When Swaps Are Dropped

Robert Jarrow and Stuart Turnbull evaluate the impact of default risk on the valuation of interest rate and foreign currency swaps

The usual ways of pricing interest rate and foreign currency swaps ignore default risk in the individual cashflows and in the determination of the swap rate. Cashflows are priced off Libor term structures for both counterparties and deals are undertaken if the total credit exposure, usually measured in terms of notional principal, falls below some value specified by the credit department. But how much difference would the inclusion of default risk make in the determination of swap rates and swap values? This article addresses this question.

A swap contract with positive value to one counterparty implies a credit exposure to the other counterparty. As a swap's value typically fluctuates between positive and negative values over its life, common sense dictates that it is necessary to consider the default characteristics of both counterparties when pricing a swap. We will explain how to handle this bilateral default risk when pricing swap contracts using our 1995 model.1

The extension of this model needed to price bilateral default in swaps is presented in detail below, but the basic procedure is as follows: first, counterparties are allocated to a credit risk class. The conditional martingale probabilities of a counterparty defaulting on its general bond obligations can be inferred from the credit spread for the assigned credit risk class. These martingale default probabilities form the basis of the remaining calculations.

Default on the swap is related to default on general bond obligations: if a counterparty defaults on its general bond obligations, it will only default on a swap contract if the contract has negative value. If a counterparty has not defaulted on its general bond obligations, it will not default on its swap obligations. Given these possible outcomes, a tree-based, recursive valuation procedure is used to determine the value of the swap. This procedure is common to option pricing and is based on the risk-neutral valuation technique. For alternative approaches, see Duffie & Huang (1994) and Sorensen & Bollier (1994).

We will first consider interest rate swaps. We will show that, for two counterparties belonging to different credit classes with a 100-basis-point yield spread on five-year, zero-coupon bonds, the swap credit spread is less than 26bp for a five-year swap. This result is relatively insensitive to the slope of the term structure.

We will then turn to foreign currency swaps. Unlike interest rate swaps, there is an exchange of principal, implying an increase in counterparty risk. This is reflected in a larger swap credit spread (5–8bp). The magnitude of this spread also depends upon the volatility of the spot exchange rate: doubling it from 8% to 16% increases the swap credit spread by about 8bp. The impact of default on both swap values and hedging is also detailed here. It is shown that taking default into account has a dramatic effect on both values and hedging.

Interest rate swaps

We first consider a plain vanilla, fixed-for-floating interest rate swap. (See chapter 14 of Jarrow & Turnbull, 1996, for a description of interest rate swaps.) Counterparty A receives fixed, semi-annual payments from counterparty B and in return makes floating-rate payments: the fixed- and floating-rate payments are made at the same time and are netted. We will let the swap payments occur at dates $T_1, T_2, \ldots, T_n$, where $N$ denotes the number of payments. Libor deposit rates are treated as default-free.2

The two counterparties each belong to a particular credit risk class. Given a term structure of interest rates for each credit class, including the default-free instruments, the conditional martingale probabilities of default can be inferred. For a given credit class, the spread above the default-free interest rate is measured. In the event of default, it is assumed that the recovery rate is known. Given the spread and the recovery rate, the conditional martingale probabilities of default can be determined. See our 1995 article for a detailed discussion.

To simplify the discussion, we will first assume that the probability of default is independent of the spot rate of interest, although this assumption will subsequently be relaxed. The swap's value is determined via backward induction, a common procedure in option valuation theory.

To start the backward induction, consider the situation at date $T_n$. The promised payment is:

$$V_n(T_n) = q_n[R_n(R_{n-1})]N_n$$

where $R_n$ is the fixed swap rate, $R(T_{n-1})$ is the floating rate set at date $T_{n-1}$, $N_n$ is the notional principal and $q_n = T_n - T_{n-1}$ is the interval between dates $T_{n-1}$ and $T_n$. The term $Y_n(T_n)$ represents the promised netted payment to counterparty A on date $T_n$.

Let $Q_n(T_n)$ (or $Q_n(T_n)$) denote the conditional martingale probability that counterparty A (or B) does not default on its bond obligations, conditional that default has not occurred at date $T_{n-1}$: The probabilities are inferred from the credit class term structure as discussed above. The state of no defaults at date $T_{n-1}$ is described as $D_n$ for counterparty A and $D_b$ for counterparty B.

There are four possible outcomes at date $T_n$ that need to be considered: no-one defaults, only B defaults, only A defaults and both A and B default.

First, if there are no defaults, the payment is:

$$V_n(T_n) = Y_n(T_n)$$

The probability of this state occurring is $Q_n(T_n)\cdot Q_b(T_n)$, assuming independence between the default processes for counterparties A and B.

The second possibility is that B defaults and A does not. If B defaults on its bond obligations at or before date $T_n$, we assume that it will default on its swap obligation – but only if this represents a negative cashflow. This is described by:

$$V_n(T_n) = \begin{cases} \delta_n Y_n(T_n) & V_n(T_n) > 0 \\ \delta_n Y_n(T_n) & V_n(T_n) \leq 0 \end{cases}$$

where $\delta_n$ is the recovery rate if B defaults on its swap obligations, $0 \leq \delta_n < 1$. The probability of this state is $Q_n(T_n)[1 - Q_b(T_n)]$.

Third, A defaults and B does not. If A defaults on its bond obligations at or before date $T_n$, we again assume that it will default on its swap obligations but only if these represent negative cashflows. This is described by:

$$V_n(T_n) = \begin{cases} \delta_n Y_n(T_n) & \delta_n Y_n(T_n) \geq 0 \\ \delta_n Y_n(T_n) & \delta_n Y_n(T_n) < 0 \end{cases}$$

where $\delta_n$ is the recovery rate if A defaults on its bond obligations.

1 See references. In 1993, we also explained the impact of bilateral default risk in the pricing of the simpler case of forward contracts. Swap contracts are more complicated because of the intermediate payments made over the life of the contract.

2 Our 1993 article shows how to model Libor and Treasury instruments in an arbitrage-free economy, where the differential spread is assumed to depend upon credit risk and a convenience yield.
its swap obligations, \( 0 \leq \delta_A < 1 \). The probability of this state is \([1-Q_A(T_w)]Q_B(T_w)\).

Finally, both counterparties could default. If this happens before date \( T_w \), counterparty B will default on its swap obligations if the swap value to counterparty A is positive. If the swap value to A is negative, then A will default on its swap obligations. This is described by:

\[
V_v(T_n) = \begin{cases} \delta_B V_B(T_n) + V_A(T_n) & \delta_B V_B(T_n) + V_A(T_n) \\ \delta_B V_B(T_n) + V_A(T_n) & < 0 \end{cases}
\] (5)

The probability of this state is \([1-Q_A(T_{w1})][1-Q_B(T_w)]\). The four different states are shown in Table 1.

We next need to determine the present value at date \( T_{n-1} \) of the date \( T_n \) payments. These depend on the current status of both counterparties. There are four possible states at time \( T_{n-1} \): no defaults, only B defaults, only A defaults and both A and B default.

Given the two counterparties have not defaulted on their general bond obligations at or before date \( T_{n-1} \), the present value at date \( T_{n-1} \) of the promised payment at date \( T_n \) is:

\[
V_v(T_{n-1}) = B(T_{n-1}, T_n) \times \left\{ \frac{Q_A(T_n)Q_B(T_n)}{V_A(T_n)} + \left[1 - Q_A(T_n)\right]Q_B(T_n) \right\} + \left[1 - Q_A(T_n)\right][1 - Q_B(T_n)V_A(T_n)]] (6)
\]

where \( B(t, T) \) denotes the date t value of definitely receiving \$1 at date T at \( t \leq T \). This is the discounted expected value of the four possible payouts at date \( T_{n-1} \) using the martingale probabilities.

At date \( T_{n-1} \), if B has defaulted on its bond obligations and A has not, then the present value at date \( T_{n-1} \) of the promised payment is:

\[
V_v(T_{n-1}) = B(T_{n-1}, T_n) \times \left\{ \frac{Q_A(T_n)V_A(T_n)}{V_A(T_n)} + \left[1 - Q_A(T_n)\right]Q_B(T_n) \right\} \] (7)

Again, this is the discounted expected value of the two possible payouts at date \( T_n \). There are only two possible payments because B has defaulted on its bond obligations. The two possible alternatives are that A defaults or A does not default (see Table 2).

Similarly, at date \( T_{n-1} \), if counterparty A has defaulted on its bond obligations and B has not, then the present value at date \( T_{n-1} \) of the promised payment is:

\[
V_v(T_{n-1}) = B(T_{n-1}, T_n) \times \left\{ \frac{Q_A(T_n)V_A(T_n)}{V_A(T_n)} + \left[1 - Q_A(T_n)\right]Q_B(T_n) \right\} + \left[1 - Q_A(T_n)\right][1 - Q_B(T_n)V_A(T_n)]] \] (8)

Finally, at date \( T_{n-1} \), if both counterparties A and B have defaulted on their bond obligations, then the present value at date \( T_{n-1} \) of the promised payment is:

\[
V_v(T_{n-1}) = B(T_{n-1}, T_n)V_A(T_n) \] (9)

This completes the date \( T_{n-1} \) valuation.

Having valued the swap payment to be made at date \( T_w \), we now consider the swap payment to be made at date \( T_{n-1} \). The netted promised swap payment at date \( T_{n-1} \) is:

\[
V_v(T_{n-1}) = \left[ Q_B(T_{n-1})R(T_{n-1}) \right]V_A(T_{n-1}) \] (10)

where \( R(T_{n-1}) \) is the floating rate set at date \( T_{n-2} \) and \( Q_{n-1} = T_{n-1} - T_{n-2} \). As before, the procedure depends on which of the four possible states at date \( T_{n-1} \) holds.

Suppose B has defaulted on its bond obligations at date \( T_{n-2} \). We want to consider the value of the swap just before date \( T_{n-1} \). For convenience we will denote the instant before date \( T_{n-1} \) as \( T_{n-1} \). The value of the swap at date \( T_{n-1} \) is:

\[
V_v(T_{n-1}, D_A, D_B) = V_v(T_{n-1}) + V_v(T_{n-1}) (11)
\]

This represents the payment at date \( T_{n-1} \), \( V_v(T_{n-1}, D_A, D_B) \), plus the present value of the promised payment at date \( T_{n-1} \). It is assumed that B will default on its swap obligation if

\[
V_v(T_{n-1}, D_A, D_B) > 0
\]

as the swap is a liability to B. If default on the swap occurs, it is assumed that the payment to counterparty A will be:

\[
\delta_B V_v(T_{n-1}) + \delta_B PV(T_{n-1}) \] (12)

where \( PV() \) denotes the present value of the remaining swap payments, assuming that there is no default on swap payments. From (7), the present value of the remaining payments is:

\[
PV(T_{n-1}) = V_v(T_{n-1}) \] (13)

In practice, default occurs, the present value of the remaining payments is calculated ignoring credit risk. In such a case, the fixed and floating payments are discounted off the Libor curve. The present value is:

\[
PV(T_{n-1}) = Q(B(T_{n-1}) - R(T_{n-1})) (14)
\]

We use this method to illustrate the effects of bilateral default on the swap rate.

At date \( T_{n-1} \), the value of the swap, conditional upon counterparty A not defaulting on its bond obligations, is:

\[
V_v(T_{n-1}, D_A, D_B) = \] (15)

\[
\begin{cases} \delta_B V_v(T_{n-1}) + \delta_B PV(T_{n-1}) \times V_v(T_{n-1}, D_A, D_B) > 0 \delta_B V_v(T_{n-1}) + \delta_B PV(T_{n-1}) \times V_v(T_{n-1}, D_A, D_B) < 0 \end{cases}
\]

If both counterparties have defaulted on their bond obligations, then:

\[
V_v(T_{n-1}, D_A, D_B) = \] (16)

\[
\begin{cases} \delta_B V_v(T_{n-1}) + \delta_B PV(T_{n-1}) \times V_v(T_{n-1}, D_A, D_B) > 0 \delta_B V_v(T_{n-1}) + \delta_B PV(T_{n-1}) \times V_v(T_{n-1}, D_A, D_B) < 0 \end{cases}
\]

Examining the case where only B has defaulted at date \( T_{n-2} \), the present value of the swap is the discounted expected payoff across the two possibilities:

\[
V_v(T_{n-2}) = B(T_{n-2}, T_n) \times \left\{ \frac{Q_A(T_{n-2})V_v(T_{n-2}, D_A, D_B)}{V_v(T_{n-2})} \right\} \] (17)

At date \( T_{n-2} \), if A has defaulted on bond obligations, then:

\[
V_v(T_{n-1}, D_A, D_B) = \] (18)

\[
\begin{cases} \delta_B V_v(T_{n-2}) + \delta_B PV(T_{n-2}) \times V_v(T_{n-2}, D_A, D_B) > 0 \delta_B V_v(T_{n-2}) + \delta_B PV(T_{n-2}) \times V_v(T_{n-2}, D_A, D_B) < 0 \end{cases}
\]

\[
V_v(T_{n-1}) = B(T_{n-1}, T_n) \times \left\{ \frac{Q_A(T_{n-1})V_v(T_{n-1}, D_A, D_B)}{V_v(T_{n-1})} \right\} \] (19)

At date \( T_{n-1} \), if B has defaulted on bond obligations, then:

\[
V_v(T_{n-1}, D_A, D_B) = \] (20)

\[
\begin{cases} \delta_B V_v(T_{n-1}) + \delta_B PV(T_{n-1}) \times V_v(T_{n-1}, D_A, D_B) > 0 \delta_B V_v(T_{n-1}) + \delta_B PV(T_{n-1}) \times V_v(T_{n-1}, D_A, D_B) < 0 \end{cases}
\]

\[
V_v(T_{n-1}) = B(T_{n-1}, T_n) \times \left\{ \frac{Q_A(T_{n-1})V_v(T_{n-1}, D_A, D_B)}{V_v(T_{n-1})} \right\} \] (21)

At date \( T_{n-1} \), if A has defaulted on bond obligations, then:

\[
V_v(T_{n-1}, D_A, D_B) = \] (22)

\[
\begin{cases} \delta_B V_v(T_{n-1}) + \delta_B PV(T_{n-1}) \times V_v(T_{n-1}, D_A, D_B) > 0 \delta_B V_v(T_{n-1}) + \delta_B PV(T_{n-1}) \times V_v(T_{n-1}, D_A, D_B) < 0 \end{cases}
\]
obligations but B has not, a similar analysis gives:

$$V_2(T_{n-2}) = B(T_{n-2}, T_{n-1})$$
$$\times \{ Q_0(T_{n-1}) \mathcal{V}(T_{n-1}, D_A, D_B)$$
$$+ [1 - Q_0(T_{n-1})] \mathcal{V}(T_{n-1}, D_A, D_B) \}$$

If both counterparties have defaulted on their bond obligations at date $T_{n-2}$, then:

$$V_2(T_{n-2}) = B(T_{n-2}, T_{n-1})$$
$$\times \{ Q_0(T_{n-1}) Q_0(T_{n-1}) \mathcal{V}(T_{n-1}, D_A, D_B)$$
$$+ [1 - Q_0(T_{n-1})] Q_0(T_{n-1}) \mathcal{V}(T_{n-1}, D_A, D_B)$$
$$+ [1 - Q_0(T_{n-1})] [1 - Q_0(T_{n-1})] \mathcal{V}(T_{n-1}, D_A, D_B) \}$$

This completes the analysis at date $T_{n-2}$.

By repeating this backward inductive procedure, the value of the swap can be determined at each point in time. For a new swap, the swap rate is set so the value of the swap is zero when initiated. Assuming that neither counterparty has defaulted on their general bond obligations when the swap is initiated, the swap rate, $\tilde{R}$, is set such that

$$V_2(0) = 0$$

This recursive procedure is easily programmed on a computer.

**The impact of default**

To illustrate the effects of bilateral default on interest rate swap values, we consider a five-year swap, with swap payments being made each six months. Fons (1994) shows that for high and medium credit classes – to which almost all counterparties to swaps belong – the credit spread is upward-sloping in maturity. Given this empirical evidence, we assume that the credit spread is linearly increasing in maturity. A similar assumption is made by Duffie & Huang. We emphasise that this assumption can easily be relaxed.

The initial term structures for two credit classes of five-year, zero-coupon bonds are summarised in Table 3. For class one, the spread between the yields on the default-free and credit-risky bonds is 100bp. For class two, three cases are considered: the spread between credit class one and credit class two is (a) 50bp, (b) 100bp or (c) 200bp.

Using data from Moody's Investors Service, Fons reports that the average recovery rate for senior unsecured debt is 48% and for senior subordinated debt 40%. For illustrative purposes, we assume an average recovery rate of 45%. Hence, if either counterparty defaults on its swap obligation, the recovery rate $\delta_0$ or $\delta_\delta$ is set at 45%.

Given the initial term structures for the different credit classes and the recovery rate, the conditional martingale probabilities of default can be inferred, using the procedure we have already described (1995, 1996). These conditional martingale probabilities are then used to price a swap, allowing for bilateral default.

The traditional method of calculating the swap rate ignores default. Let $B(0,T)$ denote the present value of being sure of receiving a dollar at date $T$. Let $T_j$ denote the date of the $j$th swap payment. The present value of the fixed payments using the data from Table A is:

$$PV = \left( \frac{\delta \delta}{2} \right) \sum_{j=1}^{10} B(0, T_j)$$

$$= \left( \frac{\delta \delta}{2} \right) 8.7139$$

where $\delta$ is the traditional swap rate, expressed on a yearly basis. The present value of the floating payments is:
implying that $R = 5.3889\%$ a year.

To illustrate the effect of bilateral default, two cases are considered. First, counterparty A, which receives fixed payments, is assumed to belong to credit class one. Counterparty B, which receives floating payments, is assumed to belong to credit class two, implying that A is of better credit quality than B. The results are shown in part 1 of table 4. In the second case, A is assumed to belong to class two, and B is assumed to belong to class one, implying that counterparty A is of lower credit quality than counterparty B. These results are shown in part 2 of table 4.

There are two important observations that can be discerned from table 4. First, the impact of bilateral default on the swap rate is less than 2bp. A similar result is found in Duffie & Huang. Second, there is an asymmetry in the results. The swap rate is more sensitive to changes in the creditworthiness of the counterparty paying floating than the counterparty paying fixed.

We next investigate whether these results depend on the shape of the term structure. In table 5, the term structure of default-free interest rates is inverted. The impact of bilateral default again affects the swap rate by less than 2bp.\(^1\)

Although bilateral default has an insignificant effect on the swap rate, this does not mean credit risk should be ignored. The differences in credit risk are reflected in different discounted present values. For example, consider an off-market, five-year swap with a swap rate of 6% a year. Let the notional principal be $100\text{ million}$ and suppose that a swap payment has just occurred. Ignoring default risk, the value of this swap is:

\[
(0.0628)\times 100\text{ million} = 2,661,700
\]

using the data from table 3. The swap has positive value to counterparty A, implying that the credit exposure to counterparty B is $2,661,700. If the default risk of counterparty A is ignored\(^4\) and counterparty B is assumed to belong to credit class two, with a spread of 100bp at year five, then the value of the swap is about $2.6 million, implying that present value of the credit exposure is $61,700.

Furthermore, the appropriate hedging technique will be different under the two formulations. Ignoring default, the Libor term structure provides the relevant hedging instruments. With default, however, this is no longer true. To hedge the swap, financial securities are needed from both counterparties. This follows because, when default occurs, alternative securities are needed to offset the price impact on the swap. These alternative securities must "jump" at the same time as the swap's value changes, hence the use of the general bond obligations of counterparties A and B. Details of the different hedging procedures can be found in our 1995 article.

**Foreign currency swaps**

This section illustrates the impact of bilateral default on the value of foreign currency swaps. These swaps usually involve an exchange of principal, which increases the credit risk compared with plain vanilla interest rate swaps.

For illustrative purposes, we consider a five-year, fixed-for-fixed swap with payments every six months. The details are shown in figure A. Let counterparty A agree to make fixed interest payments in dollars to counterparty B and promise to repay a principal of $76 million at the maturity of the swap. B agrees to make fixed interest rate payments in sterling to A and promises at maturity to repay a principal of £50 million. The current spot exchange rate is $1 = $1.52 and the domestic and foreign default-free term structures are described in table 6. The domestic term structure is the same as in table 3.

To value this swap, we start by neglecting...
default. Suppose A agrees to pay B an interest rate of 5.3889%, then:

\[
\text{Dollar value of payments to be made to counterpart B = } \sum_{t=1}^{10} (0.053889 \times 0.5) B(t, T_t) + B(t, T_{10})
\]

\[
= $100 \quad \text{per $100 face value}
\]

where B(0, T) is the value in dollars today of being certain of receiving $1 at date T. In the swap, the principal is $76 million, implying that the dollar value of payments to be made to B is $76 million.

In return, B agrees to pay A interest payments in sterling at the rate of 5.615% a year. Hence:

\[
\text{Sterling value of payments to be made to counterpart A = } \sum_{t=1}^{10} (0.056150 \times 0.5) B(t, T_t) + B(t, T_{10})
\]

\[
= £100 \quad \text{per £1,000 face value}
\]

where B(0, T) is the value in sterling of being certain of receiving one unit of sterling at date T. In the swap, the principal is £50 million, implying that the dollar value of payments to be made to A is $50 \times 1.52 = $76 million. Hence the initial value of the swap is zero.

We now want to value this swap taking into account bilateral default. We simplify the analysis by assuming that the domestic and foreign default-free term structures are deterministic. This implies that the value of the swap depends upon the volatility of the spot exchange rate, the probabilities of the counterparties defaulting and the shapes of the default-free term structures.

Let \( S(t) / S(0) = [s(t) - s(0)] e_t + \sigma dW(t) \) where \( s(0) \) is the instantaneous dollar default-free interest rate at date t, \( s(t) \) is the instantaneous sterling default-free interest rate at date t, \( \sigma \) is the volatility and \( W(0) \) is a Brownian motion. A binomial process is used to approximate the spot exchange rate process, as described in chapter 11 of our 1996 book.

To illustrate the effect of bilateral default, we consider two cases. First, A belongs to credit class one and B belongs to credit class two; and second, A belongs to class two and B to class one. The interest rate that A pays B (5.3889%) is kept fixed, while the interest rate that B pays A is varied until the value of the swap is zero.

The results are shown in table 7. In the first case (part 1), B is of lower credit class than A. Given a 100bp spread in the yields of five-year, zero-coupon bonds between credit classes one and two, this results in a default adjustment of 5bp when the spot exchange rate volatility is 8%. If A is of lower credit than B (part 2), it is seen that a 100bp spread between the two credit classes gives rise to a default adjustment of 8bp for a spot exchange rate volatility of 8%.

The difference in the two sets of results arises because of the declining forward exchange rates. In the first case, for counterpart A, the declining term structure of forward exchange rates helps to reduce the credit exposure to counterpart B. In the second case, the creditworthiness of counterpart B is fixed. These differences in swap rates are significant.

We can now consider the effect of increasing the spot rate volatility to 16% in each case. In the first case, this implies an increase in A’s credit exposure to B; to compensate, B must increase the level of interest rate payments to A. The swap rate increases to 5.7223% for a 100bp spread. In the second case, counterpart A is of lower credit than counterpart B, so doubling the volatility implies an increase in B’s credit exposure to A. To compensate, A must increase the level of interest rate pay-

\[\text{ments to counterpart B. For illustrative purposes, we have kept the interest rate which} \]

\[\text{A promises to pay B fixed. Consequently, for the swap to have zero value, the interest rate} \]

\[\text{B promises to pay A decreases to 5.4758% for the} 100 \text{bp spread.} \]

As with interest rate swaps, hedging the default risk of a foreign currency swap requires taking offsetting positions in the general obligations of the counterparties. This is in contrast to the traditional approach ignoring credit risk, which hedges foreign currency swaps using only the foreign and domestic default-free term structures.

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References:

