Contemporary Issues


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In the Black-Scholes call option pricing formula, continuous rebalancing is an abstraction. Current trading practice recognizes this and incorporates discrete trading techniques. Gilster questions whether continuous rebalancing can be approximated accurately by discrete rebalancing. This paper shows that his assertion, the risk in the Black-Scholes hedge is larger than previously computed, is false due to the upward bias inherent in his new measure.

As is well known, the theory underlying the Black-Scholes call option pricing formula is based on a continuous trading model. In this theory, traded options are priced via the cost of creating a synthetic option using a dynamic portfolio in the underlying stock and bond. This synthetic option portfolio must be rebalanced continuously through time. Unfortunately, continuous rebalancing is an abstraction, which is physically impossible in actual markets. This fact is well known and current trading practice has long incorporated discrete trading techniques (see Jarrow and Turnbull, 1996, Chapter 10 for a review).

The question studied in the paper by Gilster is whether this continuous rebalancing can be reasonably well-approximated by discrete rebalancing. That is, in the Black-Scholes setting, is a synthetic option approximately equal to a traded option, when rebalancing can only be done discretely. The previous literature suggests that it can, and this paper by Gilster argues that it cannot.

To study this question, Gilster derives a new measure for the risk of a hedged option-stock position (using the Black-Scholes hedge) over a discrete period of time, without rebalancing. The risk analysis is theoretical, done via numerical computations and without the use of market data. Using his new measure, Gilster argues that the risk of the Black-Scholes hedge is larger than previously computed. I will show that this assertion is false, by demonstrating that Gilster’s new measure is flawed because it contains a significant upward bias.

I. A Derivation of Gilster’s Measure

Gilster’s measure for computing the hedging error can be understood as follows. The notation is from Gilster’s paper. Let \( w(s,t) \) be the Black-Scholes option price at time \( t \) when the underlying stock price is \( s \). Let \( w_t(s,t) \) be the Black-Scholes hedge ratio at time \( t \) when the underlying stock price is \( s \). This quantity is often called the option’s “delta.”

The Black-Scholes hedge portfolio consists of \( w_t(s,t) \) shares of the stock and borrowings of \( [w_t(s,t)s - w(s,t)] > 0 \) dollars at the riskless rate. This portfolio is sometimes called the “synthetic option.” The Black-Scholes hedge needs to be rebalanced continuously. If it cannot, then there is a hedging error over \([t, t+\Delta t]\). The hedging error...
over \( [t, t+\Delta t] \), denoted \( E \), can be written as:

\[
E = (w_{i}(s,t)[s+\Delta s] - (1+r\Delta t)w_{i}(s,t)s - w(s,t)) - w(s+\Delta s, t+\Delta t) \tag{1}
\]

where \( r \) is the risk-free rate. The hedging error is the difference between the synthetic option’s value at time \( t+\Delta t \), \( (w_{i}(s,t)[s+\Delta s] - (1+r\Delta t)w_{i}(s,t)s - w(s,t)) \), and the traded option’s price \( w(s+\Delta s, t+\Delta t) \). The synthetic option’s position reflects both the change in the stock price from \( s \) to \( s+\Delta s \), and the additional interest at \( r\Delta t \) owed on the borrowings. The traded option’s price \( w(s+\Delta s, t+\Delta t) \) is assumed to follow the Black-Scholes formula.

This difference can be rewritten as a random component, \( H \), and a non-random component, \( (1+r\Delta t)w(s,t) - w_{i}(s,t)s \), i.e.,

\[
E = H - (1+r\Delta t)w(s,t)s - w(s,t) \tag{2}
\]

where

\[
H = w_{i}(s,t)[s+\Delta s] - w(s+\Delta s, t+\Delta t) \tag{3}
\]

Because the portfolio is not rebalanced continuously as required by the theory, the standard deviation of the hedging error \( E \) is non-zero. As the second term in Equation (2) is known at time \( t \), the standard deviation (or beta) of \( E \), as viewed from time \( t \), is equivalent to the standard deviation of \( H \). Consequently, the standard deviation of \( H \), as viewed from time \( t \), correctly measures the absolute risk of the Black-Scholes hedge over the discrete interval \( [t, t+\Delta t] \).

This measure is effectively the quantity previously computed by Boyle and Emanuel (1980), which Gilster incorrectly claims is of “limited value.”

To justify his assertion, Gilster provides an alternative measure of the risk of the hedging error. Let us consider his measure from the above perspective. Adding and subtracting \( w_{i}(s+\Delta s, t+\Delta t)[s+\Delta s] \) from both sides of Equation (3), one can decompose the random component of the hedging error \( H \) into:

\[
H = P + U \tag{4}
\]

where

\[
P = w_{i}(s+\Delta s, t+\Delta t)[s+\Delta s] - w(s+\Delta s, t+\Delta t) \tag{5}
\]

and

\[
U = -[w_{i}(s+\Delta s, t+\Delta t) - w_{i}(s,t)][s+\Delta s] \tag{6}
\]

This decomposition is the basis for Gilster’s measure.

Gilster measures the risk of the hedging error as the instantaneous standard deviation (or beta) of \( H \) over \( [t+\Delta t, t+\Delta t + \Delta t] \) as viewed from time \( t+\Delta t \), where \( \Delta t \) is instantaneous. But, from the Black-Scholes theory, the instantaneous standard deviation of \( P \) over \( [t+\Delta t, t+2\Delta t+\Delta t] \) as viewed from time \( t+\Delta t \) is zero. Thus, Gilster’s measure for the risk of the hedging error is the instantaneous standard deviation of \( U \) over \( [t+\Delta t, t+2\Delta t+\Delta t] \) as viewed from time \( t+\Delta t \). It is important to note that Gilster’s measure is identified by two characteristics: 1) it is an instantaneous standard deviation, and 2) it is computed as viewed from time \( t+\Delta t \).

II. The Bias in Gilster’s Measure

This section shows that Gilster’s measure for the risk of the hedging error is significantly upward biased. The first question one has to ask is, Why is an instantaneous standard deviation of \( H \) an interesting quantity to compute? This relates to the first characteristic of Gilster’s measure. I contend that an instantaneous standard deviation is not interesting in and of itself. As an abstraction, it is only interesting to the extent that it is a proxy for the standard deviation of the hedging error \( H \) (properly integrated) over some larger finite interval. What is this larger interval? Is it \( [t, t+\Delta t+\Delta t] \) (which is formally \( [t, t+\Delta t] \)) or is it \( [t+\Delta t, t+2\Delta t] \)? This relates to the second characteristic of Gilster’s measure.

The answer to this question concerning the relevant time interval requires an economic, not a mathematical, analysis. These computations only make economic sense for the interval \( [t, t+\Delta t] \). The interval \( [t+\Delta t, t+2\Delta t] \) considers the hedging error from the moment the hedge is initiated. As such, it considers the hedging problem as one faced by a trader, deciding whether to put on a hedge. In contrast, the interval \( [t+\Delta t, t+2\Delta t] \) characterizes the hedging problem as one faced by a trader, given a book to manage at time \( t+\Delta t \), but with an existing hedge already in place. This later situation is unlikely, and therefore uninteresting from an economic perspective. Unfortunately, as a proxy for the standard deviation of the hedging error \( H \) over \( [t, t+\Delta t] \), Gilster’s measure contains significant bias.1

To understand the bias in Gilster’s measure, note that with respect to estimating the standard deviation of \( H \) over \( [t, t+\Delta t] \) as viewed from time \( t \), Gilster’s measure 1) proxies the standard deviation of \( U \) over \( [t, t+\Delta t] \) as viewed from time \( t \), as being proportional to

1From a mathematical perspective, it can be shown that Gilster’s measure is also significantly biased as a proxy for the standard deviation of \( H \) over \( [t+\Delta t, t+2\Delta t] \) as viewed from time \( t+\Delta t \).
the instantaneous standard deviation of the stock over $[t+\Delta t, t+\Delta t + \Delta t]$, as viewed from time $t+\Delta t$ and 2) sets the risk of $P$ equal to zero. Both characteristics introduce a bias.

Gilster's proxy for the risk of $U$ over $[t, t+\Delta t]$ is crude. It measures the risk of $U$ as only being proportional to the risk of $[s+\Delta s]$, because when viewed from time $t+\Delta t$, both $w_t[s+\Delta s, t+\Delta t]$ and $w_t(s, t)$ are constants. But, when viewed from time $t$, the risk of $U$ should also include the risk of the joint product of the changing delta $w_t[s+\Delta s, t+\Delta t]$ times the changing stock price $[s+\Delta s]$, both of which are random. Ignoring the randomness of $w_t[s+\Delta s, t+\Delta t]$ introduces a bias.

The real bias, however, occurs in ignoring the contribution of the risk of $P$, the second characteristic of Gilster's measure. It is easy to show that the standard deviation of $P$ over $[t, t+\Delta t]$ is not zero, when viewed from time $t$. Using the Black-Scholes formula, we can rewrite $P$ as

$$P = N(d_1 - \sigma \sqrt{\Delta t})ke^{-r \Delta t}$$

(7)

where

$N(*)$ is cumulative normal,
$k$ is the strike price,
$\sigma$ is the volatility,
$r$ is time to maturity at $t$,

and

$$d_1 = \frac{\log((s+\Delta s)/k) + (r + \sigma^2/2)(\tau - \Delta t)}{\sigma \sqrt{\tau - \Delta t}}$$

When viewed at time $t$, the stock price at time $t+\Delta t$, written $s+\Delta s$, is random. Therefore, $d_1$ is random, $N(d_1 - \sigma \sqrt{\tau - \Delta t})$ is random, and $P$ is random, i.e., it has non-zero standard deviation. Ignoring $P$, therefore, gives a flawed measure of the risk of the hedged portfolio.

Next, it is easy to see that $P$ and $U$ are both (highly) negatively correlated. Indeed, rewriting $U$ we get

$$U = -w_t[s+\Delta s, t+\Delta t][s+\Delta s] - w_t(s, t)[s + \Delta s]$$

(8)

Using Equations (5) and (8), it is seen that the terms $P$ and $U$ differ in only their signs (positive versus negative) and the last term within parentheses. As both these last terms, $w_t[s+\Delta s, t+\Delta t]$ and $w_t(s, t)[s+\Delta s]$, are (highly) positively correlated, $P$ and $U$ will be (highly) negatively correlated.

Given that $P$ and $U$ are negatively correlated, incorrectly ignoring $P$ gives an upward bias to Gilster's measure for the hedging error. This potential upward bias, combined with the improper proxy for the risk of $U$, can explain the difference between the large magnitudes of the standard deviations in Gilster's numerical examples and the smaller ones previously obtained in the literature, thereby invalidating Gilster's assertion.

III. Conclusion

Unfortunately, as Gilster's measure of hedging risk is the central contribution of the paper, and as this measure is flawed, this paper does not provide any new contribution to the literature. Furthermore, the qualitative conclusions that Gilster draws from his biased numerical examples have no validity. The paper also has other weaknesses, but as these are second-order problems, and the bias-risk measure is a first-order problem, these additional weaknesses are not pursued here.

References


*These include: 1) the use of the CAPM for measuring the risk of an option position, 2) measuring the risk of returns with denominators close to zero rather than the risk of dollar movements, and 3) the interpretation of the mathematics surrounding his Equations (14) – (17) for discrete time intervals.