An Integrated Approach to the Hedging and Pricing of Eurodollar Derivatives

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ABSTRACT
Taking the term structure of Treasury securities and Eurodollar rates as exogenous, this article provides an integrated approach to the pricing and hedging of London Inter-bank Offer Rate (LIBOR) derivatives. Our approach allows the spread between Eurodollar and Treasury rates to reflect both the credit risk in holding Eurodollar deposits and a convenience yield from holding Treasury securities. This integrated approach includes the models of Babbs (1991), Grinblatt (1994), and Jarrow and Turnbull (1995) as special cases.

INTRODUCTION
Eurodollar futures contracts are some of the most actively traded contracts in the world. In the over-the-counter market, there is a large volume of trading in Eurodollar derivatives such as forward rate agreements, caps, and floors. All of these instruments can be used in hedging swap portfolios. At the short end of the maturity spectrum (approximately the first three years), because of liquidity considerations, Eurodollar futures contracts provide the easiest hedging vehicle for the derivatives product book. But, at the long end of the maturity spectrum, it is necessary to use both Treasury bonds and Treasury bond futures. This hedging practice implies that it is necessary to have a framework in which both LIBOR and dollar derivative contracts can be priced and hedged consistent with the absence of arbitrage.

The consideration of LIBOR and over-the-counter (OTC) derivative contracts also raises the question of credit risk, because for LIBOR there is the risk of default, and for OTC derivatives there is the additional risk that the writer of the derivative might default. Currently, there are three modeling approaches to pricing credit risky derivatives.

The first approach values these options as contingent claims on the assets of the firm (see Merton, 1974; Black and Cox, 1976; Ho and Singer, 1982; Chance,

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1990; and Cooper and Mello, 1990, 1991). This approach can also be used for valuing “vulnerable options” (see Johnson and Stulz, 1987), which are options subject to the additional risk that the option writer might default. Two major drawbacks hinder this approach. The first is that in practical applications it is necessary to simultaneously value all of the liabilities of the firm. This necessitates collecting a large amount of data and analyzing numerous contractual provisions which are often not readily accessible. The second drawback is the computational burden that results because of the complexity of all the contingent liabilities. Longstaff and Schwartz (1992) use this approach to value risky debt and argue that the computational burden can be reduced to estimating only three parameters. Unfortunately, for calibration purposes, these model prices will have too few parameters to be consistent with the observed credit risky bond prices.

The second approach to pricing credit risky derivatives is to ignore credit risk and to value risky debt options using the default-free interest rate pricing models (see Ho and Singer, 1984, and Ramaswamy and Sundaresan, 1986). Unfortunately, this approach cannot accommodate in an arbitrage-free way credit risky and default-free bonds. For Eurodollar contracts, a related approach is to consider both the Treasury and Eurodollar term structures as default free, but to add an exogenous convenience yield for holding Treasuries (see Babbs, 1991, and Grinblatt, 1994). No theoretical justification is provided for including this convenience yield, although it does create a difference between Eurodollar and Treasury rates.

The third approach to pricing credit risky derivatives uses the foreign currency analogy of Jarrow and Turnbull (1997) to explicitly model default risk. This approach takes the term structure of corporate interest rates as a given. It thus avoids the complications of the first approach and the limitations of the second approach. The details are described in Jarrow and Turnbull (1995). This article is an application and generalization of this later approach. It generalizes this model by constraining short sales of particular Treasury securities, thereby generating a convenience yield consistent with arbitrage-free pricing. This short sale restriction is often reflected via specials in the Treasury repurchase (repo) market due to shortages in the secondary markets for Treasury securities (see Cornell and Shapiro, 1989, and Chatterjea and Jarrow, 1994).1 A special case of our formulation generates the models used by Babbs (1991) and Grinblatt (1994). This integrated approach to the pricing and hedging of Eurodollar derivatives decomposes the Treasury-Eurodollar (TED) spread between Eurodollar and Treasury rates into a credit risk premium and a convenience yield. Either component could be identically zero.

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1 One scenario is as follows. Treasury securities are easily shorted in the when-issued market. Occasionally, when these shorts attempt to close out their positions, there are shortages in the secondary market. These shortages raise the prices of the Treasury securities above the prices of substitute securities. If continued short sales were unrestricted, then these higher prices would represent arbitrage opportunities. We reflect these secondary market shortages in our model via short sale restrictions.

These secondary market shortages are also reflected in repo markets, because the short traders can utilize the repo markets to temporarily cover their short positions.
This article first studies the model without short sales constraints. Using a continuous time formulation, the model is developed. Then, a description of the different types of LIBOR derivatives is given. This description is careful about the forms of the various rates (simple versus continuous) in the relevant definitions. First, we describe futures contracts and the determination of the endogenous futures price. The relationship of futures and forward prices is briefly discussed. Forward rate agreements (FRAs) are defined and the FRA rate is shown to be identical to the forward rate. Interest rate caps are described and closed-form solutions are given. These results also provide a general framework for hedging LIBOR derivatives. The last part of this section addresses the pricing of vulnerable caps, forward contracts, and swaps. After this, we augment the model to include short sale constraints on various Treasury securities and provide a model for Treasury security convenience yields.

THE MODEL

The formulation of our model structure closely follows Jarrow and Turnbull (1995). First, we only consider markets where there are no short sale constraints on either Eurodollar deposits or Treasury securities.

The Assumptions

The structure imposed in this section is general enough to illustrate the power of techniques but restrictive enough to get closed-form solutions. Closed-form solutions are desired because they facilitate understanding. The techniques presented, however, are easily generalized. We consider a continuous trading model over the time interval [0, \( \Gamma \)]. The randomness in the economy is generated by the probability space \((\Omega, \mathcal{G}, Q)\), where \(\Omega\) is the state space, \(\mathcal{G}\) is a collection of events, and \(Q\) is a probability measure. Let \(\{G_t; t \in [0, \Gamma]\}\) be the information set generated by a one-dimensional Brownian motion \(\{W(t); t \in [0, \Gamma]\}\) and an independent univariate point process \(\{N_t(t); t \in [0, \Gamma]\}\).\(^2\) The Brownian motion and the point process represent the randomness in the economy.

The point process \(N_t(t)\) models the bankruptcy process and is constructed as follows. Let \(\tau^*_1\) be the first time of bankruptcy for a Eurodollar deposit at bank ABC, which is exponentially distributed over \([0, \infty)\) with parameter \(\lambda_1\).\(^3\) Then, we define the point process \(N_t(t)\) as

\(^2\) This article extends trivially to a \(b\)-dimensional Brownian motion. For expository clarity, we concentrate only on the one-dimensional case.

\(^3\) For simplicity, our notation suppresses the functional dependence on \(\omega \in \Omega\) as \(\tau^*_1: \Omega \to \mathbb{R}\) and \(\mathcal{B}_0: \Omega \times [0, \infty) \to \mathbb{R}\), where \(\Lambda = \{(t, T): 0 \leq t \leq T \leq \tau\}\).
\[ N_i(t) \equiv 1(t \geq \tau_i^*) = \begin{cases} 1 & \text{if } t \geq \tau_i^* \\ 0 & \text{otherwise} \end{cases} . \]

If bankruptcy occurs, \( N_i(t) \) equals one; otherwise, it is zero.

Let \( B_0(t, T) \) denote the time \( t \) price of a Treasury security paying a sure dollar at time \( T \). We assume that this Treasury security can be short sold and that \( B_0(t, T) > 0, B_0(t, t) = 1 \), and \( \partial \log B_0(t, T) / \partial T \) exists for all \( 0 < t < T \leq \Gamma \).

The default-free forward rate \( f_0(t, T) \) is defined as \( f_0(t, T) = -\partial \ln B_0(t, T) / \partial T \), and the default-free spot interest rate is defined by \( r_0(t) = f_0(t, t) \).

Finally, a default-free money market account is defined by

\[ A(t) = \exp \left( \int_0^t r_0(s) ds \right) \text{ for all } t \in [0, \Gamma] . \]

The money market account has an initial investment of a dollar \( (A(0) = 1) \), and accumulates interest at the spot interest rate.

Let \( v_1(t, T) \) denote the time \( t \) price of a zero-coupon bond issued by bank ABC promising to pay a dollar at time \( T \). This can be thought of as a \( T \) maturity Eurodollar deposit at bank ABC. This promised dollar may not be repaid, so a default premium is included. With an obvious change in notation, we assume that \( \partial \ln v_1(t, T) / \partial T \) exists for all \( 0 < t < T \leq \Gamma \) and define \( f_1(t, T) = -\partial \ln v_1(t, T) / \partial T \) and \( r_1(t) = f_1(t, t) \), where \( f_1(t, T) \) represents the continuously compounded time \( t \) Eurodollar forward rate for date \( T \).

We let \( e_1(t) \equiv v_1(t, t) \) denote the payout rate on the ABC Eurodollar deposit at time \( t \). If not bankrupt at time \( t \), \( e_1(t) = 1 \); otherwise, \( e_1(t) < 1 \) will occur.

It is convenient to define a zero-coupon bond paying off in a hypothetical currency, called ABCs, that is,

\[ B_1(t, T) \equiv v_1(t, T) / e_1(t) , \]

where it is assumed that \( e_1(t) > 0 \).\(^4\) In ABCs, this zero-coupon bond is default free. (Indeed, \( B_1(T, T) = v_1(T, T) / e_1(T) = 1 \).) This transformation of the ABC bond price gives the hypothetical bond \( B_1(t, T) \) the interpretation of being a foreign currency denominated zero-coupon bond, which is default free in its own currency.

We now impose the exogenous stochastic structure directly on the forward rates \( f_0(t, T), f_1(t, T) \) and the payoff rate \( e_1(t) \). The default-free forward rates are assumed to satisfy

\[^4\]The case of \( e_1(t) = 0 \) can be easily handled, but it adds additional mathematical complexity without extra economic insights.
Assumption (A1)—Default-Free Forward Rates

\[ df_0(t, T) = \alpha_0(t, T)dt + \sigma(t, T)dW(t), \]  

(1)

where \( \alpha_0 \) is predictable, jointly measurable, and uniformly bounded, and \( \sigma \) is deterministic, jointly measurable, and uniformly bounded.\(^5\)

Making the volatility deterministic in Assumption (A1) implies that this is a Gaussian economy for default-free rates. This model is used regularly "on the street." If \( \sigma(t, T) \equiv \sigma > 0 \) is a positive constant, one gets the continuous time analogue of Ho and Lee (1986). If \( \sigma(t, T) = \sigma e^{-\xi(T-t)} \) for \( \sigma, \xi \) constants, one gets the extended Vasicek model in Hull and White (1990) and the examples in Heath, Jarrow, and Morton (1992). Assumption (A1) is easily generalized to multiple independent Brownian motions and stochastic volatilities.

The Eurodollar continuously compounded forward rates are assumed to satisfy

Assumption (A2)—Eurodollar Forward Rates

\[ df_1(t, T) = \alpha_1(t, T)dt + \sigma(t, T)dW(t) + \theta_1(t, T)1(t \leq \tau_1^*)[dN_1(t) - \lambda_1 dt], \]  

(2)

where \( \alpha_1 \) is predictable, jointly measurable, and uniformly bounded, and \( \theta_1 \) is predictable, jointly measurable, and uniformly bounded.

The process for the Eurodollar forward rates mimics the stochastic process for the default-free forward rates, with the exception of a jump occurring at the bankruptcy time \( \tau_1^* \) equal to \( \theta_1(\tau_1^*, T) \). Without loss of generality, the coefficient \( \sigma(t, T) \) preceding the Brownian motion component equals that in the default-free forward rate process (see Jarrow and Turnbull, 1995).

Finally, the payoff rate on Eurodollar deposits is described by

Assumption (A3)—Payoff Ratio

\[ e_1(t) = e^{-\delta_1 N_1(t)}, \]  

(3)

where \( \delta_1 > 0 \) is a positive constant.

The dollar payoff is unity until bankruptcy, after which deposits receive the fixed amount \( \exp(-\delta_1) < 1 \) when they mature.

Using the definition of the forward rate with expressions (1) and (2), one can derive the following stochastic processes for \( B_0(t, T) \) and \( v_1(t, T) \) (for the derivation, see Jarrow and Madan, 1995). These stochastic process representations are

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\(^5\) This is jointly measurable on \( G \times B(\Delta) \) where \( B(\Delta) \) is the Borel \( \sigma \)-algebra over \( \Delta \) (see Jacod and Shiryaev, 1987, for the definition of predictable). The random variable \( x: \Omega \rightarrow \mathbb{R} \) is uniformly bounded if \( |x(\omega)| \leq K \) for a.e. \( \omega \in \Omega \), where \( K > 0 \) is a constant.
useful for parameter estimation purposes, as well as for deriving pricing formula for LIBOR derivatives.

\[ dB_0(t, T) = B_0(t, T)[r_0(t) + \beta_0(t, T) + \frac{1}{2} a(t, T)^2]dt + B_0(t, T)a(t, T) dW(t) \quad (4) \]

where \( \beta_0(t, T) \equiv -\int_t^T \alpha_0(t, u)du \), \( a(t, T) \equiv -\int_t^T \sigma(t, u)du \), and

\[ dv_1(t, T) = v_1(t-, T)[r_1(t) + \beta_1(t, T) + \frac{1}{2} a(t, T)^2]dt + v_1(t-, T)a(t, T)dW(t) \]

\[ -v_1(t-, T)\Theta_1(t, T)I(t \leq \tau^*_1)\lambda_1 dt + v_1(t-, T)(e^{\Theta_1(t, T)}I(t \leq \tau^*_1) - 1)dN_1(t), \quad (5) \]

where \( \beta_1(t, T) \equiv -\int_t^T \alpha_1(t, u)du \), and \( \Theta_1(t, T) \equiv -\int_t^T \theta_1(t, u)du \).

To price LIBOR derivatives, we first need to find conditions on these price processes which guarantee that the economy is arbitrage-free. This is equivalent to finding conditions which guarantee the existence of a unique equivalent probability measure \( \tilde{Q} \) making \( v_1(t, T) / A(t) \) and \( B_0(t, T) / A(t) \) martingales for all \( T \). These conditions (given in Appendix A) imply the existence of market prices of risk \( \gamma(s), \mu_1(s) \) such that

\[ \tilde{W}(t) = W(t) - \int_0^t \gamma(s)ds \quad \text{and} \]

\[ \tilde{N}_1(t) = N_1(t) - \int_0^t \lambda_1 \mu_1(s)ds \]

are martingales under \( \tilde{Q} \).

There are two market prices of risk because there are two (independent) random shocks in the economy, the first due to the Brownian motion risk \( W(t) \), and the second due to the jump risk \( N_1(t) \).

Next, to obtain closed-form solutions, we add the following assumption

**Assumption (A4)—Constant Bankruptcy Risk Premium**

\[ \mu_1(t) \equiv \mu_1 > 0 \quad \text{a positive constant.} \]

This assumption implies that the market price for jump risk is constant across time. This assumption is made in order to facilitate the derivation of closed-form solutions. To calibrate the model to the initial term structure of credit risky bonds, however, it is necessary to relax this assumption and allow \( \mu_1(t) \) to be a function of \( t \). This calibration is not analyzed within this article.
The Treasury-Eurodollar Spread

Given the above assumptions, we can now characterize the Treasury-Eurodollar spread between Eurodollar and Treasury rates. Under Assumptions (A1) through (A4), we first determine the present value of the Eurodollar deposit at bank ABC to be

\[
v_1(t, T) = E_t(e_1(T) / A(T))A(t)
= E_t(e_1(T))E_t(1 / A(T))A(t)
= E_t(e_1(T))B_0(t, T).
\]  

(6a)

The present value of the Eurodollar deposit is seen to be the discounted value of the expected payoff under the martingale probability \( \tilde{Q} \).

Expression (6a) can now be used to characterize the TED spread between Treasuries and Eurodollar rates. Using

\[
\tilde{E}_t(e_1(T)) = 1 \cdot (t \geq \tau^*_1) \, e^{-\delta_1} + (t < \tau^*_1) [e^{-\lambda_1 \mu_1 [T-t]} + e^{-\delta} [1 - e^{-\lambda_1 \mu_1 [T-t]}]],
\]

we can explicitly compute this spread as a ratio. For times prior to bankruptcy, we obtain

\[
v_1(t, T)/B_0(t, T) = e^{-\lambda_1 \mu_1 [T-t]} + e^{-\delta_1} [1 - e^{-\lambda_1 \mu_1 [T-t]}].
\]  

(6b)

Expression (6b) shows that the TED spread between Eurodollar and Treasury rates is completely determined by the parameters of the bankruptcy process. As the payoff ratio \( \delta_1 \) increases, the TED spread declines. As the probability of bankruptcy \( \lambda_1 \) increases, the TED spread increases. Last, as the risk premium \( \mu_1 \) increases, the TED spread increases. All of these comparative statics are as expected.

Pricing Derivatives

The existence of a unique equivalent martingale measure \( \tilde{Q} \) also implies that the market is complete (see Jarrow and Turnbull, 1995). An implication of this fact is the following. Define a contingent claim \( X \) as a random cash flow at time \( T < \Gamma \), which depends on the information set \( G_T \), and satisfies

\[
\tilde{E}([X / A(T)]^2) < +\infty.
\]

Then, the time t "arbitrage-free" price of this contingent claim is its discounted expected value under the martingale probability; that is,

\[
\tilde{E}_t(X / A(T))A(t).
\]  

(7)
This expression provides the method for pricing options on financial securities subject to credit risk.

We illustrate this computation with an example useful for pricing LIBOR derivatives. Consider a European-type call option with exercise price $K$ and maturity $m$ on an ABC zero-coupon bond with maturity $M \geq m$. Let $c_1(t; m, K)$ denote the call’s time $t$ value. Using the risk-neutral valuation result in expression (7), Jarrow and Turnbull (1995, equation (54), p. 77) show that, for $t < t_1^*$,

$$c_1(t; m, K) = e^{-\delta t} (1 - e^{-\lambda_1 \mu_1 (m-t)}) c_0(t; m, K') + (e^{-\lambda_1 \mu_1 (M-m)} + e^{-\delta t} (1 - e^{-\lambda_1 \mu_1 (M-m)})) e^{-\lambda_1 \mu_1 (m-t)} c_0(t; m, K''),$$

where

$$c_0(t; m, L) \equiv \tilde{E}_t(\max[B_0(m, M) - L, 0]/A(m))A(t) = B_0(t, M)\Phi(h(L)) - LB_0(t, m)\Phi(h(L) - w),$$

$\Phi(\cdot)$ \equiv standard normal distribution function,

$$h(L) \equiv \ln(B_0(t, M)/B_0(t, m)L) + \frac{1}{2} w^2]/w,$$

$$w^2 \equiv \int_t^m [a(u, M) - a(u, m)]^2 ds,$$

$K' \equiv K e^{\delta t}$, and

$$K'' \equiv K/[e^{-\lambda_1 \mu_1 (M-m)} + e^{-\delta t} (1 - e^{-\lambda_1 \mu_1 (M-m)})].$$

Expression (8) gives a closed-form solution for the value of a European call option on the Eurodollar deposit in bank ABC. It is seen to be a linear combination of the values of two distinct European options on otherwise identical default-free deposits. The first term is equal to the risk-neutral probability that default occurs prior to time $m$ $(1 - e^{-\lambda_1 \mu_1 (m-t)})$ times the value of ABC’s option in that case $(e^{-\delta t} c_0(t; m, K'))$. This option is on a default-free deposit. The second term is equal to the probability that default occurs after time $m$ $(e^{-\lambda_1 \mu_1 (m-t)})$ times the value of the option in that case $(e^{-\lambda_1 \mu_1 (M-m)} + e^{-\delta t} (1 - e^{-\lambda_1 \mu_1 (M-m)})) c_0(t; m, K'')$. This option is also on a default-free deposit. Note that $c_0(t; m, K') < c_0(t; m, K'')$. 


K") as K' > K". The option on the default-free deposit is valued under Assumption (A1), and the formula is obtained from Heath, Jarrow, and Morton (1992).

Expression (8) is easily computed and easily extended to multiple factors for the Brownian motion risk.\(^6\)

For \( \mu_1 \lambda_1 \equiv 0 \), no default risk, expression (8) reduces to \( c_1(t; m, K) = c_0(t; m, K) \), which is the standard no default interest rate option pricing formula in a Gaussian economy. When \( \sigma(t, T) \equiv \sigma > 0 \), we get Ho and Lee’s (1986) model. Further, when \( \sigma(t, T) \equiv \sigma e^{-\xi(T-t)} \), we get the extended Vasicek model as in Hull and White (1990).

**Pricing Vulnerable Options**

We can extend the previous analysis to price vulnerable options, that is, options where the option writer can also default. To price these vulnerable options, the previous economy needs to be extended.

Let the writer of the option be another firm, whose risky zero-coupon bonds \((v_2(t, T))\) are also traded. Using the FX analogy of Jarrow and Turnbull (1997), we can decompose these zero-coupon bonds into

\[
v_2(t, T) = e_2(t)B_2(t, T),
\]

where \( B_2(t, t) = 1 \) for all \( t \). It is assumed that the forward rates \( f_2(t, T) \equiv -\partial \log v_2(t, T) / \partial T \) and the payoff ratio \( e_2(t) \) satisfy Assumptions (A2) and (A3) with the index "2" replacing the index "1," where \( N_2(t) \) is independent of both \( W(t) \) and \( N_1(t) \). Also defined is a new jump process under the martingale probability \( \tilde{Q} \):

\[
\tilde{N}_2(t) \equiv N_2(t) - \int_0^t \lambda_2 \mu_2(s)ds,
\]

where \( \mu_2 \) is predictable, uniformly bounded, and satisfies an extended Assumption (A4).

Next, consider an option writer who writes a European call option with exercise price \( K \) and maturity \( m \) on the Eurodollar deposit at bank ABC with maturity \( M \geq m \). This is the option valued earlier. The option’s time \( t \) price will be denoted \( c_2(t; m, K) \), with the subscript 2 indicating the fact that an option writer is involved.

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\(^6\) If there are \( b \) independent Brownian motions, \( W_i(t) \) for \( i = 1, \ldots, b \) with volatilities \( a_i(t, T) \) for \( i = 1, \ldots, b \), then \( w^2 \) in equation (8) becomes

\[
\sum_{i=1}^b \int_t^M \left[ a_i(u, M) - a_i(u, m) \right]^2 ds
\]
The option writer promises to pay \( c_1(m; m, K) \) dollars at time \( m \). However, the option writer may default. Thus, this option contract has a time \( m \) value equal to

\[
c_2(m; m, K) = v_2(m, m) \quad c_1(m; m, K) = e_2(m) \quad c_1(m; m, K).
\]

(11)

This is the option’s time \( m \) value multiplied by the writer’s payoff ratio. Using the risk-neutral valuation procedure, we obtain

\[
c_2(t;m,K) = \tilde{E}_t(e_2(m)c_1(m;m,K) / A(m))A(t).^7
\]

(12)

Using the independence of \( \tilde{W}(t) \), \( \tilde{N}_1(t) \), and \( \tilde{N}_2(t) \), expression (12) simplifies to

\[
c_2(t; m, K) = \tilde{E}_t(e_2(m))\tilde{E}_t(c_1(m; m, K)A(m))A(t)
\]

(13)

\[
= \tilde{E}_t(e_2(m))c_1(t; m, K),
\]

where

\[
\tilde{E}_t(e_2(m)) = \begin{cases} \small{e^{-\delta_2}} & \text{if } t \geq t_2^* \\ \small{e^{-\lambda_2\mu_2(m-t)} + e^{-\delta_2} (1 - e^{-\lambda_2\mu_2(m-t)})} & \text{if } t < t_2^*. \end{cases}
\]

(14)

Expressions (13) and (14) provide (along with expression [8]) a simple closed-form solution for this option’s value.

We can alternatively use expression (6) for \( v_2(t,T) \) to write

\[
\tilde{E}_t(e_2(m)) = v_2(t,m) / B_0(t,m),
\]

giving the equivalent expression

\[
c_2(t;m,K) = (v_2(t,m) / B_0(t,m))c_1(t;m,K).
\]

The value of a vulnerable option is the value of a nonvulnerable option discounted by the adjustment for the credit risk spread of the option’s writer. The larger the credit risk spread, the larger the discount.

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^7 Here, we let the subscript \( t \) denote the augmented filtration generated by \( N_1, N_2, W \).
LIBOR DERIVATIVES

This section studies the pricing of LIBOR derivatives using the previous model structure. The description of the different LIBOR derivatives is divided into two parts. The first part assumes that there is no risk of the writer of the derivative defaulting. This assumption is relaxed in the second part.

The discussion focuses on the pricing of European-type contracts, and a number of closed-form solutions are derived using the risk-neutral pricing formula given in expression (7). For the pricing of American contracts, a discrete time algorithm as described in Jarrow and Turnbull (1995) can be used. The only difficulty in the subsequent analysis is keeping straight the appropriate rates (discount versus simple versus continuous) used in the LIBOR derivative contracts. Using the wrong rate will give both an incorrect price and an incorrect hedge.

Futures Prices and Rates

We first consider LIBOR futures prices and rates as defined by market conventions. The LIBOR futures price is the delivery price set in a Eurodollar futures contract. Eurodollar futures contracts are written on q-period LIBOR spot rates. Settlement at delivery is in cash. Eurodollar futures prices are quoted based on a floating q-period rate, called the futures rate. The futures price is quoted as 100 less the floating futures rate (as a percent). Settlement at the end of each day is in terms of changes in the quoted futures price.

Let \( F(t; m, q) \) denote the quoted LIBOR futures price at time \( t \) for a contract which matures at time \( m \). The contract is written on the q-period futures rate. This futures rate \( L_{t'}(t;m,q) \) is defined as an index value:

\[
F(t; m, q) = 100 - 100L_{t'}(t; m, q), \tag{15}
\]

where \( q = M - m \). Let \( L(m; m, q) \) be the q-period spot LIBOR rate at time \( m \). It is defined by

\[
\nu(m; M) = \frac{1}{1 + L(m; m, q)q}
\]

(see Sundaresan, 1991, p. 412). Note that \( L(m; m, q) \) is a simple interest rate.

At the maturity of the futures contract, the futures price is

\[
F(m; m; q) = [1 - L(m; m, q)]100.
\]

By construction, therefore, at the maturity of the futures contract, the futures rate and spot rate are equal; that is, \( L_{t'}(m; m, q) = L(m; m, q) \). This fact enables us to derive the futures rate in terms of the model’s parameters. Indeed, given that futures contracts are marked to market, it follows from standard arguments that

\[
L_{t'}(t;m,q) = \tilde{E}_{t} [L(m;m,q)].
\]
The LIBOR futures rate is seen to be equal to the expected spot LIBOR rate, using the martingale probabilities. It is shown in Appendix B that

$$L_T(t; m, q) = \left( [B_0(0; m)/B_0(0; M)] \exp[Q(t)] \tilde{E}_1 \{ l/\tilde{E}_m[e(M)] \} \right) - 1 \left( \frac{1}{q} \right), \quad (16)$$

where

$$Q(t) = \int_{m}^{M} \int_{m}^{S} \sigma(u, s) \sigma(u, v) du dv + \frac{1}{2} \int_{m}^{M} \int_{m}^{M} \sigma(u, s) du ds$$

$$+ \int_{0}^{t} d\tilde{W}(s) \int_{m}^{M} \sigma(s, u) du, \quad (17a)$$

and

$$\tilde{E}_1 \{ l/\tilde{E}_m[e(M)] \} = \begin{cases} \exp(\delta_1); & \tau_1 < t \\ \frac{1 - \exp[-\lambda_1\mu_1(m-t)]}{\exp(-\delta_1)} & : \tau_1 < t \\ \frac{\exp[-\lambda_1\mu_1(m-t)]}{\exp(-\lambda_1\mu_1 q) + \exp(-\delta_1) [1 - \exp(-\lambda_1\mu_1 q)]} & : \tau_1 > t. \end{cases} \quad (17b)$$

This expression is easily programmed on a computer. It depends only on the default-free rates and the parameters of the bankruptcy process.

Forward Rates and Prices

This section analyzes LIBOR forward rates and prices. A Eurodollar forward contract is written on the value of a q-period Eurodollar deposit. The Eurodollar forward price for the q-period Eurodollar deposit is set at the time the contract is written. At initiation, no cash is exchanged. There is no marking to market over the life of the contract. At the delivery date, a q-period Eurodollar deposit is exchanged for the initial forward price of the contract.

Let $f(0; m, q)$ denote the forward price for a contract initiated at time zero.\(^8\) The forward contract matures at date m and is written on a Eurodollar deposit

---

\(^8\) An alternative definition of the forward price is given in Sundaresan (1991), who defines the forward rate in a way similar to that used for futures contracts. Although such a definition is useful in examining the effects of marking to market, it does not have any immediate application or use in the issues discussed in this article.
which matures at time \( M = m + q \). The value of the forward contract at maturity is by definition equal to

\[
V_f(m) = v_1(m; M) - f(0; m, q).
\]

This assumes that there is no risk that the writer of its forward contract will default. It is easy to compute the time \( t \) present value of this payoff. First, the present value of \( f(0; m, q) \) is \( f(0; m, q)B_0(t, m) \). The only risk of default is associated with the underlying LIBOR deposit. Therefore, the present value of \( v_1(m; M) \) is \( v_1(t; M) \). Combined, the time \( t \) value of the LIBOR forward contract is

\[
V_f(t) = v_1(t; M) - f(0; m, q)B_0(t; m).
\]

By market convention, the forward price is set such that the initial value of the contract is zero; that is, \( V_f(0) = 0 \), which implies

\[
v_1(0; M) = f(0; m, q)B_0(0; m).
\]

(18)

This is an example of the standard no arbitrage relationship between forward and spot prices (see Duffie, 1989, p. 130).\(^\text{9}\) It is shown in the next section that using this definition for the forward price implies that there is an equality between forward rates and the rates from forward rate agreements. This is a useful result for constructing the term structure of LIBOR forward rates.

The LIBOR forward interest rate \( L_f(0; m, q) \) is defined by the expression

\[
1 + L_f(0; m, q)(q) = 1 / f(0; m, q)
\]

\[
= [B_0(0; m)/B_0(0; M)] \{1/E_0[e(M)]\}.
\]

(19)

This is a simple interest rate.

Given expressions (16) and (19), it is easy to relate forward rates to futures rates. The forward rate will be greater than the futures rate if and only if

\[
1/E_\tilde{t}[e(M)] > \exp[Q(t)/E_\tilde{t} \{1/E_m[e(M)]\}].
\]

This yields a testable restriction.

In standard industry practice, Eurodollar forward prices/rates are calculated via

\[
f_p(0; m, q) = v_1(0; M) / v_1(0; m),
\]

(20)

---

\(^9\) For default-free Treasury bills, this gives the familiar result that the forward price is \( B_0(0; M) / B_0(0; m) \).
where the subscript \( p \) indicates practice. This implies that default risk is ignored. A comparison of this price with expression (18) reveals why. This calculation implies that the forward (simple) interest rate is

\[
1 + L_p(0; m, q)q = v_1(0; m) / v_1(0; M).
\]

Sundaresan (1991) calls \( f_p(0; m, q) \), the implied forward price. The value of the forward contract will not in general be zero when initiated if one uses this implied forward price. Comparing expressions (20) and (18) implies that, as \( B_q(0; m) \geq v_1(0; m) \), then

\[
f_p(0; m, q) = f(0; m, q).
\]

For the pricing and hedging of LIBOR derivatives, the forward price \( f(0; m, q) \) in expression (18) is the relevant price (and not expression [20]).

**Forward Rate Agreement**

This section studies forward rate agreements (FRAs). A forward rate agreement is a contract written on LIBOR which requires a cash payment at maturity based on the difference between a realized LIBOR spot rate of interest and the prespecified forward (rate agreement interest) rate.

Consider an FRA which matures at date \( m \). The contract is written on the \( q \)-period LIBOR spot rate. By market convention, the value of an FRA contract at maturity is

\[
V(m) = \text{Principal} \left( \frac{[L(m; m, q) - L_R(0; m, q)]q}{1 + L(m; m, q)q} \right),
\]

where \( L(m; m, q) \) is the \( q \)-period LIBOR spot rate, and \( L_R(0; m, q) \) is the FRA interest rate set at \( t = 0 \). Both rates are simple interest rates. For convenience, we will set the principal to be unity. For the present, assume that there is no risk of the writer of the FRA contract defaulting.

For analysis, expression (21) can be written in the form

\[
V(m) = 1 - [1 + L_R(0; m, q)q]v_1(m; M).
\]

The risk-neutral pricing formula from expression (7) gives

\[
V(t) = \mathbb{E}_t[V(m) / A(m)]A(t)
\]

\[
= \mathbb{E}_t[A(m)^{-1} - [1 + L_R(0; m, q)q]v_1(m; M) / A(m)]
\]

\[
= B_0(t; m) - [1 + L_R(0; m, q)q]v_1(t; M).
\]
Given that the FRA contract’s FRA rate is set to give the contract zero initial value—that is, \( V(0) = 0 \), then

\[
B_0(0;m) = [1 + L_R(0;m,q)q]v_1(0;M).
\]

This expression implies that the FRA rate is identical to the forward LIBOR rate, as defined by expressions (18) and (19). This is a useful result as it implies that quoted FRA rates are forward LIBOR rates.

**Interest Rate Caps**

Popular among LIBOR derivatives are interest rate caps. Consider a simple interest rate cap which matures at date \( m \) and is written on the \( q \)-period spot LIBOR rate. For the present, it is assumed that there is no risk of the cap writer defaulting. The payoff to the simple caplet is defined by

\[
cap(m; m) = \max \left\{ \frac{L(m; m, q) - k}{1 + L(m; m, q)} 0 \right\} \text{ Principal (q),}
\]

where \( k \) is the cap rate. For convenience, we set the principal to be unity.

For valuation purposes, the expression for this payoff can alternatively be written in the form

\[
cap(m; m) = [1 + kq]\max \left\{ K - v_1(m; M), 0 \right\},
\]

where \( K = 1/[1 + kq] \), \( M = m + q \), and \( v_1(m; M) = 1/[1 + L(m; m, q)] \).

This expression shows that an interest rate caplet is a put option on a Eurodollar deposit with maturity \( M = m + q \). We can now use the risk-neutral pricing formula (7) to price this caplet. From expression (11), it follows that

\[
cap(t; m) = \theta \{ \exp(-\delta_1)[1 - \exp(-\lambda_1 \mu_1(m - t))]p_0(t; m, K') + \{ \exp(-\lambda_1 \mu_1 q) \} \exp[-\lambda_1 \mu_1 (m - t)]p_0(t; m, K^\nu) \},
\]

where \( \theta = 1 + kq \) and \( p_0(t; m, K) \) is the value of an ordinary Treasury bill put option which matures at date \( m \) with strike price \( K \). The Treasury bill matures at date \( M \).

In practice, a company will purchase a cap with a series of caplets at different reset dates to provide insurance for their floating rate debt. Assume the reset dates are \( m_1, m_2, \ldots, m_n \); then the value of the interest rate cap is

\[
cap(t) = \sum_{j=1}^{n} cap(t; m_j),
\]
which can be priced using expression (24).

Vulnerable Interest Rate Caps

In this section, we want to consider valuing a vulnerable interest rate cap. Let the writer of the interest rate cap contract be another firm whose risky zero-coupon bonds \( \{v_2(t, T)\} \) are also traded. The value of a simple interest rate caplet at maturity is

\[
cap(m; m)_2 = e_2(m) \cap(m; m),
\]

and at time \( t \leq m \),

\[
cap(t; m)_2 = [v_2(t; m)/B_0(t; m)] \cap(t; m).
\]

For the cap, assume the reset dates for the caplets are \( m_1, m_2, \ldots, m_n \); then the value of a vulnerable interest rate cap is

\[
cap(t)_2 = \sum_{j=1}^{n} [v_2(t; m_j)/B_0(t; m_j)] \cap(t; m_j).
\]

A vulnerable cap has a discount related to the credit risk spread of the writer of the underlying cap. Jarrow and Turnbull (1996a) estimate the magnitude of this discount for counterparty risk on the valuation of vulnerable interest rate caps.

Vulnerable Forward Contracts

The value of an interest rate cap is non-negative, implying that, if the cap has positive value, then there is a chance that the writer of the cap may default (in the absence of any netting agreement). In contrast, the value of a forward contract can be either positive or negative. Assuming the absence of any form of netting agreements, if the value is positive, there is a chance that the writer of the contract may default; and if the value is negative, then there is a chance that the owner of the contract may default. We can price out these concerns using the analysis above.

Let the writer of the forward contract be another firm whose risky zero-coupon bonds \( \{v_2(t, T)\} \) are traded. Similarly, let the firm long the forward contract have traded risky zero-coupon bonds \( \{v_3(t, T)\} \).

The value of the forward contract at maturity is

\[
V_f(m)_1 = \begin{cases} 
  v_2(m, m)V_f(m); & V_f(m) > 0 \\
  v_3(m, m)V_f(m); & V_f(m) < 0,
\end{cases}
\]

(25)
where $V_f(m)$ refers to the value of a nonvulnerable forward contract as described by expression (18). If $V_f(m)$ is positive, the writer of the contract might default; if $V_f(m)$ is negative, the party which is long the contract might default.

Using the fact that a nonvulnerable forward contract is a simple combination of a long call and a short put, with the strike price being the forward price $f(0; m, q)_t$, then the value of the vulnerable forward contract at time $t$ is

$$V_f(t)_1 = \tilde{E}_t[e_2(m)]c(t; m, f(0; m, q)_1)$$

$$-\tilde{E}_t[e_3(m)]p(t; m, f(0; m, q)_1)$$

$$= [v_2(t; m) / B_0(t; m)]c(t; m, f(0; m, q)_1)$$

$$- [v_3(t; m) / B_0(t; m)]p(t; m, f(0; m, q)_1). \tag{26}$$

The forward price $f(0; m, q)_1$ is set such that the value of the contract is zero when initiated; that is, $V_f(0)_1 = 0$. This forward price is different from that given in expression (18). Note that in expression (26) there are three sources of default risk: the underlying asset, the party that is long the contract, and the writer of the forward contract. Also note that the volatility of the underlying asset affects the value of the forward contract.

To examine the effects of counterparty risk on the valuation of a forward contract, we consider a forward contract to buy an $m$ month Treasury bill. In the absence of counterparty default risk, the forward price is given by

$$f(0, T, t + M) = B_0(0, T + m) / B_0(0, T),$$

where $T$ is the maturity date of the forward contract.

In Table 1, Panel A, the counterparty that is long the forward contract is of lower credit than the counterparty that wrote the forward contract. This implies, using equation (26), that the value of the forward contract will be positive when initiated if the forward price $f(0, T, t + m)$ is used. For the chosen parameter values, the effect of the mispricing is small. In Panel B, it is assumed that the counterparty that wrote the forward contract is of lower credit than the counterparty that is long the forward contract. Again, it is observed that the degree of mispricing is small.

In practice, the issue of counterparty risk for forward contracts is dealt with in an ad hoc manner. Usually, the forward price is set ignoring counterparty risk, and margins may be required. Equation (26) provides a means to determine the effects of counterparty risk over the life of a forward contract.

*Counterparty Risk in Interest Rate Swaps*

The analysis of counterparty risk represented by equations (25) and (26) can be extended to many other types of contracts that can have positive or negative value, such as interest rate swaps and foreign currency swaps. Consider an interest rate
swap. Let the counterparty that receives fixed payments belong to credit class 2 and the counterparty that receives floating payments belong to credit class 3.

### Table 1

**Counterparty Risk and the Valuation of a Forward Contract**

*Panel A*: The probability of default at or before date \( T \) is modeled as \( 1 - \exp(-\lambda \mu T) \), where \( \lambda \mu \) is the probability of default per unit time. For the counterparty that is long the forward contract, \( \lambda \mu = 0.02 \), and in the event of default the payoff ratio is 0.35. For the counterparty that wrote the forward contract \( \lambda \mu = 0.005 \), and in the event of default the payoff ratio is 0.55.

<table>
<thead>
<tr>
<th>maturity (T) of Forward Contract (Years)</th>
<th>Forward Price ( f(0, T, T + 0.5) )</th>
<th>Value of Contract(^a)</th>
<th>Corrected Forward Price(^b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9755</td>
<td>25</td>
<td>0.9755</td>
</tr>
<tr>
<td>2</td>
<td>0.9738</td>
<td>63</td>
<td>0.9739</td>
</tr>
<tr>
<td>3</td>
<td>0.9723</td>
<td>103</td>
<td>0.9725</td>
</tr>
<tr>
<td>5</td>
<td>0.9698</td>
<td>176</td>
<td>0.9700</td>
</tr>
<tr>
<td>10</td>
<td>0.9653</td>
<td>278</td>
<td>0.9658</td>
</tr>
</tbody>
</table>

*Panel B*: For the counterparty that is long the forward contract, \( \lambda \mu = 0.005 \), and in the event of default the payoff ratio is 0.55. For the counterparty that wrote the forward contract \( \lambda \mu = 0.03 \), and in the event of default the payoff ratio is 0.35.

<table>
<thead>
<tr>
<th></th>
<th>Forward Price ( f(0, T, T + 0.5) )</th>
<th>Value of Contract(^a)</th>
<th>Corrected Forward Price(^b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9755</td>
<td>–29</td>
<td>0.9754</td>
</tr>
<tr>
<td>2</td>
<td>0.9738</td>
<td>–72</td>
<td>0.9737</td>
</tr>
<tr>
<td>3</td>
<td>0.9723</td>
<td>–117</td>
<td>0.9722</td>
</tr>
<tr>
<td>5</td>
<td>0.9698</td>
<td>–199</td>
<td>0.9695</td>
</tr>
<tr>
<td>10</td>
<td>0.9653</td>
<td>–305</td>
<td>0.9647</td>
</tr>
</tbody>
</table>

\(^a\) The face value of the Treasury bill is one million dollars.

\(^b\) This is the forward price that sets the initial value of the forward contract to zero.

Let the first payment occur at date \( T_1 \). Let \( \mathcal{V}(T_1) \) represent the value of the swap to the counterparty that receives fixed interest payments.

The value of the fixed payment at date \( T_1 \) is

\[
\mathcal{V}_{FD}(T_1) = c_1 + \sum_{j=2}^{N} c_j \mathcal{V}_{FD}(T_1, T_j).
\]  

(27)

where \( c_j \) is the fixed payment at date \( T_j \) for \( j = 1, \ldots, N \), and \( N \) is the number of payments.

The value of the floating payments based on LIBOR at date \( T_1 \) is
\[ V_f(T_1) = f_1 + \sum_{j=2}^{N} f_j v_2(T_1, T_j), \]  

(28)

where \( f_j = v_1(T_1, T_{j-1}) - v_1(T_1, T_j) \) for \( j = 2, \ldots, N \), and \( f_j \) is the floating rate payment based on rates determined when the contract is initiated. For \( v_1(\bullet, \bullet) \), the subscript 1 refers to the credit class associated with LIBOR.

The value of the swap at date \( T_1 \) to the party receiving fixed is

\[ V(T_1) = V_{FD}(T_1) - V_F(T_1). \]  

(29)

This payoff alternatively can be written in terms of exchange options

\[ V(T_1) = \begin{cases} V_{FD}(T_1) - V_F(T_1); & V(T_1) > 0 \\ 0; & V(T_1) < 0 \end{cases} \]

\[ = EC[V_{FD}(T_1), \ V_F(T_1), \ T_1] - EP[V_{FD}(T_1), \ V_F(T_1), \ T_1]. \]  

(30)

where the first term on the right side is a European exchange call option, and the second term is a European exchange put option. Expression (30) excludes default on the swap itself.

To include default on the swap itself, let \( V_2(T_1) \) be the value of the swap considering default at time \( T_1 \). For simplicity, it is assumed that default can only occur at or before date \( T_1 \) (for a general analysis, see Duffie and Huang, 1994, and Jarrow and Turnbull, 1996b). If \( V(T_1) \) is positive, the counterparty paying fixed may default, in the absence of any netting agreement as the value of the fixed payments is greater than the value of the floating rate payments. If \( V(T_1) \) is negative, the counterparty paying floating may default. The value of the swap at date \( T_1 \) is

\[ V_2(T_1) = \begin{cases} v_3(T_1, T_1) \ V(T_1); & V(T_1) > 0 \\ v_2(T_1, T_1) \ V(T_1); & V(T_1) < 0. \end{cases} \]  

(31)

Using equations (26), (30), and (31), the value to the counterparty receiving fixed of the swap with counterparty risk is

\[ V_2(t) = [v_3(t, T_1) / B_0(t; T_1)] EC[V_{FD}(t), \ V_F(t), \ T_1] - [v_2(t, T_1) / B_0(t, T_1)] EP[V_{FD}(t), \ V_F(t), \ T_1]. \]  

(32)

This completes our study of the pricing of LIBOR derivatives.
CONVENIENCE YIELDS ON TREASURY SECURITIES

There is a branch of the literature which argues that the differences between Eurodollar and Treasury term structures is solely due to the fact that Treasury securities have a convenience yield, because there is no default risk in Eurodollar deposits (see Babbs, 1991, and Grinblatt, 1994). The purpose of this section is to augment the previous model to include a convenience yield on (particular) Treasury securities. This convenience yield is an implication of adding short sale constraints on Treasury securities. Shortages of particular Treasury securities occasionally exist (see Cornell and Shapiro, 1989; Duffie, 1996; and Chatterjea and Jarrow, 1994). During these episodes, Treasury securities cannot be sold short, and Treasury prices consequently rise (as shorts rush to cover their positions). This shortage can create a convenience yield, just as it does in other commodities. A convenience yield can also arise when, for regulatory reasons, there are restrictions on the amount of Treasury securities a financial institution can short and these restrictions are binding at the margin.

To augment this model, we keep the previous structure as imposed above. Now, however, \( B_0(t, T) \) are interpreted as the prices of default-free instruments which can be shorted without restriction. Examples include Treasury strip prices, or those default-free rates implicit in riskless portfolios obtained from combinations of derivatives (e.g., using put/call parity on stock index options).

Next, we introduce Treasury securities for which there are restrictions on the amount that can be shorted. For example, these Treasuries may be bills, notes, or bonds for which shortages exist and which can be used as collateral in repurchase agreements. These Treasury securities' payoffs are also default-free in dollars.

Let \( b_0(t, T) \) denote the time \( t \) price of a nonshortable Treasury security which pays a sure dollar at time \( T \).

The no-arbitrage relationship between these two prices is

\[
b_0(t, T) \geq B_0(t, T) \quad \text{for all } t, T.
\]

Indeed, as \( b_0(t, T) \) cannot be short sold, \( b_0(t, T) > B_0(t, T) \) is possible. The reverse inequality cannot occur without an arbitrage opportunity existing.

Define the cumulative convenience yield discount \( Y(t, T) \leq 1 \) by

\[
b_0(t, T)Y(t, T) = B_0(t, T).
\]

Note that, when \( t = T \), both zero-coupon bonds pay a dollar for sure at maturity; that is, \( b_0(T, T) = B_0(T, T) = 1 \) and \( Y(T, T) = 1 \). For those Treasury securities which can be shorted, \( Y(t, T) = 1 \) in expression (34). Thus, this analysis can apply to one Treasury instrument as well as a whole term structure of nonshortable instruments.

\[\text{---}\]

\[10\] The term nonshortable refers to the case when there are restrictions on the amount of securities that can be shorted. The restriction may be absolute, implying that no shorting is allowed, or may arise because of the illiquidity in the secondary market.
For analysis, define the forward convenience yield \( y(t, T) \geq 0 \) by

\[
Y(t, T) = e^{\int_t^T y(t, s) ds}
\tag{35}
\]

Then, from expression \((34)\),

\[
b_0(t, T) e^{\int_t^T y(t, s) ds} = B_0(t, T),
\tag{36}
\]

where \( y(t, s) \geq 0 \) for all \( 0 \leq t \leq s \leq T \).

This is the general no-arbitrage relation between the two default-free term structures. Recall that no arbitrage with respect to \( B_0(t, T) \) and \( v_1(t, T) \) implies that \( B_0(t, T) / A(t) \) is a \( \tilde{Q} \)-martingale. This implies, using expression \((36)\), that \( b_0(t, T) \exp\left(-\int_t^T y(t, s) ds\right) / A(t) \) must be a \( \tilde{Q} \)-martingale. This condition and an exogenous process for both \( y(t, T) \) and \( B_0(t, T) \), completely determine the arbitrage-free stochastic process for \( b_0(t, T) \). Expression \((36)\) provides a potential explanation for the observed differences between Treasury strip and Treasury bill prices. Although the convenience yield is predictable, in general, it is stochastic and correlated with changes in the default-free term structure. This has important implications when the TED spread is used to infer the probability of default, as explained below.

Given the above results, the previous analysis for Eurodollar rates applies exactly as written in terms of the default-free rates \( B_0(t, T) \). If, however, only \( b_0(t, T) \) can be observed, then the stochastic process for \( y(t, s) \) needs to be estimated for use with expression \((36)\) to obtain \( B_0(t, T) \). A convenient form for estimation is to assume, for example, that the convenience yield \( y(t, s) \equiv y \) is a constant. In this special case, \( b_0(t, T) e^{-y(T-t)} = B_0(t, T) \). This simple relation can be used to infer \( y \) given simultaneous observations of \( b_0(t, T) \) and any Treasury strip price.

In the augmented model, the spread between Eurodollar rates and Treasury security (nonshortable) rates is given by the expression

\[
v_1(t, T) / b_0(t, T) = [v_1(t, T) / B_0(t, T)] e^{\int_t^T y(t, s) ds}.
\tag{38}
\]

Here, the TED spread depends on a credit risk component \([v_1(t, T) / B_0(t, T)]\) as given in expression \((6b)\) and a convenience yield discount \( \exp\left(-\int_t^T y(t, s) ds\right) \).

Note that \( Y(t, T) \) in expression \((35)\) has the properties of a default-free zero-coupon bond and \( y(t, T) \) the properties of a forward rate with the restriction of non-negativity. Consequently, the stochastic properties of the convenience yield
\( \{y(t, T)\} \) can be modeled along the lines described in Heath, Jarrow, and Morton (1992).

If the default-risk does not exist (setting \( \lambda_1 = 0 \)) as conjectured in Babbs (1991) and Grinblatt (1994), then \( v_i(t, T) = B_0(t, T) \), and the TED spread in equation (39) is attributable to only the convenience yield. This is an outstanding empirical issue.

CONCLUSION

This article applies the Jarrow and Turnbull (1995) model for pricing and hedging derivatives on the Eurodollar term structure. Various LIBOR derivatives are priced in closed form. This model includes Treasury securities which cannot be shorted. The validity of this model awaits subsequent empirical testing.

APPENDIX A

SUFFICIENT CONDITIONS FOR A UNIQUE, EQUIVALENT MARTINGALE MEASURE

First define

\[
\tilde{W}(t) = W(t) - \int_0^t \gamma(s)ds, \quad (A1)
\]

where \( \gamma: \Omega \times [0, \Gamma] \rightarrow \mathbb{R} \) is \( G_t \)-predictable and uniformly bounded, and

\[
\tilde{N}_1(t) = N_1(t) - \int_0^t \lambda_1 \mu_1(s)ds, \quad (A2)
\]

where \( \mu_1: \Omega \times [0, \Gamma] \rightarrow \mathbb{R} \) is \( G_t \)-predictable and uniformly bounded. Define a measure \( \tilde{Q} \) by

\[
d\tilde{Q}/dQ = \exp \left\{ \int_0^\Gamma \gamma(s)dW(s) - \frac{1}{2} \int_0^\Gamma \gamma^2(s)ds + \int_0^\Gamma \log \mu_1(s)dN_1(s) + \int_0^\Gamma (1 - \mu_1(s))\lambda_1(s)ds \right\}.
\]

Under the hypotheses of Assumptions (A1) and (A2), \( \tilde{Q} \) is a probability measure.

The exact specification of \( (\gamma(t), \mu_1(t)) \) is determined next.

Utilizing expressions (4) and (5) and substituting in expressions (A1) and (A2) we get
\[ d(B_0(t,T)/A(t)) = (B_0(t,T)/A(t))[\beta_0(t,T) + \frac{1}{2}a(t,T)^2 + \gamma(t)a(t,T)]dt \]

\[ + (B_0(t,T)/A(t))a(t,T)d\tilde{W}(t) \]

\[ d(v_1(t,T)/A(t)) = (v_1(t-,T)/A(t))[\gamma(t)a(t,T) - r_0(t) - \Theta_1(t,T)I(t \leq \tau_1^*)\lambda_1 \mu_1(t)I(t \leq \tau_1^*)]dt \]

\[ + \left( e^{\Theta_1(t,T)I(t \geq \tau_1^*) - \delta} - 1 \right)\lambda_1 \mu_1(t)I(t \leq \tau_1^*)dt \]

\[ + (v_1(t-,T)/A(t))a(t,T)d\tilde{W}(t) \]

\[ + (v_1(t-,T)/A(t))\left( e^{\Theta_1(t,T)I(t \geq \tau_1^*) - \delta} - 1 \right)d\tilde{N}_1(t). \]

These will be martingales if both drift terms are identically zero. Thus, we add the following assumption:

\[ \left( e^{\Theta_1(t,T)I(t \geq \tau_1^*) - \delta} - 1 \right) \neq 0 \text{ for all } (t, T) \in \Delta \text{ and a.e. } \omega \in \Omega. \]

Under this assumption, the following system of equations have unique solutions \((\gamma(t), \mu_1(t))\) for all \((t, T) \in \Delta\) and a.e. \(\omega \in \Omega\).

\[ \beta_0(t,T) + \frac{1}{2}a(t,T)^2 + \gamma(t)a(t,T) = 0 \]

\[ r_1(t) - r_0(t) + \frac{1}{2}a(t,T)^2 + \gamma(t)a(t,T) \]

\[ -\Theta_1(t,T)I(t \leq \tau_1^*)\lambda_1 \mu_1(t)I(t \leq \tau_1^*) = 0. \]

Let \(\gamma(t)\) and \(\mu_1(t)\) be the solutions to expressions (A5) and (A6). They exist and are unique by Assumptions (A1) through (A4). Expressions (A5) and (A6), combined with expressions (A3) and (A4), guarantee that \(B_0(t,T)/A(t)\) and \(v_1(t,T)/A(t)\) are \(\tilde{Q}\)-martingales, because expressions (A5) and (A6) guarantee that the drift terms are identically zero.
APPENDIX B

From equations (15) and (16),

\[ 1 + L(m;m,q)(q) = [v_1(m,M)^{-1}], \]

and

\[ v_1(m,M) = B_0(m,M) \tilde{E}_m [e(M)]. \]

Given the assumption of independence, consider first \( \tilde{E}_t [B_0(m,M)^{-1}] \). Now

\[
B_0(m,M) = [B_0(0,M)/B_0(0,m)] \exp \left[ - \int \frac{m}{0} ds \int \frac{m}{u} \sigma(u,s) \int \frac{S}{v} \sigma(u,v) dv du \right. \\
 \left. - \int \frac{M}{m} ds \int \frac{m}{t} \sigma(u,s) d\tilde{W}(u) \right],
\]

so that

\[ L_t \equiv \tilde{E}_t [B_0(m,M)^{-1}] \]

\[ = [B_0(0,m)/B_0(0,M)] \exp \left[ \int \frac{m}{0} ds \int \frac{m}{u} \sigma(u,s) \int \frac{S}{v} \sigma(u,v) dv du + \frac{1}{2} \int \frac{m}{t} du \int \frac{M}{m} \sigma(u,s) ds \right]^2 \\
+ \int \frac{m}{0} d\tilde{W}(u) \int \frac{m}{M} \sigma(u,s) ds \]

\[ = [B_0(0,m)/B_0(0,M)] \exp [Q(t)], \]

where

\[ Q(t) = \int \frac{m}{0} ds \int \frac{m}{u} \sigma(u,s) \int \frac{S}{v} \sigma(u,v) dv du + \frac{1}{2} \int \frac{m}{t} du \int \frac{M}{m} \sigma(u,s) ds \right]^2 \\
+ \int \frac{m}{0} d\tilde{W}(u) \int \frac{m}{M} \sigma(u,s) ds .
\]

For use later, it is useful to record the result
\[ \frac{dL_1}{L_1} = d\tilde{W}(t) \int_m^M \sigma(t,s) ds. \]

Now we want to evaluate

\[ L_2 = \tilde{E}_t \{ 1/\tilde{E}_m [e(M)] \}. \]

Let

\[ X = \tilde{E}_m [e(M)] \]

\[ = I(\tau^* \leq m) \exp(-\delta_1) + I(\tau^* > m) \{ \exp(-\lambda_1 \mu_1 q) + \exp(-\delta_1) [1 - \exp(-\lambda_1 \mu_1 q)] \}, \]

where \( q = M - m \), so that

\[ \tilde{E}_t [X^{-1}] = I(\tau^* \leq t) \exp(\delta_1) \]

\[ + I(\tau^* > t) \left[ \exp(-\lambda_1 \mu_1 (m - t)) \right] \exp(\delta_1) \]

\[ + \exp(-\lambda_1 \mu_1 (m - t)) \left[ \exp(-\lambda_1 \mu_1 q) + \exp(-\delta_1) \left[ 1 - \exp(-\lambda_1 \mu_1 q) \right] \right]^{-1} \]

\[ = N_1(t) \left( \frac{1}{\exp(-\delta_1)} - \frac{1}{\exp(-\lambda_1 \mu_1 q) + \exp(-\delta_1) \left[ 1 - \exp(-\lambda_1 \mu_1 q) \right]} \right) \exp(-\lambda_1 \mu_1 (m - t)) \]

\[ + \left[ \frac{1 - \exp(-\lambda_1 \mu_1 (m - t))}{\exp(-\delta_1)} + \frac{\exp(-\lambda_1 \mu_1 (m - t))}{\exp(-\lambda_1 \mu_1 q) + \exp(-\delta_1) \left[ 1 - \exp(-\lambda_1 \mu_1 q) \right]} \right] \]

\[ = N_1(t)(a - b) \exp(-\lambda_1 \mu_1 (m - t)) + a \left[ 1 - \exp(-\lambda_1 \mu_1 (m - t)) \right] + b \exp(-\lambda_1 \mu_1 (m - t)) \]

where \( a = 1/\exp(-\delta_1) \); and \( b = 1/\{ \exp(-\lambda_1 \mu_1 q) + \exp(-\delta_1) [1 - \exp(-\lambda_1 \mu_1 q)] \} \). Therefore,

\[ d\tilde{E}_t (X^{-1}) = (a - b) \exp(-\lambda_1 \mu_1 (m - t)) [dN_1(t) - \lambda_1 \mu_1 l(t \leq \tau^*) dt]. \]

Combining the results, we have
\[ L_1 L_2 = B_0(0, M) \exp(Q_t) \begin{cases} \frac{1 - \exp[-\lambda_1 \mu_1 (m - t)]}{\exp(-\delta_1)} \\
\frac{\exp[-\lambda_1 \mu_1 (m - t)]}{\exp(-\lambda_1 \mu_1 q) + \exp(-\delta_1)[1 - \exp(-\lambda_1 \mu_1 q)]} \end{cases} \]

\[ \exp(\delta_1); \quad \tau_1^* \leq t. \]

and

\[ dF(t; m, q) = -100 dL_F(t; m, q) \]

\[ = -100[L_2 \text{d}L_1 + L_1 \text{d}L_2](1/q) \]

\[ = -100 \left\{ L_1 L_2 d\tilde{W}(t) \int_m^M \sigma(t, s) \text{d}s \right\} (1/q) \]

\[ + L_1 (a - b) \exp[-\lambda_1 \mu_1 (m - t)] [dN_1(t) - \lambda_1 \mu_1 I(t \leq \tau_1^*) \text{d}t] (1/q). \]

**FORWARD CONTRACTS**

From footnote 6 it is necessary to evaluate

\[ L = E_1 [A(m)^{-1} B_0(m; M)^{-1}] A(t). \]

Let

\[ B_0(m; M)^{-1} = \left[ B_0(0; m) / B_0(0; M) \right] \exp(\mu_2 + Z_2), \]

where

\[ \mu_2 = - \int_m^M d\int_m^0 \sigma(u, s) a(u, s) du + \int_0^t d\tilde{W}(s) \int_m^M \sigma(s, v) dv, \]

\[ Z_2 = \int_t^M d\tilde{W}(s) \int_m^M \sigma(s, v) dv, \]

and

\[ A(m)^{-1} = B_0(0; m) \exp(\mu_1 + Z_1), \]
where
\[ \mu_1 = -\frac{1}{2} \int_0^m a(s,m)^2 \, ds + \int_0^t a(s,m) \, d\tilde{W}(s) \]
\[ Z_1 = \int_t^m a(s,m) \, d\tilde{W}(s). \]

Therefore,
\[ L = [B_0(0;m)^2 / B_0(0;M)] \exp[\mu_1 + \mu_2 + (\sigma_{11} + 2\sigma_{12} + \sigma_{22}) / 2] \, A(t), \]

where \( \sigma_{ij} = \tilde{E}_t (Z_i Z_j) \). Now
\[ B_0(t;m) = \tilde{E}_t [A(m)^{-1}]A(t) \]
\[ = B_0(0;m) \exp(\mu_1 + \sigma_{11}/2)A(t), \]

so that
\[ L = [B_0(0;m) / B_0(0;M)] \exp(\mu_2 + \sigma_{12} + \sigma_{22}) / 2 \) \( B_0(t;m). \]

Also,
\[ B_0(t;M) = \tilde{E}_t [B_0(m;M)A(m)^{-1}]A(t) \]
\[ = B_0(0;M) \exp[\mu_1 - \mu_2 + (\sigma_{11} - 2\sigma_{12} + \sigma_{22}) / 2]A(t) \]
\[ = [B_0(0;M)/B_0(0;m)] \exp(-\mu_2 - \sigma_{12} + \sigma_{22}/2)B_0(t;m). \]

Therefore,
\[ L = [B_0(t;m) / B_0(t;M)] \exp \sigma_{22}. \]
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