Option Pricing Using a Binomial Model with Random Time Steps (A Formal Model of Gamma Hedging)

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Abstract. This paper provides a new option pricing model which justifies the standard industry implementation of the Black-Scholes model. The standard industry implementation of the Black-Scholes model uses an implicit volatility, and it hedges both delta and gamma risk. This industry implementation is inconsistent with the theory underlying the derivation of the Black-Scholes model. We justify this implementation by showing that these ad-hoc adjustments to the Black-Scholes model provide a reasonable approximation to valuation and delta hedging in our new option pricing model.

Keywords: options, gamma hedging, Black-Scholes model, Binomial model, implicit volatility, Poisson process

The Black-Scholes model is the industry standard for pricing equity options. For valuation purposes, implicit volatilities rather than historic volatilities are used. For risk management purposes, discretely adjusted delta hedging is augmented with gamma hedging, and sometimes even vega hedging.

These standard adjustments to the Black-Scholes model are ad-hoc, and inconsistent with the underlying theory. Indeed, the Black-Scholes formula is based on a derivation where all risk is removeable from an option position via a continuously rebalanced delta hedge. Of course, in practice, continuously rebalancing a delta hedged portfolio is impossible. Consequently, a binomial approximation is employed, where discretely adjusted delta hedging now yields the same result—a riskless portfolio. This implication of the underlying theory is inconsistent with the need to both gamma and vega hedge.

Industry practice has modified the implementation of the Black-Scholes model in a manner inconsistent with the underlying theory, but yet the implementation has been successful. Why is this true? That is, why is the binomial approximation to the Black-Scholes model using implicit volatilities and augmented with gamma (and perhaps vega) hedging, a successful pricing and risk management tool?

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This paper provides a rational explanation for this enigma. It does this by deriving a new and simple option pricing model which provides a better characterization of observed stock price movements. The new model has stock prices jumping discretely, either up or down, at each time step as in the binomial model. But, in contrast, the time between stock price changes is now random. This provides an additional risk. Using this new model, it is shown that the Black-Scholes (or the binomial) model with an implicit volatility is an approximation to the new option's price. The new option's price is based on a historic volatility, not an implicit volatility. Secondly, delta and gamma hedging with the Black-Scholes model is also shown to be an approximation to delta hedging in the new option pricing model. The new option pricing model has two risks which need to be hedged, not just one, as in the Black-Scholes case. These two observations combined, provide the explanation for the enigma, i.e. the Black-Scholes model with implicit volatilities, gamma hedging, and vega hedging is an approximation to pricing and hedging in our new model. Furthermore, they also provide the first formal justification for gamma hedging within the Black-Scholes model which is consistent with standard option pricing theory.

Our new model for pricing equity options has the following characteristics:

1. The stock price process exhibits discrete price changes.

2. The market is **incomplete** in the stock and the money market account alone, that is, a dynamic trading strategy in a stock and money market account cannot replicate an option.

3. The stock price process has as a limit geometric Brownian motion. That is, in the limit, one obtains the Black-Scholes economy.\(^3\)

4. The market is **completed using a single traded call option on the stock**. Therefore, to hedge an option, one needs the underlying stock, the money market account, and another (distinct) option on the stock.

Our model can be interpreted as a **binomial model with random time steps**, or alternatively and perhaps more precisely, a Poisson directed process with two distinct jump amplitudes. Other models using discrete time multinomial lattices, instead of a Poisson directed process, include Boyle (1988) and Ho, Stapleton, and Subrahmanyam (1995).

This paper makes three contributions to the literature. The first is the presentation and derivation of a binomial option pricing model with random time steps, which provides a rigorous justification for the standard industry usage of the Black-Scholes model with implicit volatilities and gamma hedging. Existing justifications are ad-hoc, and inconsistent with the underlying theory (see Hull, 1993). We show analytically that delta and gamma hedging with the standard binomial model is an approximation to delta hedging in our new model. Of course, delta hedging with our model is superior in this context, because our pricing formula includes the "gamma" risk. This is not true in the Black-Scholes model, and it is the advantage of using our new model (both in theory and practice).

The second is the technique we use to price an option. We price an option relative to the price of the underlying stock and another (distinct) option on that stock. Hypothesized, therefore, are the stochastic processes for the underlying stock and a "basis" option.\(^4\) In
the derivation, we solve a “fixed-point” problem for the stochastic process of the derived option. The “fixed-point” problem is that the derived process for the option needs to be of the same form as the stochastic process exogenously given for the “basis” option. Otherwise, the model would be internally inconsistent. Our solution to this “fixed-point” problem is generic, and applicable to other models beyond our own. In particular, this approach to solving the fixed point problem can be applied to stochastic volatility option pricing models.

The third contribution of our paper relates to the convergence theory of discretized option values and option deltas to continuous limits. We extend the convergence results of He (1990), and Duffie and Protter (1992) to sequences of economics \( n = 1, 2, 3, \ldots \), where for finite \( n < +\infty \), the economy is **incomplete**, but where the limiting economy \( n = +\infty \) is complete.

An outline for our paper is as follows. Section 1 presents the model structure, both for the approximating and limit economies. Section 2 presents the pricing models and hedging procedures. This section solves the “fixed-point” problem discussed previously. Section 3 demonstrates that delta and gamma hedging in the Black-Scholes model is approximately equal to delta hedging in our new model. Section 4 discusses vega hedging in the context of our model. Finally, Section 5 completes the paper. All proofs are contained in the appendix.

1. The Model

This section presents the model analyzed in this paper. The idea is to construct a sequence of security market models \( (n = 1, 2, \ldots) \) consisting of a binomial model with random time steps such that for large enough \( n \), they approximate the Black-Scholes economy.

1.1. The Random Time Step Model

Let \( \{N^n\}_{n \in \mathbb{N}} \) and \( \{M^n\}_{n \in \mathbb{N}} \) be sequences of independent Poisson processes on the time interval \([0, \tau^*]\) large with intensities \( n(1 + \frac{\mu}{\sigma \sqrt{2n}}) \) and \( n(1 + \frac{\mu}{\sigma \sqrt{2n}}) \), respectively. They are defined on the probability spaces \((\Omega^n, \mathcal{F}^n, P^n)\) with \( (\mathcal{F}^n)_{0 \leq t \leq \tau^*} \), the filtration\(^5\) generated by \( N^n \) and \( M^n \). The intensity of these processes are selected so that as \( n \to \infty \), jumps occur infinitely often.

Define the factors:

\[
F^{n,1}_t = N^n_t - n \left( 1 + \frac{\mu}{\sigma \sqrt{2n}} \right) t \quad \text{and} \quad (1)
\]

\[
F^{n,2}_t = M^n_t - n \left( 1 + \frac{\mu}{\sigma \sqrt{2n}} \right) t. \quad (2)
\]

These factors are martingales, and they represent the two risks in the economy.

The stock price process is given by the following equivalent expressions:

\[
\frac{dS^n_t}{S^n_t} = (\mu + r) dt + \frac{\sigma}{\sqrt{2n}} dF^{n,1}_t - \frac{\sigma}{\sqrt{2n}} dF^{n,2}_t \quad (3)
\]
and
\[ S^n_t = S_0 \exp(rt) \left( 1 + \frac{\sigma}{\sqrt{2n}} \right)^{N^n_t} \left( 1 - \frac{\sigma}{\sqrt{2n}} \right)^{M^n_t}. \] (4)

One can think of this stock price process as a binomial model with random time steps. The stock price process moves at random times, which are determined by the Poisson factors \( F^n_{1} \) and \( F^n_{2} \). At jump times, the stock price moves proportionally up or down with a jump magnitude of \( \frac{\sigma}{\sqrt{2n}} \). The jump magnitude is selected such that as \( n \to \infty \), the up and down movements approach zero. Thus, for large \( n \), \( S^n_t \) jumps very often and only by small amounts. Intuitively, for large \( n \), \( S^n_t \) can be used to approximate a geometric Brownian motion or lognormal process.

This stock price process can alternatively be interpreted as a binomial model with "random volatility". This interpretation follows because the return volatility of expression (3), when measured over fixed time-steps, is random (in comparison to the standard binomial).

The expected return on the stock per unit time is \((\mu + r)\). The quantity \(\mu\) is the excess return on the stock above the risk-free rate \(r\). The money market account’s value is given by the following equivalent expressions. The money market account’s value is independent of \(n\).

\[ \frac{dB_t}{B_t} = rdt \quad \text{and} \quad B_t = \exp(rt). \] (5)

The money market account’s rate of return is constant at the risk-free rate \(r\) per unit time.

1.2. **The Black-Scholes Model**

Let \(W\) be a standard Brownian motion on the time interval \([0, \tau^*]\). It is defined on the probability space \((\Omega, F, P)\) with \((F_t)_{0 \leq t \leq \tau^*}\) the filtration generated by \(W\). The Black-Scholes economy has only one factor, \(W_t\). That is, there is only one risk in the Black-Scholes economy.

The stock price process is given by a geometric Brownian motion, defined by the following equivalent expressions:

\[ \frac{dS_t}{S_t} = (\mu + r)dt + \sigma dW_t \quad \text{and} \quad S_t = S_0 \exp \left( \left( \mu + r - \frac{\sigma^2}{2} \right) t + \sigma W_t \right). \] (7)

The stock price process has an expected return of \((\mu + r)\) per unit time and a constant volatility of \(\sigma\) per unit time. As in the random time step model, the money market account’s process is given by

\[ B_t = \exp(rt). \]

This model is the Black-Scholes economy.
1.3. Convergence

As discussed before, for large \( n \), \( S^n_t \) in the random time step model experiences very frequent jumps (both ups and downs) of small magnitudes (\( \pm \sigma / \sqrt{2n} \)). In fact, \( S^n_t \) is constructed so that it converges weakly to the geometric Brownian motion given in expression (7) and (8).

**Theorem 1** (Convergence to the Black-Scholes Economy) *The random time step stock price process \( S^n_t \) converges weakly (under \( P^n \), \( P \)) as \( n \to \infty \) to the geometric Brownian motion stock price process \( S_t \).*

All proofs are contained in the appendix.

Thus, for large enough \( n \), the stock price process given by the random time step model \( S^n_t \) will have an approximate lognormal distribution. This convergence is guaranteed under the original probability measures.

2. Pricing

The purpose of this section is twofold; first, to value a traded option on the stock in the random time step model, and second, to show that as \( n \) gets large, the random time step option's value and "deltas" converge to those of the Black-Scholes model. This last step requires us to show that the "gamma" in the approximating model converges to zero. For simplicity, we restrict our attention to European call options with exercise time \( \tau \) and strike price \( \kappa > 0 \).

This section also solves the "fixed-point" problem discussed in the introduction. That is, we uniquely value a call option using another (distinct) "basis" call option such that the exogenously given "basis" call price process and the derived call price processes are identical (up to exercise time \( \tau \) and strike price \( \kappa \)). Hence, the model is internally consistent.

2.1. The Random Time Step Model

For each \( n \), the random time step economy is incomplete in the underlying stock and money market account alone. This follows because there are two distinct factors to hedge, \( F^n_{t,1} \) and \( F^n_{t,2} \), but only a single security for hedging these risks (the stock). Intuitively, the two risks correspond to (i) the jump magnitude and (ii) the time of the jump. The standard binomial model does not have the second random time step risk.

Using a result from Harrison and Pliska (1981), the market is incomplete if and only if there is a non-unique price for the call option. We also know from Harrison and Pliska (1981) that an arbitrage-free price system is equivalent to the existence of an equivalent martingale measure for the discounted stock price process

\[
\beta S^n_t \text{ with } \beta_t = \frac{1}{B_t}.
\]  

(9)
The possible equivalent martingale measures are those under which both $N^n$ and $M^n$ have the same intensity $\lambda(n, \omega, t)$ for $n \in N$, $\omega \in \Omega^n$, $t \in [0, \tau^*]$. This condition follows via expression (3), and recognizing the fact that for $S^n_t / B_t$ to be a martingale under a new probability measure, the new intensities must negate each other.

We restrict ourselves to those martingale measures under which $N^n$ and $M^n$ have the same intensity $n\lambda_n$ where $\lambda_n \in R^+$. The additional restriction here is that $\lambda_n$ is independent of the state and the date. This is imposed for simplicity and could be easily relaxed. Hence, for each economy $n$, there are a continuum of equivalent martingale measures $Q^{n, \lambda_n}$ indexed by $\lambda_n \in R^+$.

Consider a European call option with exercise time $\tau$ and exercise price $\kappa$. From the standard risk-neutrality results (see Harrison and Pliska, 1981), the call’s value is given in expression (10).

$$v^{\tau, \kappa}(0, S^0_0) \quad \text{with} \quad v^{\tau, \kappa}(s, x) = \beta_{t \to s} E_{Q^{n, \lambda_n}} \left( (S^n_t - \kappa)^+ \mid S^n_s = x \right). \quad (10)$$

The right side of expression (10) corresponds to the discounted expected value of the call’s time $\tau$ payoff, under an equivalent martingale measure $Q^{n, \lambda_n}$. The call’s value is not unique, and it is indexed by the parameter $\lambda_n \in R^+$. For simplicity, we suppress the functional dependence of $(n, \lambda_n)$ in the call’s price.

Given the stock process in expression (3) and (4), expression (10) can be rewritten as:

$$v^{\tau, \kappa}(s, x) = \beta_{t \to s} \sum_j \sum_t \exp(-\mu_n^j(t-s)) \frac{\mu_n^j(t-s)!}{j!^t} \exp(-\mu_n^j(t-s)) \left( x \exp(\tau \mu) \left( 1 + \frac{\sigma_n^j}{\sqrt{2\nu}} \right)^j \left( 1 - \frac{\sigma_n^j}{\sqrt{2\nu}} \right)^j - \kappa \right)^+ \quad (11)$$

This represents the expected value of the call’s payoff at expiration using two independent Poisson processes, each with the same intensities, but different jump amplitudes. We define the call’s value process by:

$$V^{\tau, \kappa}_t = v^{\tau, \kappa}(t, S^0_t). \quad (12)$$

To determine a unique price for this call option, we need to complete the market by introducing another traded security. We let this security be another (distinct) European call option on the same underlying stock with exercise time $T > \tau$ and exercise price $K$. As the analysis underlying expression (12) is independent of $\tau$ and $K$, we have shown that this additional call’s price process is also given by:

$$V^{T, K}_t = v^{T, K}(t, S^0_t), \quad (13)$$

where only the parameters $T, K$ are different.

Given a fixed economy $n$, and an observed price $v^*(t, S^0_t)$ for the call option, the martingale measure $Q^{n, \lambda_n}$ is uniquely identified, i.e., $\lambda_n$ is uniquely determined as the solution to expression (14). This follows because the call value is strictly increasing in $\lambda_n$ (see Lemma 2 in the appendix).

$$v^*(t, S^0_t) = v^{T, K}(t, S^0_t). \quad (14)$$
**Theorem 2** (Complete Markets) Fix \( n \). The unique martingale measure for \( \beta S^u \) and \( V^{T,K} \) is \( Q^{n,\lambda_n} \) where \( \lambda_n \) is determined as the unique solution to expression (14).

Theorem 2 implies that the model is complete in \( S \) and \( V^{T,K} \), that is, we can hedge any contingent claim using the stock \( S \) and the money market account \( B \). With this choice of \( \lambda_n \) we fix the call price process \( V^{T,K} \) and henceforth the price system of the economy. Since \( \lambda_n \) uniquely determines the call value process \( V^{T,K} \), this theorem also provides the solution to the fixed point problem. The derived call price process is identical, given the parameter \( \lambda_n \), as the exogenously given call price process.

To determine the hedge ratios for constructing a delta-neutral portfolio, we first rewrite expression (13) in terms of the factors \( (F^{n,1}, F^{n,2}) \). Then, combined with expression (3), we will be able to compute portfolio holdings which set the coefficients preceding the two factor terms to zero. Equivalent expressions for (13) in terms of the factors are:

\[
V_t^{T,K} - V_0^{T,K} = \int_0^t \alpha^{T,K}(s, S^n_s) dF^{n,1}_s + \int_0^t \beta^{T,K}(s, S^n_s) dF^{n,2}_s + \int_0^t \gamma^{T,K}(s, S^n_s) ds
\]

(15)

where

\[
\alpha^{T,K}(s, x) \equiv v^{T,K}(s, x \left(1 + \frac{\sigma}{\sqrt{2n}}\right)) - v^{T,K}(s, x)
\]

(16)

\[
\beta^{T,K}(s, x) \equiv v^{T,K}(s, x \left(1 - \frac{\sigma}{\sqrt{2n}}\right)) - v^{T,K}(s, x)
\]

(17)

\[
\gamma^{T,K}(s, x) = \alpha^{T,K}(s, x) n \left(1 + \frac{\mu}{\sigma\sqrt{2n}}\right) + \beta^{T,K}(s, x) n \left(1 - \frac{\mu}{\sigma\sqrt{2n}}\right)
\]

\[
- n \lambda_n \left(v^{T,K}(s, x \left(1 + \frac{\sigma}{\sqrt{2n}}\right)) + v^{T,K}(s, x \left(1 - \frac{\sigma}{\sqrt{2n}}\right))\right)
\]

\[
- 2 v^{T,K}(s, x)
\]

(18)

Using expression (15), the next theorem gives the hedge ratio processes for the European call with respect to both the stock \( S^u \) and the "basis" call option \( V^{T,K} \). These hedge ratios are the coefficients preceding the random shocks \( dS^n_t \) and \( dV^{T,K}_t \), respectively.

**Theorem 3** (Deltas for the Binomial Model with Random Time Steps)

\[
V_t^{T,K} - V_0^{T,K} = \int_0^t h^{T,K}(s, S^n_s) dS^n_s + \int_0^t g^{T,K}(s, S^n_s) dV^{T,K}_s
\]

\[
- \int_0^t h^{T,K}(s, S^n_s) \beta S^n_s dB_s,
\]

(19)
where

\[ h^{r,k}(s, x) = \frac{1}{x} \sqrt{2n} \left\{ u^{r,k}(s, x) - u^{T,K}(s, x) \frac{z^{r,k}(s, x)}{z^{T,K}(s, x)} \right\} \]

(20)

\[ g^{r,k}(s, x) = \frac{z^{r,k}(s, x)}{z^{T,K}(s, x)} \]

(21)

\[ u^{r,k}(s, x) = u^{r,k} \left( s, x \left( 1 + \frac{\sigma}{\sqrt{2n}} \right) \right) - u^{r,k} \left( s, x \left( 1 - \frac{\sigma}{\sqrt{2n}} \right) \right) \]

(22)

and

\[ z^{r,k}(s, x) = u^{r,k} \left( s, x \left( 1 + \frac{\sigma}{\sqrt{2n}} \right) \right) \]

\[ + u^{r,k} \left( s, x \left( 1 - \frac{\sigma}{\sqrt{2n}} \right) \right) - 2u^{r,k}(s, x) \]

(23)

Theorem 3 shows that a position in the option \( V^{r,k} \) can be replicated by holding \( h^{r,k} \) shares of the stock \( S^n \), \( g^{r,k} \) shares of the “basis” option \( V^{T,K} \), and short \( h^{r,k} S^n \) units of the money market account B. These holdings must be continuously rebalanced across time as given in expression (19).

The holdings \( h^{r,k} \) in the stock \( S^n \) can be decomposed into those holdings in the stock which hedge changes in the option directly due to changes in the stock itself \( \tilde{h}^{r,k} \), and the reduction in those holdings necessary to account for changes in the “basis” option due to the stock \( h^{r,k} \). In this regard, let

\[ \tilde{h}^{r,k}(s, x) = \frac{1}{x} \sqrt{2n} \frac{1}{\sigma} u^{r,k}(s, x) \]

(24)

and

\[ \tilde{h}^{r,k}(s, x) = \tilde{h}^{T,K}(s, x) g^{r,k}(s, x) \]

(25)

Then we can rewrite (19) as

\[ V^{r,k}_t - V^{r,k}_0 = \int_0^t \tilde{h}^{r,k} \left( s, S^n \right) dS^n - \int_0^t \tilde{h}^{r,k} \left( s, S^n \right) \beta_s S^n dB_s \]

\[ - \int_0^t \tilde{h}^{r,k} \left( s, S^n \right) dS^n + \int_0^t g^{r,k} \left( s, S^n \right) dV^{T,K} \]

\[ - \int_0^t \tilde{h}^{r,k} \left( s, S^n \right) \beta_s S^n dB_s \]

(26)

The first quantity in expression (26) involving \( \tilde{h} \) corresponds to the hedge ratio component directly due to changes in the stock itself. This is interpreted as “delta” hedging in the standard Black-Scholes model. The second term is the adjustment in the money market account, due to the first term, to make the position self-financing. The third and fourth
quantities combined, \((-\hat{h} + g)\) capture changes due to the “basis” option. It is decomposed into the adjustment in \(\hat{h}^n\) due to the fact that the “basis” call option also increases when the stock increases, plus the adjustment due to the “basis” call option itself \(g\). The component \(\hat{h}\) is, thus, the reduction in \(h\) necessary due to the correlation between the stock and the “basis” option. We interpret the last component, \(g\), as “gamma” hedging in the standard Black-Scholes model. A rigorous justification for these statements is provided in Section 3 below, see expressions (38), (44)-(46). The fifth term is the adjustment in the money market account, due to the third term, to make the position self-financing. Expression (26) will prove useful for comparison with the Black-Scholes model.

2.2. The Black-Scholes Model

This model is complete, so by Harrison and Pliska (1981), there is a unique price for a given contingent claim. Consider a European call option with exercise time \(\tau\) and exercise price \(x\). The call’s value is given by:

\[
C^r_\tau (s, x) = \beta_{\tau-t} E_Q \left( (S^r_t - x)^+ \mid S_t = x \right),
\]

where \(Q\) is the unique equivalent martingale measure for the discounted stock price process \(\beta S\). The call price process is given by

\[
C^r_{\tau, x} = C^r_{\tau} (\tau, S_t) = S_t \Phi(q_1) - Ke^{-r(\tau-t)} \Phi(q_2)
\]

where \(\Phi\) is the standard cumulative normal distribution function,

\[
q_1 = \frac{\log \left( \frac{S_t}{Ke^{-r(\tau-t)}} \right) + \left( (1/2)\sigma^2(\tau-t) \right)}{\sigma(\tau-t)^{1/2}}
\]

and

\[
q_2 = q_1 - \sigma(\tau-t)^{1/2}.
\]

Since this model is complete, we can hedge the call with the underlying stock alone. The following theorem gives the hedge ratio process with respect to the stock \(S\) in a form comparable to expression (26).

**Theorem 4** (The Black-Scholes Delta)

\[
C^r_{\tau, x} - C^r_0 = \int_0^\tau \eta^r_{\tau, x}(s, S_s) dS_s - \int_0^\tau \eta^r_{\tau, x}(s, S_s) \beta_s S_s dB_s,
\]

where

\[
\eta^r_{\tau, x}(s, x) = \beta_{\tau-t} E_Q \left( \frac{1}{\tau} S_{\tau} 1_{\{S_{\tau} > x\}} \mid S_t = x \right) = \Phi(q_1).
\]
2.3. Convergence

Next, we study the conditions under which the call values and deltas of the random time step model converge to those of the Black-Scholes model. Given the random time step model's stock price process $S^n$ was constructed to converge to a geometric Brownian motion, one would also expect that the call values and deltas converge as well. This is true, but an additional hypothesis is needed, as given in theorem 5.

**Theorem 5** (Sufficient Condition for Convergence to Black-Scholes) *If* $\lambda_n \to 1$, *then* $V^{\tau,K}$ *converges weakly (under* $P^n$, $P$ *to the Black-Scholes formula.***

Theorem 5 shows that the condition $\lambda_n \to 1$ is sufficient for the sequence of call values to converge to the Black-Scholes value. In the following result, we show that this condition is also necessary.

**Theorem 6** (Necessary Condition for Convergence to Black-Scholes) *If* $V_0^{\tau,K}$ *converges weakly (under* $P^n$, $P$ *to Black-Scholes, then* $\lambda_n \to 1$. *The reason underlying the additional restriction for convergence* ($\lambda_n \to 1$) *can now be clarified. Theorem 1 shows that without this restriction, $S^n_t$ converges to $S_t$ under the sequence of martingale measures. The condition* ($\lambda_n \to 1$) *guarantees weak convergence of $S^n_t$ to $S_t$ under these martingale measures as well, as the following three theorems show.*

Alternatively, one can understand this condition via the "basis" call option $V^{\tau,K}$. We introduced the "basis" call to complete the market, but imposed no conditions to ensure that its price converged to the appropriate limit. Recall that the call's price is determined under the martingale measure, not $P$, and that this measure is uniquely characterized by $\lambda_n$ of Theorem 2. To guarantee that the constructed economies (both basis assets, i.e., the stock and the call) converge to the appropriate limit, we need to add ($\lambda_n \to 1$). This follows via Theorem 6.

This condition is, in fact, strong enough to ensure convergence of all the appropriate deltas as well.

**Theorem 7** (Convergence of the Call's Delta to the Black-Scholes Delta) *Assume* $\lambda_n \to 1$. *Then*

$$\hat{h}^{\tau,K}(\bullet, S^n_t) \text{ converges weakly (under } P^n, P \text{) to the Black-Scholes delta.}$$  \hspace{1cm} (33)

**Theorem 8** (Convergence of the Call's Gains Process Resulting From "Delta" Hedging Alone) *Assume* $\lambda_n \to 1$. *Then*

$$\int_0^t \hat{h}^{\tau,K}(s, S^n_s) dS^n_s - \int_0^t \hat{h}^{\tau,K}(s, S^n_s) \beta_sds \text{ converges weakly}$$  \hspace{1cm} (34)

*under* $P^n, P$ *to the change in the Black-Scholes value.*
Theorem 9 (Convergence of the “Gamma” and Adjustment Term to Zero) Assume $\lambda_n \to 1$. Then
\[
\left\{
\begin{array}{l}
- \int_0^* \tilde{h}_{t,x} (s, S^u_s) dS^u_s - \int_0^* \tilde{h}_{t,x} (s, S^u_s) \beta_t S^u_{x} dB_s \\
+ \int_0^* g_{t,x} (s, S^u_s) dV^u_{s}
\end{array}
\right\}
\text{converges weakly (under $P^n$, $P$) to zero.} \quad (35)
\]

Theorem 7 shows that as $\lambda_n \to 1$, for large $n$, the hedge ratio for “delta” hedging in the random time step model is close to that in the Black-Scholes formula (i.e. $\eta \approx \tilde{h}$ when $n$ is large).

As might be expected then, Theorem 8 shows that for large $n$, the gains process from using “delta” hedging alone in the random time step model (the first two terms in expression (26)) is close to the gains process from using “delta” hedging as in the Black-Scholes model, and thus to the call price process in the limiting model (see expression (31)).

Finally, theorem 9 shows that for $n \to \infty$ and $\lambda_n \to 1$, “gamma” hedging is neglectable or stated differently, the gains process from “delta” hedging alone is close to the call price process in the approximating model. This is necessary so that the last three terms in expression (26) converge to zero.

2.4. Discussion

Theorems 7–9 collectively demonstrate that for large enough $n$, our binomial model with random time steps is a good approximation to the Black-Scholes model. All relevant quantities converge weakly to the appropriate limits. Theorem 9 is particularly interesting. Theorem 9 argues that for large enough $n$, “gamma” hedging (hedging with the basis call) is unnecessary. Industry experience with the Black-Scholes model (or the standard binomial model), however, indicates that this is not the case. Gamma hedging is standard practice. Two assets, a stock and “basis” call, are needed to create a riskless portfolio. Consequently, one can conclude from this that $n$ is not large, and the approximation error is relevant.

Given this fact, the binomial model with random time steps is distinct from Black-Scholes, and needs to be implemented with finite $n$. This implies that the quantities $(\lambda^*, \sigma^*) = (n\lambda, \sigma/\sqrt{2n})$ need to be estimated (implicitly or historically) using market data, and the model implemented using the finite $n$ representation. The sequence index $n$ does not need to be estimated separately from either $\lambda_n$ or $\sigma$ as it always enters the relevant formula in these fixed relationships. Our model is, therefore, an alternative to and an improvement over the Black-Scholes model for both pricing and hedging.

3. Gamma Hedging

The purpose of this section is to demonstrate that the market practice of using the standard binomial model with gamma hedging is approximately equal to delta hedging in our bino-
mial model with random time steps. Thus, this section provides a rigorous justification for gamma hedging with the standard binomial model.

Let us review the standard binomial model using our notation. To distinguish the stock price in the standard binomial model from our \( S^m_t \), we denote it by \( Y^m_t \). Let \([0, \tau^*] \) for \( \tau^* > \tau \) be divided into \( m \) steps of equal size. Let the pseudo-probabilities be set equal to \((1/2)\). This is without loss of generality. The standard binomial’s stock price process is given (in difference form) by:

\[
\begin{align*}
Y^m_{\lfloor t \rfloor} &= \begin{cases} 
Y^m_{\lfloor \frac{t}{m} \rfloor} e^r (1 + \sigma / \sqrt{m}) & \text{with probability } (1/2) \\
Y^m_{\lfloor \frac{t}{m} \rfloor} e^r (1 - \sigma / \sqrt{m}) & \text{with probability } (1/2),
\end{cases}
\end{align*}
\]  

(36)

where \( \lfloor t \rfloor \) denotes the largest integer less than or equal to \( t \).

For large \( m \), \( Y^m_t \) approaches \( S_t \) in expression (8) under the equivalent martingale measure. Let \( C^{m,t,\kappa}(t, Y^m_t) \) be the price process for the call under the standard binomial model. A formula for \( C^{m,t,\kappa}(t, Y^m_t) \) can be found in Jarrow and Rudd (1983). It is well-known that as \( m \to \infty \), \( C^{m,t,\kappa}(t, Y^m_t) \) converges weakly to \( C^{t,\kappa} \). At an arbitrary \( (t, x = Y^m_t) \), the stock’s delta in the binomial model is:

\[
\eta^{m,t,\kappa}(t, x) = \frac{1}{\sqrt{m}} \frac{1}{\sigma} \left[ \frac{\sqrt{m}}{2} C^{m,t,\kappa}(t, x \left( 1 + \sigma / \sqrt{m} \right)) - C^{m,t,\kappa}(t, x \left( 1 - \sigma / \sqrt{m} \right)) \right]
\]

(37)

Compare this to \( \tilde{h}^{t,\kappa} \) in expression (25). These expressions are nearly identical in form. As both deltas converge to the Black-Scholes delta, we see that for large \( m \), these terms will be approximately equal, i.e.,

\[
\tilde{h}^{t,\kappa}(t, x) \approx \eta^{m,t,\kappa}(t, x) \quad \text{for large } m, n \text{ and } \lambda_n \text{ close to } 1.
\]

(38)

Gamma hedging the call option \((t, \kappa)\) using the stock and call option \((T, K)\) is best explained using the continuous time limit, the Black-Scholes formula. Let \( \lambda(t, S_t), \xi(t, S_t) \) be positions in the stock and call option, respectively, such that

\[
\gamma(t, S_t) + \xi(t, S_t) \frac{\partial C^{t,K}_T}{\partial S_t} = \frac{\partial C^{t,K}_T}{\partial S_t}, \quad (\text{delta neutral}) \quad \text{and,}
\]

(39)

\[
\xi(t, S_t) \frac{\partial^2 C^{t,K}_T}{\partial S_t^2} = \frac{\partial^2 C^{t,K}_T}{\partial S_t^2}, \quad (\text{gamma neutral}).
\]

(40)

Expression (39) makes the hedging portfolio delta neutral, and expression (40) makes the hedging portfolio gamma neutral. Expression (40) implies that

\[
\xi(t, S_t) = \frac{\partial^2 C^{t,K}_T}{\partial S_t^2} \left( \frac{\partial^2 C^{t,K}_T}{\partial S_t^2} - \frac{\partial C^{t,K}_T}{\partial S_t} \frac{\partial \xi(t, S_t)}{\partial S_t} \right)
\]

(41)

and

\[
\gamma(t, S_t) = \frac{\partial C^{t,K}_T}{\partial S_t} - \frac{\partial C^{t,K}_T}{\partial S_t} \xi(t, S_t).
\]

(42)
Expression (42) shows the delta in the stock needs to be reduced to account for changes in the call used to gamma hedge.

In the standard binomial model evaluated at time \( s \) and \( x = Y^m_s \) we approximate the second derivative via expression (43).

\[
\frac{\partial^2 C^s_{s,x}(x)}{\partial S_s^2} \approx \frac{1}{x} \frac{m}{\sigma^2} \left[ C^{m,r,K}_s(s, x (1 + \sigma/\sqrt{m})) + C^{m,r,K}_s(s, x (1 - \sigma/\sqrt{m})) \right. \\
\left. - 2C^{m,r,K}_s(s, x) \right] \text{ for large } m.
\]  

(43)

This follows from Rudin (1976, page 115, Exercise 11). Consequently,

\[
\xi(s, x) \approx \xi^{m,r,K}(s, x) \\
\approx \frac{[C^{m,r,K}_s(s, x (1 + \sigma/\sqrt{m})) + C^{m,r,K}_s(s, x (1 - \sigma/\sqrt{m})) - 2C^{m,r,K}_s(s, x)]}{[C^{m,T,K}_s(s, x (1 + \sigma/\sqrt{m})) + C^{m,T,K}_s(s, x (1 - \sigma/\sqrt{m})) - 2C^{m,T,K}_s(s, x)]} \text{ for large } m
\]

(44)

and

\[
\gamma(s, x) \approx \gamma^{m,r,K}(s, x) \\
\approx \eta^{m,T,K}(s, x) - \eta^{m,T,K}(s, x) \xi^{m,r,K}(s, x) \text{ for large } m
\]

(45)

Expression (44) is the gamma hedge and expression (45) is the adjustment to the delta hedge in the standard binomial model. Comparing these expressions to (21)—(25) yields:

\[
g^{r,K}(s, x) \approx \xi^{m,r,K}(s, x),
\]

(46)

\[
h^{r,K}(s, x) \approx \gamma^{m,r,K}(s, x), \text{ and}
\]

(47)

\[
h^{r,K}(s, x) \approx \eta^{m,T,K}(s, x) \xi^{m,r,K}(s, x) \text{ for large } m, n \text{ and } \lambda, \text{ close to one.}
\]

(48)

That is, the delta position in the option in the binomial model with random time steps is approximately equal to the gamma hedge in the standard binomial model. Furthermore, the delta adjustment in the standard binomial model due to the gamma hedge is also approximately equal to the adjustment to the delta hedge in the binomial model with random time steps. This completes the analytic demonstration.

4. Vega Hedging

Industry usage of the standard binomial model (with \( m \) large) sometimes employs vega hedging. Vega hedging is the practice of making an option position neutral with respect to changes in implicit volatilities (see Hull, 1993). This practice can be understood in the context of our model.

Suppose, as we have argued, that the binomial model with random time steps and \( n \) small is the correct model. Also suppose that the standard binomial model with \( m \) large is utilized (wrongly). In this situation, gamma hedging with the standard binomial model and using historic volatilities will not perfectly eliminate the second (random-time) risk from an option portfolio. This is because the approximating relationship in expression (46) is not
exact for \( n \) small (and \( m \) large). The error is due to the difference in the pricing formulas \( C^{m,t,x} \) and \( V^{t,x} \).

One way to partially eliminate the risk of the differences between \( C^{m,t,x} \) and \( V^{t,x} \) is to use implicit volatilities. This revision will bring \( C^{m,t,x} \) closer to \( V^{t,x} \). It will not remove all the difference, however, as implicit volatilities only make \( C^{m,t,x} \) and \( V^{t,x} \) identical at the current date and \textbf{before a jump}, but not \textbf{after the jump} as well. Note that the gamma hedging formulas also require equality after the jump (see expression (44)). This difference will imply that some residual risk remains in a gamma neutral portfolio due to changing implicit volatilities. This remaining residual risk can be effectively eliminated by vega hedging in the standard binomial model. This completes our analysis of the industry practice of vega hedging.

5. Conclusion

This paper develops a new option pricing model. It can be interpreted as a binomial model with random time steps. The random time between jumps in this model implies that the economy is incomplete in the money market account and the stock alone, and this necessitates hedging any call with an additional traded asset. We choose for this additional traded asset another distinct call on the same underlying stock. The solution for the option price requires a solution to a fixed-point problem, to ensure that the exogenously given option and the derived one have identical stochastic processes (up to strike price and maturity date differences).

This new model is shown to converge to the Black-Scholes formula as the number of jumps per unit time gets infinitely large and the jump magnitude gets infinitely small. Furthermore, we show that delta and gamma hedging with the standard binomial model for large \( n \) (or Black-Scholes) is an approximation to delta hedging in this new model. Although standard practice adjusts hedging with the binomial to account for the additional gamma risk, they don’t adjust pricing. Thus, valuation and hedging are inconsistent. In contrast, our model has both valuation and hedging consistent with this additional risk. Consequently, we conjecture that this new model will perform better in practice than does the industry usage of the Black-Scholes model. This conjecture, however, awaits subsequent empirical testing.

Appendix: Mathematical Proofs

This appendix references the book by Jacod and Shiryaev (1987) for all undefined terms and notation in the following proofs, e.g. finite variation, tight sequence, square-integrable martingale, etc.

Let \( \tau^* \equiv 1 \).

**Theorem 1** \((S^n \mid P^n) \Rightarrow (S \mid P)\).

The symbol “\( \Rightarrow \)” means weak convergence, i.e. \( E_{P^n}(f(S^n)) \rightarrow E_P(f(S)) \) for all bounded real-valued continuous functions (see Jacod and Shiryaev, 1987; p. 312). We need to indicate the measure in the notation, since we will be using different measures.
Define the return processes $R^n$ and $R$ by

$$R^n_t = \frac{\sigma}{\sqrt{2n}} (N^n_t - M^n_t) + rt$$

(A.1)

$$R_t = \sigma W_t + (\mu + r)t$$

(A.2)

Define $\varepsilon(R^n)$, as the unique semimartingale $Z$ that is a solution of $Z_t = 1 + \int_0^t Z_s - dR^n_s$. This is the stochastic exponential. Similarly, define $\varepsilon(R)_t$.

Note that $S^n = \varepsilon(R^n)$ and $S = \varepsilon(R)$. Therefore $\frac{dS^n}{S^n} = dR^n_t$ and $\frac{dS}{S} = dR_t$.

**Proof of Theorem 1:** Let $\tilde{R}^n$ and $\tilde{R}$ be defined by $\tilde{R}^n_t = R^n_t - (\mu + r)t$ and $\tilde{R}_t = R_t - (\mu + r)t$ respectively. $\tilde{R}^n$ is under $P^n$ a square-integrable martingale. Since $\tilde{R}^n_0 = 0$, $(\tilde{R}^n, \tilde{R}^n)_n$ is tight and $C$-tight respectively. Theorem 4.13 on page 322 of Jacod and Shiryaev (1987) states that this is a sufficient condition for the sequence $(\tilde{R}^n_n)_n$ to be tight. An application of the Lindeberg-Feller theorem shows for fixed $t, s \in [0, 1]$

$$\left( \tilde{R}^n_t - \tilde{R}^n_s \mid P^n \right) \Rightarrow \sigma \Phi(0, t-s),$$

which is the distribution of $\tilde{R}_t - \tilde{R}_s$ under $P$. Since both $\tilde{R}^n$ and $\tilde{R}$ are processes with independent increments, this implies that the finite dimensional distributions of $\tilde{R}^n$ converge to $\tilde{R}$ (see Lemma 1.3 on page 350 of Jacod and Shiryaev (1987)) and it is shown that $(\tilde{R}^n \mid P^n) \Rightarrow (\tilde{R} \mid P)$. Since $f(t) = (\mu + r)t$ is a continuous real function $(R^n \mid P^n) \Rightarrow (R \mid P)$, by Proposition 3.17 on page 314 of Jacod and Shiryaev (1987). Since the jumps of $R^n$ are uniformly bounded in $n, t$ and $\omega$, Corollary 6.6 on page 342 of Jacod and Shiryaev (1987) implies that $((R^n, [R^n, R^n]) \mid P^n) \Rightarrow ((R, [R, R]) \mid P)$. Theorem 1 of Avram (1988) states that $((R^n, [R^n, R^n]) \mid P^n \Rightarrow ((R, [R, R]) \mid P)$ is a sufficient condition for $(\varepsilon(R^n) \mid P^n) \Rightarrow (\varepsilon(R) \mid P)$. Hence, by definition, $(S^n \mid P^n) \Rightarrow (S \mid P)$.

**Theorem 2** Assume $n$ is fixed. Let $\lambda_n$ be the solution to expression (14). Then, the unique equivalent martingale measure for $\beta S^n$ and $V^{n, \lambda_n, T, K}$ is $Q^{n, \lambda_n}$.

**Proof of Theorem 2:** The equivalent martingale measure for $\beta S$ and $V^{n, \lambda_n, T, K}$ is determined by the intensities $\mu_n$ and $\delta_n$ of $N^n$ and $M^n$ respectively. Recalling the definitions of $\beta$, $S^n$ and $V^{n, \lambda_n, T, K}$ (see (9), (3) and (15)),

\[
\beta_S S^n_t - \beta_0 S^n_0 = \int_0^t \frac{\sigma}{\sqrt{2n}} \beta_S S^n_s - d(N^n_s - \mu_s) - \int_0^t \frac{\sigma}{\sqrt{2n}} \beta_S S^n_s - d(M^n_s - \delta_s) + \int_0^t \frac{\sigma}{\sqrt{2n}} \beta_S S^n_s - (\mu_s - \delta_s) ds
\]

(A.3)
\[ V_{n, n_0, T, K} - V_{n, n_0, T, K}^0 = \int_0^t \alpha_{n, n_0, T, K}(s, \Delta_{n_0}^n) d\left( N^n_t - \mu_n s \right) + \int_0^t \beta_{n, n_0, T, K}(s, \Delta_{n_0}^n) d\left( M^n_t - \delta_n s \right) + \int_0^t \gamma_{n, n_0, T, K}(s, \Delta_{n_0}^n) ds \] (A.4)

where

\[ \gamma_{n, n_0, T, K}(s, x) = \alpha_{n, n_0, T, K}(s, x) \mu_n + \beta_{n, n_0, T, K}(s, x) \delta_n - n \lambda_n \left( v_{n, n_0, T, K}(s, x \left( 1 + \frac{\sigma}{\sqrt{2n}} \right)) \right) + v_{n, n_0, T, K}(s, x \left( 1 - \frac{\sigma}{\sqrt{2n}} \right)) - 2v_{n, n_0, T, K}(s, x) \] (A.5)

Under the equivalent martingale measure \( \beta, S_t^n - \beta_0 S_0^n, V_{n, n_0, T, K} - V_{n, n_0, T, K}^0 \) are martingales. So are \( N^n_t - \mu_n t \) and \( M^n_t - \delta_n t \) and since they are both square-integrable so are furthermore integrals with respect to \( N^n_t - \mu_n t \) and \( M^n_t - \delta_n t \). Thus, (A.3) and (A.4) show that

\[ \int_0^t \frac{\sigma}{\sqrt{2n}} \beta_s S_{n_0}^n (\mu_n - \delta_n) ds \] (A.6)

and

\[ \int_0^t \gamma_{n, n_0, T, K}(s, \Delta_{n_0}^n) ds \] (A.7)

are martingales under the equivalent martingale measure as well. Since they are, however, continuous martingales of finite variation, they must be equal to zero. Thus, \( \mu_n \) and \( \delta_n \) must satisfy

\[ \frac{\sigma}{\sqrt{2n}} \beta_s S_{n_0}^n (\mu_n - \delta_n) = 0 \] (A.8)

and

\[ \alpha_{n, n_0, T, K}(s, x) \mu_n + \beta_{n, n_0, T, K}(s, x) \delta_n - n \lambda_n \left( v_{n, n_0, T, K}(s, x \left( 1 + \frac{\sigma}{\sqrt{2n}} \right)) \right) + v_{n, n_0, T, K}(s, x \left( 1 - \frac{\sigma}{\sqrt{2n}} \right)) - 2v_{n, n_0, T, K}(s, x) = 0 \] (A.9)

Using the definitions of \( \alpha_{n, n_0, T, K} \) [see (16)] and \( \beta_{n, n_0, T, K} \) [see (17)], the unique solution to the system of linear equations is given by \( \mu_n = \delta_n = n \lambda_n \).
Proof of Theorem 3: The main difficulty in proving this result is that we cannot apply Ito's lemma to \( v^{n,\lambda, t, \kappa} \), since on

\[
\left\{ (s, x): \text{there is some } k, j \in \mathbb{N} \text{ such that} \right. \\
x \exp(r(t - s)) \left( 1 + \frac{\sigma}{\sqrt{2n}} \right)^k \left( 1 - \frac{\sigma}{\sqrt{2n}} \right)^j = \kappa \left. \right\}
\]

\( \frac{\partial^{n+1}_{\lambda, t, \kappa}}{\partial s} (s, x) \) and \( \frac{\partial^n_{\lambda, t, \kappa}}{\partial x} (s, x) \) do not exist, which can be seen by using expression (A.31) for \( v^{n,\lambda, t, \kappa} (s, x) \) to obtain that

\[
v^{n,\lambda, t, \kappa} (s, x) = \beta_{t-s} \sum_i \sum_j \left[ \frac{\exp(-n\lambda(t-s))}{i!} \frac{\exp(-n\lambda(t-s))}{j!} \right] \\
\left[ x \exp(r(t - s)) \left( 1 + \frac{\sigma}{\sqrt{2n}} \right)^i \left( 1 - \frac{\sigma}{\sqrt{2n}} \right)^j \right]^+ \tag{A.10}
\]

Fix \( \omega \). Let \( t_1, \ldots, t_{m-1} \) be the jump times of \( N^\omega_n(t) \) and \( M^\omega_n(t) \) on \((0, \tau)\) and let \( t_0 = 0, t_m = t \). Let

\[
v^{n,\lambda, t, \kappa} (s, x) = \beta_{t-s} \sum_i \sum_j \frac{\exp(-n\lambda(t-s))}{i!} \frac{\exp(-n\lambda(t-s))}{j!} \left[ x \exp(r(t - s)) \left( 1 + \frac{\sigma}{\sqrt{2n}} \right)^i \left( 1 - \frac{\sigma}{\sqrt{2n}} \right)^j \right]^+ \tag{A.11}
\]

and note that \( \frac{\partial^n_{\lambda, t, \kappa}}{\partial s} (s, x) \) exists on \([0, \tau]\). Comparing (A.10) with (A.11) shows that

\[
v^{n,\lambda, t, \kappa} (s, x) = \hat{v}^{n,\lambda, t, \kappa} (s, \exp(-rs)x) \tag{A.12}
\]

Since now for \( t_{k-1} < s < t_k, 1 \leq k \leq m \),

\[
S^\omega_n(t) = \exp(rs) \left. \left( 1 + \frac{\sigma}{\sqrt{2n}} \right)^{N^\omega_{t-1}(\omega)} \left( 1 - \frac{\sigma}{\sqrt{2n}} \right)^{M^\omega_{t-1}(\omega)} \right.
\]

(A.12) implies that for \( t_{k-1} \leq s < t_k, 1 \leq k \leq m \),

\[
v^{n,\lambda, t, \kappa} (s, S^\omega_n(t)) = \hat{v}^{n,\lambda, t, \kappa} (s, S^\omega_n(t)) \left( 1 + \frac{\sigma}{\sqrt{2n}} \right)^{N^\omega_{t-1}(\omega)} \left( 1 - \frac{\sigma}{\sqrt{2n}} \right)^{M^\omega_{t-1}(\omega)} \right.
\]

(A.13)

Now

\[
v^{n,\lambda, t, \kappa} (t, S^\omega_n(t)) = v^{n,\lambda, t, \kappa} (0, 1) \\
= \sum_{k=1}^{m} \left( v^{n,\lambda, t, \kappa} (T, S^\omega_n(t)) - v^{n,\lambda, t, \kappa} (t_{k-1}, S^\omega_n(t)) \right)
\]
\[
= \sum_{k=1}^{m} \left\{ v^{n, \lambda_n, T, K} \left( t_k, S^n_{h_{n-1}}(\omega) \right) - v^{n, \lambda_n, T, K} \left( t_{k-1}, S^n_{h_{n-1}}(\omega) \right) \right\} \\
+ \sum_{k=1}^{m} \left\{ v^{n, \lambda_n, T, K} \left( t_k, S^n_{h_{n}}(\omega) \right) - v^{n, \lambda_n, T, K} \left( t_k, S^n_{h_{n-1}}(\omega) \right) \right\}
\]

Using (A.13), the first sum in the last expression is

\[
= \sum_{k=1}^{m} \left\{ v^{n, \lambda_n, T, K} \left( t_k, \left( 1 + \frac{\sigma}{\sqrt{2n}} \right) N^n_{q_{k-1}}(\omega), \left( 1 - \frac{\sigma}{\sqrt{2n}} \right) M^n_{q_{k-1}}(\omega) \right) \\
- v^{n, \lambda_n, T, K} \left( t_{k-1}, \left( 1 + \frac{\sigma}{\sqrt{2n}} \right) N^n_{q_{k-1}}(\omega), \left( 1 - \frac{\sigma}{\sqrt{2n}} \right) M^n_{q_{k-1}}(\omega) \right) \right\}
\]

\[
= \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \left[ \frac{\partial v^{n, \lambda_n, T, K}}{\partial S} \right] \left( s, \left( 1 + \frac{\sigma}{\sqrt{2n}} \right) N^n_{q_{k-1}}(\omega), \left( 1 - \frac{\sigma}{\sqrt{2n}} \right) M^n_{q_{k-1}}(\omega) \right) ds
\]

Therefore

\[
v^{n, \lambda_n, T, K} \left( t, S^n_{h_{n}} \right) - v^{n, \lambda_n, T, K} \left( 0, 1 \right) = \sum_{0 \leq s \leq t} \left\{ v^{n, \lambda_n, T, K} \left( s, S^n_{h_{n}} \right) - v^{n, \lambda_n, T, K} \left( s, S^n_{h_{n-1}} \right) \right\}
\]

\[
+ \int_0^t \left[ \frac{\partial v^{n, \lambda_n, T, K}}{\partial S} \right] \left( s, \exp(-rs)S^n_{h_{n-1}} \right) ds \quad (A.14)
\]

To evaluate the right-hand side of (19), note first that

\[
\int_0^t h^{n, \lambda_n, T, K}(s, S^n_{h_{n-1}}) dS^n_{h_{n}} - \int_0^t \beta_n S^n_{h_{n-1}} h^{n, \lambda_n, T, K}(s, S^n_{h_{n-1}}) dB_s
\]

\[
= \int_0^t B_s h^{n, \lambda_n, T, K}(s, S^n_{h_{n-1}}) d\beta_n(S^n_{h_{n}}) \quad (A.15)
\]

Since \( \frac{d(B_t, S^n_t)}{dS^n_t} = \frac{\sigma}{\sqrt{2n}}(N^n_t - M^n_t) \), using the definition of \( h^{n, \lambda_n, T, K} \) [see (20)], the right-hand side of (A.15) is

\[
\int_0^t \frac{1}{2} \left( u^{n, \lambda_n, T, K}(s, S^n_{h_{n-1}}) - u^{n, \lambda_n, T, K}(s, S^n_{h_{n-1}}) \right) d(N^n_t - M^n_t) \quad (A.16)
\]

Using the expression (15) for \( V^{n, \lambda_n, T, K} \) and the definition of \( u^{n, \lambda_n, T, K} \) [see (18)] and \( z^{n, \lambda_n, T, K} \) [see (23)] gives that

\[
dV^{n, \lambda_n, T, K} = a^{n, \lambda_n, T, K}(t, S^n_{h_{n}}) dN^n_t + \beta^{n, \lambda_n, T, K}(t, S^n_{h_{n}}) dM^n_t - n\lambda_n z^{n, \lambda_n, T, K} dt \quad (A.17)
\]
Using now (A.17) and the definition of \( g^{n, \lambda_n, T, K} \) [see (21)] shows that
\[
\int_0^t g^{n, \lambda_n, T, K}(s, S^n_{s-}) dV^n_{s} = \int_0^t \alpha^{n, \lambda_n, T, K}(s, S^n_{s-}) \frac{z^{n, \lambda_n, T, K}(s, S^n_{s-})}{z^{n, \lambda_n, T, K}(s, S^n_{s-})} dN^n_s
+ \int_0^t \beta^{n, \lambda_n, T, K}(s, S^n_{s-}) \frac{z^{n, \lambda_n, T, K}(s, S^n_{s-})}{z^{n, \lambda_n, T, K}(s, S^n_{s-})} dM^n_s
- \int_0^t n\lambda_n z^{n, \lambda_n, T, K}(s, S^n_{s-}) ds \quad (A.18)
\]

Combining (A.16) and (A.18) shows that the right-hand side of (19) is
\[
\int_0^t \frac{1}{2} \left( u^{n, \lambda_n, T, K}(S^n_{s-}) - u^{n, \lambda_n, T, K}(s, S^n_{s-}) \frac{z^{n, \lambda_n, T, K}(s, S^n_{s-})}{z^{n, \lambda_n, T, K}(s, S^n_{s-})} \right) d(N^n_s - M^n_s)
+ \int_0^t \alpha^{n, \lambda_n, T, K}(s, S^n_{s-}) \frac{z^{n, \lambda_n, T, K}(s, S^n_{s-})}{z^{n, \lambda_n, T, K}(s, S^n_{s-})} dN^n_s
+ \int_0^t \beta^{n, \lambda_n, T, K}(s, S^n_{s-}) \frac{z^{n, \lambda_n, T, K}(s, S^n_{s-})}{z^{n, \lambda_n, T, K}(s, S^n_{s-})} dM^n_s
- \int_0^t n\lambda_n z^{n, \lambda_n, T, K}(s, S^n_{s-}) ds \quad (A.19)
\]

Note that for \( s \) such that \( \Delta N^n_s \neq 0 \) and \( \Delta M^n_s = 0 \), \( S^n_s = S^n_{s-}(1 + \frac{\Delta S^n_s}{\sqrt{2n}}) \). Therefore, using the definitions of \( u^{n, \lambda_n, T, K} \) [see (22)], \( \alpha^{n, \lambda_n, T, K} \) [see (16)] and \( z^{n, \lambda_n, T, K} \) [see (23)], for those \( s \)
\[
\frac{1}{2} u^{n, \lambda_n, T, K}(s, S^n_{s-}) = \left\{ u^{n, \lambda_n, T, K}(s, S^n_{s-}) - u^{n, \lambda_n, T, K}(s, S^n_{s-}) \frac{z^{n, \lambda_n, T, K}(s, S^n_{s-})}{z^{n, \lambda_n, T, K}(s, S^n_{s-})} \right\} - \frac{1}{2} z^{n, \lambda_n, T, K}(s, S^n_{s-}) \quad (A.20)
\]

and
\[
\alpha^{n, \lambda_n, T, K}(s, S^n_{s-}) = u^{n, \lambda_n, T, K}(s, S^n_{s-}) - u^{n, \lambda_n, T, K}(s, S^n_{s-}) \quad (A.21)
\]

Similarly for \( s \) such that \( \Delta N^n_s \neq 0 \) and \( \Delta M^n_s = 0 \), \( S^n_s = S^n_{s-}(1 - \frac{\Delta S^n_s}{\sqrt{2n}}) \) and for those \( s \)
\[
-\frac{1}{2} u^{n, \lambda_n, T, K}(s, S^n_{s-}) = \left\{ u^{n, \lambda_n, T, K}(s, S^n_{s-}) - u^{n, \lambda_n, T, K}(s, S^n_{s-}) \frac{z^{n, \lambda_n, T, K}(s, S^n_{s-})}{z^{n, \lambda_n, T, K}(s, S^n_{s-})} \right\} - \frac{1}{2} z^{n, \lambda_n, T, K}(s, S^n_{s-}) \quad (A.22)
\]

and
\[
\beta^{n, \lambda_n, T, K}(s, S^n_{s-}) = u^{n, \lambda_n, T, K}(s, S^n_{s-}) - u^{n, \lambda_n, T, K}(s, S^n_{s-}) \quad (A.23)
\]

Note further that if \( \Delta N^n_s \neq 0 \) then \( \Delta N^n_s = 1 \) and similarly is \( \Delta M^n_s \neq 0 \) then \( \Delta M^n_s = 1 \) and
that $\Delta S^n_t \neq 0$ if and only if either $\Delta N^n_t \neq 0$ or $\Delta M^n_t \neq 0$. Thus, using (A.20) and (A.22), (A.16) is equal to

$$\sum_{0 < t \leq T} \left\{ n^{n, \lambda_n, T.K} (s, S^n_s) - n^{n, \lambda_n, T.K} (s, S^n_{s-}) \right\}$$

$$- \sum_{0 < t \leq T} \left\{ (n^{n, \lambda_n, T.K} (s, S^n_s) - n^{n, \lambda_n, T.K} (s, S^n_{s-})) \right\} z^{n, \lambda_n, T.K} (s, S^n_{s-})$$

(A.24)

and, using (A.21) and (A.23), the right-hand side of (A.18) is equal to

$$\sum_{0 < t \leq T} \left\{ (n^{n, \lambda_n, T.K} (s, S^n_s) - n^{n, \lambda_n, T.K} (s, S^n_{s-})) \right\}$$

$$- \int_0^t n^{n, \lambda_n, T.K} (s, S^n_{s-}) ds$$

(A.25)

Combining (A.24) and (A.25) shows that (A.19) and thus the right-hand side of (19) is equal to

$$\sum_{0 < t \leq T} \left\{ (n^{n, \lambda_n, T.K} (s, S^n_s) - n^{n, \lambda_n, T.K} (s, S^n_{s-})) \right\} - \int_0^t n^{n, \lambda_n, T.K} (s, S^n_{s-}) ds$$

(A.26)

Comparing (A.14) and (A.26) gives that

$$n^{n, \lambda_n, T.K} (t, S^n_t) - n^{n, \lambda_n, T.K} (0, 1)$$

$$= \int_0^t h^{n, \lambda_n, T.K} (s, S^n_{s-}) dS^n_s - \int_0^t h^{n, \lambda_n, T.K} (s, S^n_{s-}) \beta_b S^n_s dB_s$$

$$+ \int_0^t g^{n, \lambda_n, T.K} (s, S^n_{s-}) dV^n_{s, \lambda_n, T.K}$$

$$+ \int_0^t \left[ \frac{\partial n^{n, \lambda_n, T.K}}{\partial s} (s, \exp(-r_s) S^n_{s-}) + n^{n, \lambda_n, T.K} (s, S^n_s) \right] ds$$

(A.27)

Since now $n^{n, \lambda_n, T.K} (t, S^n_t) - n^{n, \lambda_n, T.K} (0, 1), \int_0^t g^{n, \lambda_n, T.K} (s, S^n_{s-}) dV^n_{s, \lambda_n, T.K}$ and, using (A.15), $
\int_0^t h^{n, \lambda_n, T.K} (s, S^n_{s-}) dS^n_s - \int_0^t h^{n, \lambda_n, T.K} (s, S^n_{s-}) \beta_b S^n_s dB_s$ are $\mathcal{Q}^{n, \lambda_n}$ martingales,

$$\int_0^t \left[ \frac{\partial n^{n, \lambda_n, T.K}}{\partial s} (s, \exp(-r_s) S^n_{s-}) + n^{n, \lambda_n, T.K} (s, S^n_s) \right] ds$$

is a continuous $\mathcal{Q}^{n, \lambda_n}$ martingale of finite variation and hence equal to zero. Thus, since $V^n_{t, \lambda_n, T.K} = n^{n, \lambda_n, T.K} (t, S^n_t)$ [see (A.30)], (A.27) proves (23).

**Proof of Theorem 4:** For the appendix, $C^{T.K} \equiv v^{T.K}, c^{T.K} (s, x) \equiv v^{T.K} (s, x)$, and $\eta^{T.K} (s, x) \equiv h^{T.K} (s, x)$. 

Let \( t < \tau \). Then, since \( S_t = \exp(\sigma (W_t + \frac{\mu^*}{\sigma^2} t) - \frac{\sigma^2}{2} t + rt) \) [see (8)] with \((W_t + \frac{\mu^*}{\sigma^2} t)\) a standard Brownian motion under \( Q \),

\[
v^{\tau,\kappa}(s, x) = \beta_{t-\tau} E_Q ((S_\tau - \kappa)^+ | S_t = x)
\]

\[
= \beta_{t-\tau} \int_{\ln x + \left( \frac{\sigma^2}{2} + r \right) (\tau - s)}^\infty \frac{1}{\sigma \sqrt{2\pi(\tau - s)}} \exp\left( -\frac{\left( \ln y + \left( \frac{\sigma^2}{2} + r \right) \tau - \kappa \right)^2}{\sigma^2(\tau - s)} \right) \times \exp\left( y - \left( \frac{\sigma^2}{2} + r \right) \tau - \kappa \right) dy
\]

From the last formula it follows easily that a \( C^2 \) function on \([0, \tau) \times R \). Since \( dS_t = S_t dR_t \) [see (10) for the definition of \( R \)] and the quadratic variation does not change under equivalent change of measure, \( d[S_t, S_t] = (S_t)^2 \sigma^2 ds \) (see Jacod and Shiryaev, 1987, page 156). Ito's formula gives for \( 0 \leq t < \tau \) (see Jacod and Shiryaev, 1987, p. 57).

\[
v^{\tau,\kappa}(t, S_t) - v^{\tau,\kappa}(0, 1) = \int_0^t \frac{\partial v^{\tau,\kappa}}{\partial t} (s, S_s) ds + \int_0^t \frac{\partial v^{\tau,\kappa}}{\partial x} (s, S_s) dS_s + \frac{1}{2} \int_0^t \frac{\partial^2 v^{\tau,\kappa}}{\partial x^2} (s, S_s)(S_s)^2 \sigma^2 ds
\]

\[
= \int_0^t \frac{\partial v^{\tau,\kappa}}{\partial x} (s, S_s) dS_s - \int_0^t \frac{\partial v^{\tau,\kappa}}{\partial x} (s, S_s) ^2 S_s dB_s
\]

\[
+ \int_0^t \frac{\partial v^{\tau,\kappa}}{\partial x} S_s dS_s + \int_0^t \frac{\partial v^{\tau,\kappa}}{\partial t} (s, S_s) ds + \frac{1}{2} \int_0^t \frac{\partial^2 v^{\tau,\kappa}}{\partial x^2} (s, S_s)(S_s)^2 \sigma^2 ds
\]

Now \( \int_0^t \frac{\partial v^{\tau,\kappa}}{\partial x} (s, S_s) dS_s - \int_0^t \frac{\partial v^{\tau,\kappa}}{\partial x} \beta_s S_s dB_s = \int_0^t B_s \frac{\partial v^{\tau,\kappa}}{\partial x} (s, S_s) d(\beta_s S_s) \) and \( v^{\tau,\kappa}(s, S_s) - v^{\tau,\kappa}(0, 1) \) \( Q \)-martingales. Therefore

\[
\int_0^t \left[ \frac{\partial v^{\tau,\kappa}}{\partial x} S_s r + \frac{\partial v^{\tau,\kappa}}{\partial t} (s, S_s) + \frac{1}{2} \frac{\partial^2 v^{\tau,\kappa}}{\partial x^2} (s, S_s)(S_s)^2 \sigma^2 \right] ds
\]

is a continuous martingale of finite variation and hence equal to zero. So

\[
v^{\tau,\kappa}(t, S_t) - v^{\tau,\kappa}(0, 1) = \int_0^t \frac{\partial v^{\tau,\kappa}}{\partial x} (s, S_s) dS_s - \int_0^t \frac{\partial v^{\tau,\kappa}}{\partial x} (s, S_s) \beta_s S_s dB_s \quad \text{(A.28)}
\]
By the dominated convergence theorem $\frac{\partial v^{t,x}}{\partial x}(s, x)$ is equal to
\[
\int_{\ln \kappa + (\frac{\kappa}{2} + r)}^{\infty} \beta_{t-s} \left[ \frac{1}{\sigma \sqrt{2\pi(\tau - s)}} \left( \exp \left( y - \left( \frac{\sigma^2}{2} + r \right) \tau \right) - \kappa \right) \times \frac{1}{x} \left( -\frac{\ln x + \left( \frac{\sigma^2}{2} + r \right) s - y}{\sigma^2(\tau - s)} \right) \times \exp \left( -\frac{\left( \ln x + \left( \frac{\sigma^2}{2} + r \right) s - y \right)^2}{2\sigma^2(\tau - s)} \right) \right] dy
\]

Using integration by parts,
\[
\frac{\partial v^{t,x}}{\partial x}(s, x) = \int_{\ln \kappa + (\frac{\kappa}{2} + r)}^{\infty} \beta_{t-s} \left[ \frac{1}{\sqrt{2\pi(\tau - s)}} \exp \left( y - \left( \frac{\sigma^2}{2} + r \right) \tau \right) \times \frac{1}{x} \right] \times \exp \left( -\frac{\left( \ln x + \left( \frac{\sigma^2}{2} + r \right) s - y \right)^2}{2\sigma^2(\tau - s)} \right) dy,
\]

which, recalling the definition of $h^{t,x}$ [see (20)], shows that
\[
\frac{\partial v^{t,x}}{\partial x}(s, x) = \beta_{t-s} E_Q \left( \frac{1}{x} \mathbf{1}_{S_t, S_t > \kappa} | S_t = x \right) = h^{t,x}(s, x) \tag{A.29}
\]

Comparing (A.28) with (A.29) shows that the conclusion of the theorem holds for $t < \tau$. To extend the result to $t = \tau$ note that $v^{t,x}(t, S_t)$ is continuous in $t$, $V^{t,x}(\tau, S_\tau) = \lim_{\tau \to t} v^{t,x}(t, S_t)$. By the definition of $h^{t,x}(\tau, S_\tau(\omega)) = 1_{[S_\tau(\omega) > \kappa]}$. Using in addition that $\ln S$ is a process with independent identically distributed increments, we have for $t < \tau$,
\[
h^{t,x}(s, S_\tau(\omega)) = \beta_{t-s} E_Q S_t^{2} \mathbf{1}_{[S_\tau(\omega) > \kappa]}.
\]

Now $E_Q S_t^2 = \exp(2t(\sigma^2 - r))$, which implies that $S_{t-s}$ is uniformly integrable under $Q$ and thus that $h^{t,x}(s, S_t)$ is uniformly bounded in $s$ and $\omega$. The last result together with the continuity of $S$ and $B$ shows that
\[
\left| \int_0^t h^{t,x}(s, S_t) dS_t - \int_0^t h^{t,x}(s, S_t) \beta_t S_t dB_t \right|
\]
\[
\leq \| h^{t,x} \|_{\infty}(S_t - S_t) + \| h^{t,x} \|_{\infty}(B_t - B_t) \to 0 \text{ for } t \to \tau
\]

The norm $\| \cdot \|_{\infty}$ is the sup-norm, see Jacod and Shiryaev (1987, page 2). Hence, for $0 \leq t \leq \tau$, $v^{t,x}(t, S_t) - v^{t,x}(0, 1) = \int_0^t h^{t,x}(s, S_t) dS_t - \int_0^t h^{t,x}(s, S_t) \beta_t S_t dB_t. \quad \blacksquare$

**Theorem 5** Assume $\lambda_n \to 1$. Then $(V^{n,\lambda_n^{t,x}} \mid P^n) \Rightarrow (V^{t,x} \mid P)$. 
Let $\lambda \in (0, \infty)$. We define $S^\lambda$ to be $e^{\lambda(R_t - rt)}$ [see (10) for a definition of $R$],
that is
\begin{equation}
S^\lambda_t = \exp(rt) \exp \left( \sqrt{\lambda} \left( W_t + \frac{\mu t}{\sigma^2} \right) - \frac{\lambda \sigma^2}{2} t \right).
\end{equation}
(A.30)

Note that, under $Q$, $W_t + \frac{\mu t}{\sigma^2}$ is a standard Brownian motion and $\beta S^\lambda$ is the stochastic exponential of a Brownian motion with variance $\lambda \sigma^2$ and in particular is $S^1 = S$ [see (8)].

**Proposition 1** Let $\lambda_n \to \lambda \in (0, \infty)$. Then
\[ (S^n \mid Q^{n, \lambda_n}) \Rightarrow (S^\lambda \mid Q) \]

**Proof:** The proof is essentially as the proof of Theorem 1. Let $\hat{R}^n$ and $\hat{R}$ be defined by $\hat{R}^n_t = R^n_t - rt$ and $\hat{R}_t = R_t - rt$. $\hat{R}^n$ is under $Q^{n, \lambda_n}$ a square-integrable martingale. Since $\hat{R}^n_0 = 0$, $(\hat{R}^n, \hat{R}^n)_t = \lambda_n t$ for all $n$ and $\lambda_n \to \lambda$, the sequences $(\hat{R}^n_t)_{t \in \mathbb{N}}$ and $(\hat{R}_t, \hat{R}^n_t)_{t \in \mathbb{N}}$ are tight and $C$-tight respectively. Therefore, the sequence $(\hat{R}^n)_{n \in \mathbb{N}}$ is tight by Theorem 4.13 on page 322 of Jacod and Shiryaev (1987). An application of the Lindeberg-Feller theorem shows for fixed $t, s \in [0, 1]$
\[ (\hat{R}^n_t - \hat{R}^n_s \mid Q^{n, \lambda_n}) \Rightarrow \sqrt{\lambda \sigma} \Phi(0, t - s), \]
which is the distribution of $\sqrt{\lambda} \hat{R}$ under $Q$. Since both $\hat{R}^n$ and $\hat{R}$ are processes with independent increments the finite dimensional distributions of $\hat{R}^n$ converge to $\sqrt{\lambda} \hat{R}$, by Lemma 1.3 on page 350 of Jacod and Shiryaev (1987) and it is shown that $(\hat{R}^n \mid Q^{n, \lambda_n}) \Rightarrow (\sqrt{\lambda} \hat{R} \mid Q)$. Since $f(t) = rt$ is a continuous real function, $(\hat{R}^n \mid Q^{n, \lambda_n}) \Rightarrow (\sqrt{\lambda} \hat{R} + rt \mid Q)$ by Proposition 3.17 on page 314 of Jacod and Shiryaev (1987). Since the jumps of $R^n$ are uniformly bounded in $n, t$ and $\omega$, Corollary 6.6 on page 342 of Jacod and Shiryaev (1987) states that this is a sufficient condition for
\[ ((R^n, [R^n, R^n]) \mid Q^{n, \lambda_n}) \Rightarrow ((\sqrt{\lambda} \hat{R} + rt, [\sqrt{\lambda} \hat{R} + rt, \sqrt{\lambda} \hat{R} + rt]) \mid Q) \]
Theorem 1 of Avram (1986) shows that this implies the stochastic exponential converges, i.e. $(e(R^n) \mid Q^{n, \lambda_n}) \Rightarrow (e(\sqrt{\lambda} \hat{R} + rt) \mid Q)$, that is $(S^n \mid Q^{n, \lambda_n}) \Rightarrow (S^\lambda \mid Q)$.

Define
\begin{equation}
V^{\lambda, \tau, \kappa}(s, x) = \beta_{t-s} E_Q \left( (S^\lambda_t - \kappa)^+ \mid S^\lambda_s = x \right)
\end{equation}
(A.31)
and
\begin{equation}
V^{\lambda, \tau, \kappa}_t = V^{\lambda, \tau, \kappa}(t, S^\lambda_t)
\end{equation}
(A.32)

$V^{\lambda, \tau, \kappa}$ is the price process for the call with exercise time $\tau$ and exercise price $\kappa$ on the stock $S^\lambda$ [see (A.30)]. In particular, $V^{\tau, \kappa} = V^{\tau, \kappa}_0$ [see (A.30)].
Note that, since \( \ln S^n \) and \( \ln S^\lambda \) are processes with independent identically distributed increments and \( S^n_0 = S^\lambda_0 = 1 \),

\[
v^{n, \lambda, t, x}(s, x) = \beta_{t-s} E Q^{s,t} (x S^n_{t-s} - \kappa)^+
\]

and

\[
v^{\lambda, t, x}(s, x) = \beta_{t-s} E Q (x S^\lambda_{t-s} - \kappa)^+
\]

The following lemma is the main tool to prove Proposition 2 and to establish later in the same fashion weak convergence of hedge ratio processes.

**Lemma 1** Let, for each \( n \), \( f^n(t, x) \) and \( f(t, x) \) be a continuous real valued function on \([0, 1] \times R^d\). Let \( A \subseteq [0, 1] \times R^d \) be such that if \((t, x) \in [0, 1] \times R^d - A \) and \( t_n \rightarrow t, x_n \rightarrow x \) then \( f^n(t_n, x_n) \rightarrow f(t, x) \). Under this assumption, it follows that

(a) For any compact set \( C \subseteq [0, 1] \times R^d - A \),

\[
\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \sup_{(t, x) \in C} |f^n(t, x) - f(t, x)| = 0
\]

(b) If \( S^n \Rightarrow S \) a.s. it follows that \( \|f^n(t, S^n_t) - f(t, S_t)\| \rightarrow 0 \) a.s.

(c) If \( S^n \Rightarrow S \) it follows that \( f^n(\bullet, S^n_\bullet) \Rightarrow f(\bullet, S_\bullet) \).

**Proof:** (a) Assume the statement is not true. Then there is some \( \delta > 0 \) such that for every \( m \) there is some \( n_m > m \) and \((t_m, x_m, y_m)\) with

\[
(t_m, y_m) \in C, |x_m - y_m| \leq \frac{1}{m} \text{ and } |f^n(t_m, x_m) - f(t_m, y_m)| > \delta.
\]

Now there exists a convergent subsequence \((t_{m_k}, x_{m_k}, y_{m_k}) \rightarrow (t, x, y)\), for which, since \( |x_m - y_m| < \frac{1}{m_k} \), \( x = y \). Since \( f^{n_{m_k}}(t_{m_k}, x_{m_k}) \rightarrow f(t, x) \) and by assumption \( f \) is continuous,

\[
|f^{n_{m_k}}(t_{m_k}, x_{m_k}) - f(t_{m_k}, y_{m_k})| \rightarrow 0,
\]

a contradiction to the assumption, that \( |f^{n_{m_k}}(t_{m_k}, x_{m_k}) - f(t_{m_k}, y_{m_k})| > \delta \) for all \( k \).

(b) Let \( \Omega_1 = \{\|S^n - S\| \rightarrow 0\} \). Let \( C(\omega) = \text{closure of } \{(t, S_t(\omega)) : 0 \leq t \leq 1\} \) and \( \Omega_2 = \{\omega : C(\omega) \text{ is a compact subset of } [0, 1] \times R^d - A\} \). For any \( \omega \in \Omega_1 \cap \Omega_2 \) (a) gives

\[
\|f^n(t, S^n_t) - f(t, S_t)\| \rightarrow 0
\]
Now $\Omega_1$ has measure one since $S^n \to S$ a.s. and $\Omega_2$ has measure one since $\bar{S}$ is continuous and $B$ has measure 0. Hence, so does $\Omega_1 \cap \Omega_2$, which proves the claim.

(c) Let, for each $n$, $\bar{S}^n$ and $\bar{S}$ be stochastic processes given by the Skorokhod representation theorem. Now (b) gives in particular $f^n(\bullet, \bar{S}^n) \to (\bullet, \bar{S})$ a.s. Thus, for any real, bounded and continuous function $g$ on $(D[0, 1], R)$

$$E_{\bar{P}}g(f^n(\bullet, \bar{S}^n)) \to E_{\bar{P}}g(f(\bullet, \bar{S})).$$

Therefore, also

$$E_{\bar{P}}g(f^n(\bullet, S^n)) \to E_{\bar{P}}g(f(\bullet, S)).$$

which shows $f^n(\bullet, S^n) \Rightarrow f(\bullet, S)$. ■

**Proposition 2** Let $\lambda_n \to \lambda$. Then $(V^{n, \lambda_n, r, \kappa} \mid P^n) \Rightarrow (V^{\lambda, r, \kappa} \mid P)$.

**Proof:** (i) Let $t < \tau, t_n \to t$ and $x_n \to x$. By Proposition 1 $(S^n \mid Q^{n, \lambda_n}) \Rightarrow (S^\lambda \mid Q)$. Let, for each $n$, $S^n$ and $S^\lambda$ be processes given by the Skorokhod representation theorem. Since $S^\lambda$ is continuous, $\bar{S}^\lambda_{t-t_n} \to \bar{S}^\lambda_{t-t}$ a.s. Furthermore, since $E_{Q^{n, \lambda_n}}(S^n_{t-t_n})^2 = \exp((\lambda_n r^2 + 2r)(\tau - t_n)) \to \exp((\lambda r^2 + 2r)(\tau - t))$, the sequence $\{(x_n \bar{S}^n_{t-t_n} - K)^+\}_{n \in N}$ is uniformly integrable under $\bar{P}$. Using the expressions (A.33) and (A.34) for $V^{n, \lambda_n, r, \kappa}$ and $V^{\lambda, r, \kappa}$ respectively together with the continuity of $\beta$, it follows therefore that

$$V^{n, \lambda_n, r, \kappa}(t_n, x_n) = \beta_{t-t_n}E_{\bar{P}}\left(x_n \bar{S}^n_{t-t_n} - K\right)^+ \to \beta_{t-t}E_{\bar{P}}\left(x \bar{S}^\lambda_{t-t} - K\right)^+ = V^{\lambda, r, \kappa}(t, x)$$

Let $t = \tau, t_n \to t$ and $x_n \to x$.

$$Q^{n, \lambda_n}(\mid S^n_{t-t_n} - S^\lambda_0 \mid > \epsilon) \leq \frac{1}{\epsilon^2}E_{Q^{n, \lambda_n}}(S^n_{t-t_n} - 1)^2 \leq \frac{1}{\epsilon^2}(1 - 2 \exp(r(\tau - t_n))) + \exp((\lambda_n r^2 + 2r)(\tau - t_n)) \to 0.$$

So in particular $(S^n_{t-t_n} \mid Q^{n, \lambda_n}) \Rightarrow 1$. Let $(\bar{S}^n_{t-t_n})_{n \in N}$ be a sequence of processes given by the Skorokhod representation theorem which converges to $1$ almost surely. This, together with the uniform integrability of the sequence $\{(x_n \bar{S}^n_{t-t_n} - K)^+\}_{n \in N}$ gives $E_{\bar{P}}(x_n \bar{S}^n_{t-t_n} - K)^+ \to (x - K)^+$. Using the expressions (A.31) and (A.32) for $V^{n, \lambda_n, r, \kappa}$ and $V^{\lambda, r, \kappa}$ respectively, the last result implies $V^{n, \lambda_n, r, \kappa}(t_n, x_n) \to V^{\lambda, r, \kappa}(\tau, x)$.

(ii) Now (c) of Lemma 1 with $A = 0$ together with Theorem 1 shows that $(V^{n, \lambda_n, r, \kappa} \mid P^n) \Rightarrow (V^{\lambda, r, \kappa} \mid P)$. ■

**Proof of Theorem 5 (continued):** Taking $\lambda = 1$ shows Theorem 5 as a corollary of Proposition 2. ■
Theorem 6 Assume $V_{0}^{n,\lambda_{n},r,x} \rightarrow V_{0}^{1}$. Then $\lambda_{n} \rightarrow 1$.

Lemma 2 For fixed $n$, the call price is strictly increasing in $\lambda_{n}$.

Proof: By the definition of $V_{0}^{n,\lambda_{n},r,x}$ [see (12)],

$$
V_{0}^{n,\lambda_{n},r,x} = \beta_{r} E_{\mathcal{Q}^{n}}(S_{t}^{n} - \kappa)^{+} = \beta_{r} E_{\mathcal{Q}^{n}}\left(\exp(r\tau)\left(1 + \frac{\sigma}{\sqrt{2n}}\right)^{N_{s}^{n}}\left(1 - \frac{\sigma}{\sqrt{2n}}\right)^{M_{s}^{n}} - \kappa\right)^{+} = \beta_{r} E_{\mathcal{Q}^{n}}\left(\exp(r\tau)\left(1 + \frac{\sigma}{\sqrt{2n}}\right)^{N_{s}^{n}}\left(1 - \frac{\sigma}{\sqrt{2n}}\right)^{M_{s}^{n}} - \kappa\right)^{+} = \beta_{r} E_{\mathcal{Q}^{n}}\left(\exp(r(\tau - \lambda_{n}))S_{t}^{n} - \kappa\right)^{+}.
$$

Note that for fixed $t > 0$ and $x$, since $Q^{n}(\exp(r(\tau - \lambda_{n})))S_{t}^{n} > \frac{x}{\kappa} > 0$, $Q^{n}(\exp(r(\tau - \lambda_{n})))S_{t}^{n} < \frac{x}{\kappa} > 0$, and $E_{\mathcal{Q}^{n}}S_{t}^{n} = \exp(rt) > 1$,

$$
E_{\mathcal{Q}^{n}}(x \exp(r(\tau - \lambda - n)))S_{t}^{n} - \kappa)^{+} > (x \exp(r(\tau - \lambda_{n})))E_{\mathcal{Q}^{n}}S_{t}^{n} - \kappa)^{+} > (x \exp(r(\tau - \lambda_{n}))) - \kappa)^{+}.
$$

Since, for $t > s$,

$$
E_{\mathcal{Q}^{n}}\left(\left(\exp(r(\tau - \lambda_{n}))S_{t}^{n} - \kappa\right)^{+} | S_{t}^{n} = x\right) = E_{\mathcal{Q}^{n}}\left(x \exp(r(\tau - \lambda_{n}))S_{t}^{n} - \kappa\right)^{+},
$$

the last expression shows that

$$
E_{\mathcal{Q}^{n}}(\exp(r(\tau - \lambda_{n})))S_{t}^{n} - K)^{+} > E_{\mathcal{Q}^{n}}(\exp(r(\tau - \lambda_{n})))S_{t}^{n} - K)^{+},
$$

that is, $E_{\mathcal{Q}^{n}}(\exp(r(\tau - \lambda_{n})))S_{t}^{n} - K)^{+}$ is increasing in $\lambda_{n}$ and therefore so is $V_{0}^{n,\lambda_{n},r,x}$.

Proof of Theorem 6: Since for $\lambda_{n} \rightarrow \infty$, $V_{0}^{n,\lambda_{n},r,x} \rightarrow \infty$, we can assume without loss of generality that $(\lambda_{n})_{n \in \mathbb{N}} \in [0, M]$ for some $M > 0$. So any subsequence of $(\lambda_{n})_{n \in \mathbb{N}}$ has a further convergent subsequence, here denoted by $(\lambda_{k})_{k \in \mathbb{N}}$ with $\lambda_{k} \rightarrow \lambda^{*}$ for some $\lambda^{*} \in [0, M]$. Proposition 2 shows then in particular that $V_{0}^{n,\lambda_{k},r,x} \rightarrow V_{0}^{\lambda^{*},r,x}$ (see Proposition 3.14 on page 313 of Jacod and Shiryaev). Since by definition [see (2.8)] $V_{0}^{\lambda} = \beta_{r} E_{\mathcal{Q}}(S_{t}^{\lambda} - K)^{+}$ and by the Brownian scaling relation: $[S_{t}^{\lambda}$ is distributed like $\exp(r(\tau - \lambda))S_{t}^{\lambda}]$, then as in the proof of Lemma 2, $V_{0}^{\lambda,r,x}$ is strictly increasing in $\lambda$. A proof of the Brownian scaling relation can be found in Durrett (1991, page 334). Since now in particular $V_{0}^{n,\lambda_{k},r,x}$ converges to $V_{0}^{1,r,x}$ therefore necessarily $\lambda^{*} \rightarrow 1$, that is, $\lambda_{k} \rightarrow 1$ and hence also $\lambda_{n} \rightarrow 1$. 

$\blacksquare$
Proof of Theorem 7: (i) Let $0 \leq s < \tau$. Let $s_n \to s, x_n \to x$. Using expression (A.33) for $v^{n, \lambda_0, \tau, \kappa}$,

$$v^{n, \lambda_0, \tau, \kappa}(s_n, x_n) \left(1 + \frac{\sigma}{\sqrt{2n}}\right) - v^{n, \lambda_0, \tau, \kappa}(s_n, x_n) \left(1 - \frac{\sigma}{\sqrt{2n}}\right)$$

$$= \beta_{\tau-s} \sum_y \mathcal{Q}^{n, \lambda_0}(S^n_{\tau-s} = y) \left(x_n \left(1 + \frac{\sigma}{\sqrt{2n}}\right) y - \kappa\right) +$$

$$- \beta_{\tau-s} \sum_y \mathcal{Q}^{n, \lambda_0}(S^n_{\tau-s} = y) \left(x_n \left(1 - \frac{\sigma}{\sqrt{2n}}\right) y - \kappa\right) +$$

$$= \beta_{\tau-s} \sum_{x_n \left(1 - \frac{\sigma}{\sqrt{2n}}\right) \kappa > 0} \sqrt{\frac{2}{\pi}} \sigma x_n y Q^{n, \lambda_0}(S^n_{\tau-s} = y)$$

$$\beta_{\tau-s} \sum_{x_n \left(1 + \frac{\sigma}{\sqrt{2n}}\right) \kappa > 0} \left(x_n \left(1 - \frac{\sigma}{\sqrt{2n}}\right) y - \kappa\right) Q^{n, \lambda_0}(S^n_{\tau-s} = y)$$

Therefore, using the definition of $\hat{h}^{n, \lambda_0, \tau, \kappa}$ [see (24)],

$$\hat{h}^{n, \lambda_0, \tau, \kappa}(s_n, x_n) = \frac{\sqrt{\frac{n}{2}} \sigma x_n}{\left(1 + \frac{\sigma}{\sqrt{2n}}\right)} \left(v^{n, \lambda_0, \tau, \kappa}(s_n, x_n) \left(1 + \frac{\sigma}{\sqrt{2n}}\right)\right)$$

$$- v^{n, \lambda_0, \tau, \kappa}(s_n, x_n) \left(1 - \frac{\sigma}{\sqrt{2n}}\right)\right)$$

$$= \beta_{\tau-s} E_{Q^{n, \lambda_0}} S^n_{\tau-s}$$

$$\times 1 \left\{ \frac{S^n_{\tau-s} > \left(\frac{\kappa}{1 - \frac{\sigma}{\sqrt{2n}}}\right)}{\left(\frac{\kappa}{1 - \frac{\sigma}{\sqrt{2n}}\right)} \right\}$$

$$+ \beta_{\tau-s} E_{Q^{n, \lambda_0}} \sqrt{\frac{n}{2}} \sigma x_n \left(x_n \left(1 + \frac{\sigma}{\sqrt{2n}}\right) S^n_{\tau-s} - \kappa\right)$$

$$\times 1 \left\{ \frac{x_n \left(1 + \frac{\sigma}{\sqrt{2n}}\right) \kappa < S^n_{\tau-s} < x_n \left(1 - \frac{\sigma}{\sqrt{2n}}\right) \right\}$$

By Proposition 1, $(S^n | Q^{n, \lambda_0}) \Rightarrow (S | Q)$. Let, for each $n$, $\tilde{S}^n$ and $\tilde{S}$ be stochastic processes given by the Skorohod representation theorem. Since $\tilde{S}$ is continuous, this implies $\tilde{S}^n_{\tau-s} \to \tilde{S}_{\tau-s}$ a.s. Since furthermore the sequence $(\tilde{S}^n_{\tau-s})_{n \in \mathbb{N}}$ is uniformly integrable under $P$ and $\tilde{P}(\tilde{S}_{\tau-s} = \frac{x}{x}) = 0$,

$$E_{\tilde{P}} \tilde{S}^n_{\tau-s} 1 \left\{ \tilde{S}^n_{\tau-s} > \left(\frac{\kappa}{1 - \frac{\sigma}{\sqrt{2n}}\right)} \right\} \to E_{\tilde{P}} \tilde{S}_{\tau-s} 1 \left\{ \tilde{S}_{\tau-s} > \frac{x}{x} \right\}$$
which implies
\[
\beta_{t-s_n} E_{Q^n \lambda_x} S_{t-s_n} \begin{cases} S_{t-s_n} > \frac{\epsilon}{x_n(1 - \frac{\sigma}{\sqrt{2n}})} \\ S_{t-s_n} \leq \frac{\epsilon}{x_n(1 - \frac{\sigma}{\sqrt{2n}})} \end{cases} \rightarrow \beta_{t-s} E_Q S_{t-s} \begin{cases} S_{t-s} > \frac{\epsilon}{x(1 - \frac{\sigma}{\sqrt{2n}})} \\ S_{t-s} \leq \frac{\epsilon}{x(1 - \frac{\sigma}{\sqrt{2n}})} \end{cases}
\] (A.36)

For the second term of (A.35) let
\[
G^n = \left\{ \frac{\kappa}{x_n \left(1 + \frac{\sigma}{\sqrt{2n}}\right)} < \frac{\kappa}{x_n \left(1 - \frac{\sigma}{\sqrt{2n}}\right)} \right\}
\]
\[
= \left( x_n \left(1 - \frac{\sigma}{\sqrt{2n}}\right) S_{t-s_n} - \kappa < 0 < x_n \left(1 + \frac{\sigma}{\sqrt{2n}}\right) S_{t-s_n} - \kappa \right)
\]
and note that on \(G^n\)
\[
x_n \left(1 + \frac{\sigma}{\sqrt{2n}}\right) S_{t-s_n} - \kappa < \sqrt{\frac{2}{n}} \sigma x_n S_{t-s_n} < \sqrt{\frac{2}{n}} \sigma \kappa \left(1 - \frac{\sigma}{\sqrt{2n}}\right)^{-1}
\]
Therefore
\[
0 \leq E_{\tilde{P}} \sqrt{\frac{n}{2 \sigma}} \frac{1}{x_n} \left( x_n \left(1 + \frac{\sigma}{\sqrt{2n}}\right) S_{t-s_n} - \kappa \right) 1_{G^n} \leq E_{\tilde{P}} \frac{\kappa}{x_n} \left(1 - \frac{\sigma}{\sqrt{2n}}\right)^{-1} 1_{G^n}
\] (A.37)

Since \(S_{t-s_n} \rightarrow S_{t-s} \text{ a.s., } \tilde{P}(G^n) \rightarrow \tilde{P} \left( S_{t-s} = \frac{\epsilon}{x} \right) = 0 \) and therefore
\[
E_{\tilde{P}} \frac{\kappa}{x_n} \left(1 - \frac{\sigma}{\sqrt{2n}}\right)^{-1} 1_{G^n} \rightarrow 0
\]

Using (A.37) and the definition of \(G^n\), it follows that
\[
\beta_{t-s_n} E_{Q^n \lambda_x} \sqrt{\frac{n}{2 \sigma}} \frac{1}{x_n \sigma} \left( x_n \left(1 + \frac{\sigma}{\sqrt{2n}}\right) S_{t-s_n} - \kappa \right) 1_{\left\{ \frac{\epsilon}{x_n(1 - \frac{\sigma}{\sqrt{2n}})} < S_{t-s_n} \leq \frac{\epsilon}{x_n(1 - \frac{\sigma}{\sqrt{2n}})} \right\}} \rightarrow 0
\] (A.38)

Combining (A.35), (A.36) and (A.38) and the definition of \(h^{\tau, \kappa}\) [see (32)] gives
\[
\tilde{f}_{n, \lambda_x}^{\tau, \kappa} (s_n, x_n) \rightarrow \beta_{t-s} E_Q S_{t-s} \begin{cases} S_{t-s} > \frac{\epsilon}{x(1 - \frac{\sigma}{\sqrt{2n}})} \\ S_{t-s} \leq \frac{\epsilon}{x(1 - \frac{\sigma}{\sqrt{2n}})} \end{cases} = h^{\tau, \kappa} (s, x)
\]

Let \(s = \tau\). Let \(s_n \rightarrow \tau, x_n \rightarrow x \neq \kappa, \ Q^n_{\tau, \lambda_x} (|1 - S_{t-s_n}| > \epsilon) \rightarrow 0, \) so in particular there are processes \(S_{t-s_n}^n\), which are defined on a common probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\), have the same distribution as \(S_{t-s_n}\), and converge almost surely to 1. Since the sequence \(\{S_{t-s_n}^n\}_{n \in N}\) is uniformly integrable and \(x \neq \kappa, \)
\[
E_{\tilde{P}} \tilde{S}_{t-s_n} \begin{cases} \tilde{S}_{t-s_n} > \frac{\epsilon}{x_n(1 - \frac{\sigma}{\sqrt{2n}})} \\ \tilde{S}_{t-s_n} \leq \frac{\epsilon}{x_n(1 - \frac{\sigma}{\sqrt{2n}})} \end{cases} \rightarrow 1_{[x > \kappa]},
\]
which implies
\[
\beta_{t_{x}} E_{Q^{\kappa}} S_{t_{x}}^{n} \frac{1}{\left\{ S_{t_{x}}^{n} < \left( \frac{x}{x_{\kappa}} \right) \right\}} \to 1_{[x > x_{\kappa}]} \tag{A.39}
\]

Let \(G^{n}\) be as before. Since now \(\tilde{S}_{t_{x}}^{n} \to 1\) a.s. and \(x \neq x_{\kappa}\), \(\tilde{P}(G^{n}) \to 0\). Thus (A.37) gives
\[
E_{\tilde{P}} \frac{1}{\sqrt{2 n x_{\kappa} \sigma}} \left( x_{n} \left( 1 + \frac{\sigma}{\sqrt{2 n}} \right) \tilde{S}_{t_{x}}^{n} - \kappa \right) \frac{1}{G^{n}} \leq E_{\tilde{P}} \frac{\kappa}{x_{n}} \left( 1 - \frac{\sigma}{\sqrt{2 n}} \right)^{-1} 1_{G^{n}} \to 0,
\]

which, together with the definition of \(G^{n}\) implies
\[
\beta_{t_{x}} E_{Q^{\kappa}} \frac{1}{\sqrt{2 n x_{\kappa} \sigma}} \left( x_{n} \left( 1 + \frac{\sigma}{\sqrt{2 n}} \right) S_{t_{x}}^{n} - \kappa \right) \left\{ \left( \frac{x}{x_{\kappa}} \right) < S_{t_{x}}^{n} < \frac{x_{\kappa}}{n} \left( \frac{x}{x_{\kappa}} \right) \right\} \to 0 \tag{A.40}
\]

Combining (A.35), (A.39) and (A.40) and the definition of \(h^{r, x}\) gives that
\[
\tilde{h}^{n, \lambda, r, x}(s, x_{n}) \to 1_{[x > x_{\kappa}]} = h^{r, x}(\tau, x).
\]

(ii) The above together with Theorem 1 and (c) of Lemma 1, with \(A = \{(r, \kappa)\}\), shows that \(\tilde{h}^{n, \lambda, r, x}(\bullet, S_{n}^{n}) \to h^{r, x}(\bullet, S_{\bullet})\). \(\blacksquare\)

**Proof of Theorem 8:** By Theorem 4, the statement of the theorem is equivalent to
\[
\int_{0}^{*} \tilde{h}^{n, \lambda, r, x}(s, S_{t_{x}}^{n}) dS_{t_{x}}^{n} - \int_{0}^{*} \tilde{h}^{n, \lambda, r, x}(s, S_{t_{x}}^{n}) \beta_{x} S_{t_{x}}^{n} dB_{x} = 0
\]

\[
\Rightarrow \int_{0}^{*} h^{r, x}(s, S_{t_{x}}) dS_{t_{x}} - \int_{0}^{*} h^{r, x}(s, S_{t_{x}}) \beta_{x} S_{t_{x}} dB_{x} \tag{A.41}
\]

Since now
\[
\int_{0}^{*} \tilde{h}^{n, \lambda, r, x}(s, S_{t_{x}}^{n}) dS_{t_{x}}^{n} - \int_{0}^{*} \tilde{h}^{n, \lambda, r, x}(s, S_{t_{x}}^{n}) \beta_{x} S_{t_{x}}^{n} dB_{x} = \int_{0}^{*} B_{x} \tilde{h}^{n, \lambda, r, x}(s, S_{t_{x}}^{n}) d(\beta_{x} S_{t_{x}}^{n}), \tag{A.42}
\]

and
\[
\int_{0}^{*} h^{r, x}(s, S_{t_{x}}) dS_{t_{x}} - \int_{0}^{*} h^{r, x}(s, S_{t_{x}}) \beta_{x} S_{t_{x}} dB_{x} = \int_{0}^{*} B_{x} h^{r, x}(s, S_{t_{x}}) d(\beta_{x} S_{t_{x}}), \tag{A.43}
\]

(A.41) and thus the statement of the theorem is equivalent to
\[
\int_{0}^{*} B_{x} \tilde{h}^{n, \lambda, r, x}(s, S_{t_{x}}^{n}) d(\beta_{x} S_{t_{x}}^{n}) \Rightarrow \int_{0}^{*} B_{x} h^{r, x}(s, S_{t_{x}}) d(\beta_{x} S_{t_{x}}) \tag{A.44}
\]
(i) By Theorem 1, $S^n \Rightarrow S$. Let $\tilde{S}^n$ and $\tilde{S}$ be stochastic processes given by the Skorohod representation theorem. $\|\tilde{S}^n - \tilde{S}\|_{\infty} \to 0$, which implies also $\|\delta_n \tilde{S}^n - \delta \tilde{S}\|_{\infty} \to 0$. Recalling the proof of (b) of Lemma 1 and using (i) of Theorem 7 shows that there is a set of measure zero outside of which, for $\lambda_n \to 1$,

$$\left\| B_n h^{n, \lambda_n, t, \kappa} \left( \bullet, \tilde{S}^n(\omega) \right), \delta_n \tilde{S}_n(\omega) \right\| - \left( B_n h^{t, \kappa} \left( \bullet, \tilde{S}(\omega) \right), \delta \tilde{S} \right) \| \to 0.$$

This implies, following the same argument as outlined in the proof of (c) of Lemma 1, that

$$\left( B_n h^{n, \lambda_n, t, \kappa} \left( \bullet, S^n \right), \delta_n \tilde{S}_n \right) \Rightarrow \left( B_n h^{t, \kappa} \left( \bullet, S \right), \delta \tilde{S} \right).$$

(ii) $\beta S^n = \varepsilon(\tilde{R}^n)$ with $\tilde{R}^n = \frac{\sigma}{\sqrt{2\pi}} (N^n - M^n)$. The decomposition of $\tilde{R}^n$ into a martingale $M^{\tilde{R}^n}$ with $M^{\tilde{R}^n}_0 = 0$ and a predictable process of finite variation $A^{\tilde{R}^n}$ is given by

$$M^{\tilde{R}^n}_t = \frac{\sigma}{\sqrt{2\pi}} (N^n_t - M^n_t) - \mu t, \quad A^{\tilde{R}^n}_t = \mu t.$$

Now $E_{\mathbb{P}_n} \left[ \|M^{\tilde{R}^n}, A^{\tilde{R}^n}\|_\tau \right] = \sigma^2 \tau$ and $E_{\mathbb{P}_n} \|A^{\tilde{R}^n}\|_\tau = \mu \tau$, where $\|A^{\tilde{R}^n}\|_\tau$ denotes the total variation of $A^{\tilde{R}^n}$ in time $\tau$. This checks condition $A$ of Duffie and Proter (1992) is satisfied, so using their Theorem 4.1 shows that, for any $g^\alpha$, weak convergence of $(g^\alpha, \tilde{R}^n)$ to $(g^\alpha, \tilde{R})$ implies weak convergence of $(g^\alpha, \tilde{R}^n, \int_0^t g^\alpha \tilde{R}_s ds)$ to $(g^\alpha, \tilde{R}, \int_0^t g^\alpha \tilde{R}_s ds)$. Further, using Theorem 4.4 of Duffie and Proter (1992) with $f^\alpha(s, x) = f(s, x) = x$ and $H^n = H = 1$ shows that the same is true for $\beta S^n$. Thus, (i) implies

$$\int_0^t B_n h^{n, \lambda, t, \kappa} (s, S^n_s) d(\beta_n S^n_s) \Rightarrow \int_0^t B_n h^{t, \kappa} (s, S_s) d(\beta S_s),$$

which proves the theorem [see (A.44)].

Proof of Theorem 9: By Theorem 3 and the definitions of $\hat{h}^{n, \lambda_n, t, \kappa}$ [see (24)] and $\tilde{h}^{n, \lambda, t, \kappa}$ [see (25)],

$$- \left\{ \int_0^t \hat{h}^{n, \lambda_n, t, \kappa} (s, S^n_s) dS^n_s - \int_0^t \hat{h}^{n, \lambda_n, t, \kappa} (s, S^n_s) \beta_n S^n_s dB_s \right\}$$

$$+ \left\{ \int_0^t \hat{g}^{n, \lambda_n, t, \kappa} (s, S^n_s) dV^n_{s, t}, \hat{V}^{n, \lambda_n, t, \kappa} \right\}$$

$$= \left\{ \hat{V}^{n, \lambda_n, t, \kappa} - \hat{V}^{n, \lambda, t, \kappa} \right\}$$

$$- \left\{ \int_0^t \hat{h}^{n, \lambda_n, t, \kappa} (s, S^n_s) dS^n_s - \int_0^t \hat{h}^{n, \lambda_n, t, \kappa} (s, S^n_s) \beta_n S^n_s dB_s \right\}$$

Using (A.42), the last expression equals

$$\hat{V}^{n, \lambda_n, t, \kappa} - \hat{V}^{n, \lambda, t, \kappa} - \int_0^t B_n \hat{h}^{n, \lambda_n, t, \kappa} (s, S^n_s) d(\beta_n S^n_s)$$
Thus, the statement of the theorem is equivalent to

\[
\{V^{\lambda_n, \tau, \kappa}_n - V^0_{\lambda_n, \tau, \kappa}\} = \left\{ \int_0^\tau B_t \hat{h}^{n, \lambda_n, \tau, \kappa}(s, S^n_{x-}) d(\beta_s S^n_s) \right\} \Rightarrow 0 \quad (A.45)
\]

As was shown in (ii) of the proof of Theorem 8, for any \(g^n\), weak convergence of \((g^n, \beta S^n)\) to \((g, \beta S)\) implies weak convergence of \((g^n, \beta_s S^n_s, \int_0^\tau g^n_{z-s} d(\beta_s S^n_s))\) to \((g, \beta S_s, \int_0^\tau g_{z-s} d(\beta_s S_s))\) and thus in particular weak convergence of \((\beta_s S^n_s, \int_0^\tau g^n_{z-s} d(\beta_s S^n_s))\) to \((\beta_s S_s, \int_0^\tau g_{z-s} d(\beta_s S_s))\). Thus, using (i) of the proof of Theorem 7, for \(\lambda_n \to 1\),

\[
\left( \beta_s S^n_s, \int_0^\tau B_t \hat{h}^{n, \lambda_n, \tau, \kappa}(s, S^n_{x-}) d(\beta_s S^n_s) \right) \Rightarrow \left( \beta_s S_s, \int_0^\tau B_t h^{\tau, \kappa}(s, S_{x-}) d(\beta_s S_s) \right)
\]

Since \(B = \frac{1}{\beta}\) is a continuous nonrandom function, the last result implies

\[
\left( S^n_s, \int_0^\tau B_t \hat{h}^{n, \lambda_n, \tau, \kappa}(s, S^n_{x-}) d(\beta_s S^n_s) \right) \Rightarrow \left( S_s, \int_0^\tau B_t h^{\tau, \kappa}(s, S_{x-}) d(\beta_s S_s) \right) \quad (A.46)
\]

Using (i) of the proof of Proposition 2 gives that for \((t_n, x_n, y_n) \to (t, x, y)\) such that \((t, x, y) \notin (\tau, \kappa) \times R\), for \(\lambda_n \to 1\), \(v^{n, \lambda_n, \tau, \kappa}(t_n, x_n) - y_n \to v(t, x) - y\). Let \(A = (\tau, \kappa) \times R\). Then \(\left\{ (\tau, S_t, \int_0^\tau B_t h^{\tau, \kappa}(s, S_s) d(\beta_s S_s) \right\} \in A\) has probability zero, since \(\{S_t = \kappa\}\) has probability zero. Therefore, (c) of Lemma 1 together with (A.46) shows that, for \(\lambda_n \to 1\),

\[
v^{n, \lambda_n, \tau, \kappa}(\bullet, S^n_s) - \int_0^\tau B_t \hat{h}^{n, \lambda_n, \tau, \kappa}(s, S^n_{x-}) d(\beta_s S^n_s) \Rightarrow v^{\tau, \kappa}(\bullet, S_s) - \int_0^\tau B_t h^{\tau, \kappa} d(\beta_s S_s) \quad (A.47)
\]

Since in particular \(v^{n, \lambda_n, \tau, \kappa}(0, 1) \to v^{\tau, \kappa}(0, 1)\), (A.47) implies

\[
\left\{ v^{n, \lambda_n, \tau, \kappa}(\bullet, S^n_s) - v^n(0, 1) \right\} - \int_0^\tau B_t \hat{h}^{n, \lambda_n, \tau, \kappa}(s, S^n_{x-}) d(\beta_s S^n_s) \\
\Rightarrow \left\{ v^{\tau, \kappa}(\bullet, S_s) - v^{\lambda_n, \tau, \kappa}(0, 1) \right\} - \int_0^\tau B_t h^{\tau, \kappa} d(\beta_s S_s) \quad (A.48)
\]

Using (A.42) and the definition of \(V^{\tau, \kappa}\), Theorem 4 shows that \(\left\{ v^{\tau, \kappa}(\bullet, S_s) - v^{n, \lambda_n, \tau, \kappa}(0, 1) \right\} = \left\{ \int_0^\tau B_t h^{\tau, \kappa} d(\beta_s S_s) \right\} \) and thus that, for \(\lambda_n \to 1\),

\[
\left\{ v^{n, \lambda_n, \tau, \kappa}(\bullet, S^n_s) - v^n(0, 1) \right\} - \int_0^\tau B_t \hat{h}^{n, \lambda_n, \tau, \kappa}(s, S^n_{x-}) d(\beta_s S^n_s) \Rightarrow 0
\]

This proves (A.45) by the definition of \(v^{n, \lambda_n, \tau, \kappa}\) and thus the theorem. 

\[\Box\]

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Notes

1. This model has also been applied to stock indices, foreign currencies, interest rate options, and commodities.
2. For the rest of this introduction, the term "Black-Scholes model" is meant to include both the continuous time Black-Scholes model and its discrete time binomial approximation.
3. Thus, in the limit, the economy is complete in the stock and the money market account alone.
4. Although Jones (1984) prices options using other options, he does not consider the "fixed-point" problem discussed below.
5. The definitions of a probability space, filtration, and Poisson process can be found in Protter (1990: 1–20).
6. A random process $S^n_t$ converges weakly (under $P^n$, $P$) to $S_t$ if $E_P(f(S^n_t)) \to E_P(f(S_t))$ for all bounded real-valued continuous functions (see Jacod and Shiryaev, 1987: p. 312). This is called convergence in law or distribution.
7. The functional form that is selected for the intensity $\lambda(n) = n\lambda_0$ is without loss of generality.
8. It also shows that for the "basis" option, $V^T_t$, its value will converge to Black-Scholes.
9. The standard binomial model with finite $m$ is not an acceptable substitute because it implies only a single delta, as does its limit (as $m \to \infty$). Its limit is the Black-Scholes model.
10. Note that if $\xi = 0$, this gives the Black-Scholes delta.

References