar{D}) = 40
\end{align*}
\]

\[
\begin{align*}
t = 2 & \\
State B
& \\
\quad v(2,2,\bar{D}) = 40
\end{align*}
\]

\[
\begin{align*}
t = 0 & \\
State A
& \\
\quad v(0,2,\bar{D}) = 89.7056 \\
\quad \mu(0) = 4.72\%
\end{align*}
\]

\[
\begin{align*}
t = 1 & \\
State A
& \\
\quad v(1,2,\bar{D}) = 38.10 \\
\quad r(1) = 4.87\% \\
\quad v(2,2,\bar{D}) = 40
\end{align*}
\]

\[
\begin{align*}
t = 2 & \\
State A
& \\
\quad v(2,2,\bar{D}) = 40
\end{align*}
\]

Figure 8.6 Two-period credit risky debt: summary of results
where the expected pay-off is

\[ E_0^Q[e(1)] = 1 - \mu(0) + \delta \mu(0) \]
\[ = 0.9964 \]

Hence

\[ B_T(0, 1)E_0^Q[e(1)] = 95.3921 \times 0.9964 \]
\[ = 95.0486 \]

which agrees with Table 8.1.

Equation (6) can be written in the form

\[ v(0, 2, \overline{D}) = B_T(0, 2)E_0^Q[e(2)] \]  \hspace{1cm} (8)

where the expected pay-off is

\[ E_0^Q[e(2)] = [1 - \mu(0)][1 - \mu(1) + \delta \mu(1)] + \delta \mu(0) \]
\[ = 0.989840 \]

Hence

\[ B_T(0, 2)E_0^Q[e(2)] = 90.6264 \times 0.989840 \]
\[ = 89.7056 \]

which agrees with Table 8.1.

In general one can write

\[ v(0, T; \overline{D}) = B_T(0, T)E_0^Q[e(T)|\overline{D}] \]  \hspace{1cm} (9)

Equation (9) gives the value of the zero coupon bond if the firm is not in default. Equation (9) is an important and intuitive result. It is important because (i) it provides a practical way of computing the martingale probabilities of default using market data, and (ii) it can be used for pricing derivatives on credit risky cash flows. It is intuitive because the second term in equation (9), \( E_0^Q[e(T)|\overline{D}] \), can be interpreted as the date 0 present value of the promised pay-off at date \( T \). We can rewrite equation (9) in the form

\[ E_0^Q[e(T)|\overline{D}] = v(0, T, \overline{D})/B_T(0, T) \]  \hspace{1cm} (10)

where the right-hand side can be interpreted as a credit spread.

### 8.3 Pricing Options on Credit Risky Bonds

Consider a put option written on debt issued by the ABC company. The maturity of the option is one year. At maturity the option allows you to sell, for a strike price of 94, a two-year bond with a coupon of USD 3 paid annually and face value of USD 100, issued by the ABC firm. To price this option, we start by considering the value of the option at its maturity date. For simplicity of exposition, we maintain our assumption that the
length of the lattice interval is one year, so that we can use all the results summarized in Figure 8.6. In practice, one would use intervals of shorter length than a year.

At the maturity of the option, at date $t = 1$, there are four possible values of the underlying bond depending on interest rates and whether default has occurred or not. If default has not occurred, then the value of the coupon bond is

$$v_c(1; \bar{D}) = 3B(1, 2)E_t^Q[e(2)|\bar{D}] + (3 + 100)B(1, 3)E_t^Q[e(3)|\bar{D}]$$

using equation (9), where $B(1, T)$ denotes the value at date $1$ of receiving one dollar for sure at date $T$. If default has occurred, then the value of the coupon bond is

$$v_c(1; D) = 3B(1, 2)E_t^Q[e(2)|D] + (3 + 100)B(1, 3)E_t^Q[e(3)|D]$$

Relevant values are shown in Figure 8.7.

---

**Figure 8.7** Part A: default-free term structure, Part B: default/no default states
The values of the coupon bond in the four possible states are shown in Figure 8.8, along with the values of the put option. To determine the option value today, we must calculate the expected value of the option at date \( t = 1 \) and discount back at the risk-free rate of interest.

In States A and B default has occurred on the underlying ABC zero coupon bond. The value of the option varies over these two states because the value of the underlying asset varies due to the interest rate risk. In States C and D the underlying asset at \( t = 1 \) is not in default. The value of the option today is

\[
p(0) = 0.9539 \times \left[ (1 - \mu(0)) \left( \pi[0.5267 + (1 - \pi)0] + \mu(0)[\pi56.01 + (1 - \pi)55.6111] \right) \right]
\]

(11)

Given that \( \pi = 0.5 \) and \( \mu(0) = 0.006 \) then

\[
p(0) = 0.9539 \times \left[ (1 - \mu(0))0.2634 + \mu(0)55.8105 \right]
\]

\[= 0.5692\]

This methodology can easily be extended to price American options on credit risky bonds.
8.4 PRICING VULNERABLE DERIVATIVES

This section studies the pricing of vulnerable derivatives. **Vulnerable derivatives** are derivative securities subject to the additional risk that the writer of the derivative might default. Consider an example of an over-the-counter (OTC) option written on a Treasury bill. There is no default risk associated with the underlying asset—the Treasury bill. However, the writer of the option is a financial institution which may default, so that there is the risk that if the option is exercised the writer may be unable to fulfil the obligation to make the required payment to the option owner. The methodology that we have developed can handle this problem. A simple example is used to illustrate the procedure.

First, let us assume there is no risk of the writer defaulting. Consider a call option written on a Treasury bill. The maturity is one year, and at expiration the option holder can purchase a one-year Treasury bill at a strike price of 92. The option is valued using the information summarized in Figure 8.9. The lattice of interest rates comes from Figure 8.8. The date 0 value of the call option is

\[
c(0) = 0.9539[0.5 \times 2.76 + 0.5 \times 3.40]
= 2.94
\]

Now assume that the financial institution that wrote the option belongs to the ABC risk class. When the option matures there are four possible states depending on whether interest rates go up or down and whether the writer defaults. The four states are shown in Figure 8.10. This figure is similar in nature to Figure 8.8. In States A and B the writer defaults. By assumption, claim holders receive as a pay-off ratio 40 per cent of the value of their option.

The value of the vulnerable option today is

\[
c_v(0) = 0.9539 \times [(1 - \mu(0))[\pi 2.76 + (1 - \pi)3.40]
+ \mu(0)[\pi 2.76 + (1 - \pi)3.40] \times 0.40]
\]

<table>
<thead>
<tr>
<th>$t = 0$</th>
<th>$t = 1$</th>
<th>Value of One-Period Zero Coupon Bonds, $B_p(1,2)$</th>
<th>Value of Option*</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r(0) = 4.72%$</td>
<td>94.7612</td>
<td>2.76</td>
<td></td>
</tr>
<tr>
<td>$r(1) = 5.38%$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\pi$</td>
<td>95.4028</td>
<td>3.40</td>
<td></td>
</tr>
</tbody>
</table>

* Strike Price is 92 where $\pi = 0.5$ represents the martingale probability of an up-state.

**Figure 8.9** Pricing a default-free Treasury bill call option
\[ t = 0 \quad t = 1 \quad \text{Value of Option} \]

- State D
  - No Default
  - \(1 - \mu(0)\)
    - State C
      - No Default
      - \(1 - \mu(0)\)
        - State B
          - Default
          - \(\mu(0)\)
          - \(2.76 \times 0.40 = 1.10\)
        - State A
          - Default
          - \(\mu(0)\)
          - \(3.40 \times 0.40 = 3.75\)

Where \(\mu(0) = 0.006\) represents the martingale probability of default occurring at date \(t = 1\), conditional upon not being in default at date \(t = 0\).

**Figure 8.10** Pricing a vulnerable call option

Given that \(\pi = 0.5\) and \(\mu(0) = 0.006\), then

\[
c_V(0) = (1 - \mu(0))c(0) + \mu(0)d(0) \times 0.40
\]

\[= 2.93\] (12)

The difference in the option prices is small, only 1 cent, which is to be expected given that the martingale probability of default is small.

### 8.4.1 Formalization

This section formalizes the analysis in the previous example. This involves little more than replacing numerical values with symbols. Let \(c(1)\) represent the value of the option at date \(t = 1\) in the absence of default on the part of the writer, and \(c_V(1)\) the value of the vulnerable option. At maturity the pay-off to the option holder is given by

\[
c_V(1) = \begin{cases} 
c(1); & \text{no default} \\
\delta c(1); & \text{default}
\end{cases}
\]

where \(\delta\) represents the pay-off fraction of the option the holder receives if default occurs.

The date 0 value of the option in the absence of default is \(c(0)\) and the value of the vulnerable option is, using equation (12),

\[
c_V(0) = (1 - \mu(0))c(0) + \mu(0)d(0)
\]

\[= E^G_0[e(1)]c(0)\] (13)

because \(E^G_0[e(1)] = (1 - \mu(0)) + \mu(0)\delta\).
This result has an important implication. Given that there is a positive probability of default, then

\[ E_0^Q[e(1)] < 1 \]

which implies that a vulnerable option must always be worth less than a non-vulnerable option,

\[ c_V(0) < c(0) \]  \hspace{1cm} (14)

Equation (14) generalizes in a natural way for a European option that matures at date \( T \):

\[ c_V(0) = E_0^Q[e(T)c(0)] \]  \hspace{1cm} (15)

Using equation (9) this can be written in the form

\[ c_V(0) = \frac{v(0, T; D)}{B_r(0, T)}c(0) \]  \hspace{1cm} (16)

Table 8.2  Pricing a vulnerable cap

Part A: Term structure data

<table>
<thead>
<tr>
<th>Maturity (Years)</th>
<th>Default-Free</th>
<th>Credit Class A</th>
<th>Credit Class B</th>
<th>Credit Class C</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>97.7098</td>
<td>97.4460</td>
<td>97.4069</td>
<td>97.2334</td>
</tr>
<tr>
<td>1.0</td>
<td>95.3513</td>
<td>94.8364</td>
<td>94.7542</td>
<td>94.4131</td>
</tr>
<tr>
<td>1.5</td>
<td>92.9414</td>
<td>92.1883</td>
<td>92.0598</td>
<td>91.5571</td>
</tr>
<tr>
<td>2.0</td>
<td>90.4954</td>
<td>89.5169</td>
<td>89.3397</td>
<td>88.6816</td>
</tr>
<tr>
<td>2.5</td>
<td>88.0269</td>
<td>86.8356</td>
<td>86.6079</td>
<td>85.8011</td>
</tr>
<tr>
<td>3.0</td>
<td>85.5478</td>
<td>84.1563</td>
<td>83.8770</td>
<td>82.9279</td>
</tr>
<tr>
<td>3.5</td>
<td>83.0689</td>
<td>81.4894</td>
<td>81.1579</td>
<td>80.0733</td>
</tr>
<tr>
<td>4.0</td>
<td>80.5994</td>
<td>78.8440</td>
<td>78.4602</td>
<td>77.2466</td>
</tr>
<tr>
<td>4.5</td>
<td>78.1475</td>
<td>76.2278</td>
<td>75.7919</td>
<td>74.4562</td>
</tr>
</tbody>
</table>

Part B: Pricing the caplets

<table>
<thead>
<tr>
<th>Maturity (Years)</th>
<th>Value of Caplet*</th>
<th>Credit Class A</th>
<th>Credit Class B</th>
<th>Credit Class C</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>70</td>
<td>69.81</td>
<td>69.78</td>
<td>69.66</td>
</tr>
<tr>
<td>1.0</td>
<td>1092</td>
<td>1086.10</td>
<td>1085.16</td>
<td>1081.26</td>
</tr>
<tr>
<td>1.5</td>
<td>3212</td>
<td>3185.97</td>
<td>3181.53</td>
<td>3164.16</td>
</tr>
<tr>
<td>2.0</td>
<td>5877</td>
<td>5813.45</td>
<td>5801.95</td>
<td>5759.21</td>
</tr>
<tr>
<td>2.5</td>
<td>8709</td>
<td>8591.14</td>
<td>8515.19</td>
<td>8488.79</td>
</tr>
<tr>
<td>3.0</td>
<td>11484</td>
<td>11297.20</td>
<td>11259.71</td>
<td>11132.30</td>
</tr>
<tr>
<td>3.5</td>
<td>14094</td>
<td>13826.01</td>
<td>13769.77</td>
<td>13585.75</td>
</tr>
<tr>
<td>4.0</td>
<td>16472</td>
<td>16113.25</td>
<td>16034.81</td>
<td>15786.79</td>
</tr>
<tr>
<td>4.5</td>
<td>18393</td>
<td>18136.26</td>
<td>18032.55</td>
<td>17714.76</td>
</tr>
<tr>
<td>Total</td>
<td>79603</td>
<td>78119.19</td>
<td>77750.45</td>
<td>76782.60</td>
</tr>
<tr>
<td>Difference</td>
<td>1483.81</td>
<td>1852.55</td>
<td>2820.32</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.86%</td>
<td>2.33%</td>
<td>3.54%</td>
<td></td>
</tr>
</tbody>
</table>

*Volatility Reduction Factor \( 1.2 \) per cent
Volatility Reduction Factor \( 0.15 \)
Cap Rate \( 7.00 \) per cent
Principal \( USD 10 \) million
This form of the expression is useful in practice because it involves pricing a vulnerable option in terms of a credit risk spread for the writer, and the price of a non-vulnerable option.

8.4.2 Example

A firm wants to buy a five-year interest rate cap on the six-month default-free interest rate. Three institutions offer to sell the firm a cap. The institutions, however, have different credit ratings. Institution A belongs to credit class A, institution B belongs to credit class B, and institution C belongs to credit class C. Credit class A has a lower risk of default than credit class B, and credit class B has a lower risk of default than credit class C.

The term structure details are given in Table 8.2, Part A, for default-free interest rates and the three credit classes. The value of the caplets, assuming no counterparty risk, is calculated using the Heath et al. (1991) model, assuming interest rates are normally distributed. The prices of the caplets are given in Table 8.2, Part B.

To incorporate the effects of counterparty risk, equation (16) is used. Consider the last caplet. The value in the absence of counterparty risk is USD 18,593. For institution A, belonging to credit class A, using the figures from the last row of Table 8.2, Part A:

\[ v_A(0, 4.5, \bar{D})/B_F(0, 4.5) = 76.2278/78.1475 \]

\[ = 0.9754 \]

Therefore using equation (16) the value of the caplet is

\[ \text{USD 18,593} \times 0.9754 = \text{USD 18,136.26} \]

as shown in Table 8.2, Part B. The values of the other caplets are calculated in a similar way. For institution A, its credit risk lowers the value of the cap by approximately 1.86 per cent; for institution B, 2.33 per cent; and for institution C, 3.54 per cent.

8.5 CREDIT DEFAULT SWAP

We now examine the pricing of a simple credit default swap. Consider a one-year credit default swap referenced to two credits. The basic structure is shown in Figure 8.11. The bank is buying protection from the counterparty on the first of two credits to experience a default. The counterparty has a liability to pay the bank in the event that one of the two reference credits defaults. The counterparty’s exposure is to two names or reference

![Figure 8.11](A simple credit default swap)
credits and the exposure is limited to the first name to default. After the first default, any exposure to subsequent defaults is terminated. In the event of a default by one of the two names, the counterparty pays a fixed amount to the bank. In return for this default insurance, the bank pays a premium to the counterparty.

To illustrate how to price this form of swap, the data in Table 8.2 will be used. It is assumed that the counterparty belongs to credit class A and the two reference credits belong to credit class C. Conditional on no defaults by the two reference credits at date \( t - 1 \), payment by the counterparty at date \( t \) is described by one of four mutually exclusive and exhaustive events:

1. First credit defaults, second credit does not default.
2. First credit does not default, second credit defaults.
3. First credit defaults, second credit defaults.
4. First credit does not default, second credit does not default.

If one of the first three events occurs, then the counterparty makes a fixed payment, \( F \), to the bank. If event four occurs, then no payment occurs. The probability that first (second) credit does not default at date \( t \), conditional upon no default at date \( t - 1 \) is \( [1 - \mu_c(t - 1)h] \), where \( \mu_c(t - 1) \) is the (martingale) conditional probability of default occurring at date \( t \) for a firm in credit class C, conditional upon no default at date \( t - 1 \), and \( h \) is the length of the interval between dates \( t - 1 \) and \( t \). Assuming independence between the event of default for the first credit and the second credit, the conditional probability of event four occurring is \( [1 - \mu_c(t - 1)h]^2 \).

To summarize the payment by the counterparty to the bank, it will prove useful to define the following indicator function. Conditional upon no default at date \( t - 1 \),

\[
e_1(t) = \begin{cases} 
0; & \text{probability } [1 - \mu_c(t - 1)h]^2 \\
1; & \text{probability } 1 - [1 - \mu_c(t - 1)h]^2
\end{cases}
\]  

(17)

If \( e_1(t) = 0 \) at date \( t \), then this implies event four has occurred and no payment is made by the counterparty to the bank; if \( e_1(t) = 1 \) at date \( t \), then this implies that either event one, two, or three has occurred and the counterparty makes a payment, \( F \), to the bank. If a default has occurred at or prior to date \( t - 1 \), then define

\[
e_1(t) = 0
\]  

(18)

implying that the counterparty’s exposure is terminated. In this example the credit swap has maturity of one year. For the sake of simplicity, we have divided the one year into two half year intervals. In practice, shorter intervals would be used. The default payment process over the two intervals is shown in Figure 8.12.

Referring to Figure 8.12, if no defaults have occurred at date \( t = 1 \), then the value of the swap is

\[
V(1; D) = B(1, 2)[0 \times q_1 + F \times (1 - q_1)]
\]  

(19)

where \( B(1, 2) \) is the value at date \( t = 1 \) of a default-free zero coupon bond that pays one dollar at date 2. If one or more defaults occur at date \( t = 1 \), then

\[
V(1; D) = F
\]  

(20)

Today the value of the swap is

\[
V(0; D) = B(0, 2)q_0[0 \times q_1 + F \times (1 - q_1)] + B(0, 1)(1 - q_0)F
\]  

(21)
Using the values in Tables 8.2 and 8.3

\[ V(0; \bar{D}) = F[0.9049 \times 0.9858 \times (1 - 0.9854)] + F[0.9535 \times (1 - 0.9858)] \]

\[ = F0.0266 \]

This analysis implicitly assumes that the counterparty does not default. This assumption can be relaxed using the analysis given in Section 8.4.

8.6 SUMMARY

We take as exogenous the term structure of zero coupon corporate bonds for firms within a given risk class and the term structure of zero coupon default-free bonds. Using standard arguments, we show how to extract the conditional martingale probabilities of default.
Given these probabilities we show how to price options on credit risky bonds, how to price vulnerable options, and how to price credit default swaps.

We have made the simplifying assumption that the martingale default probabilities are independent of the martingale probabilities for the default-free spot interest rates. This assumption can be relaxed and generalized in numerous ways. Jarrow et al. (1994) let the default probabilities for firm ABC be dependent on a current credit rating given by an external agency, such as Standard & Poors, Inc. or Moody's. This creates a Markov chain in credit ratings, in which historical default frequency data can be utilized. Lando (1994) allows the default probability to be dependent on the level of spot interest rates. This last modification appears promising in the area of Eurodollar contracts (see Jarrow et al. (1995)).

8.7 ENDNOTES

1. The existence and uniqueness of these martingale probabilities of default is discussed in Jarrow and Turnbull (1995).
2. For different types of bonds, average recovery rates are given in Moody's Special Report (1992).

8.8 REFERENCES