The arbitrage-free valuation and hedging of demand deposits and credit card loans

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Abstract

Using a market segmentation argument, this paper uses the interest rate derivative’s arbitrage-free methodology to value both demand deposit liabilities and credit card loan balances in markets where deposits/loan rates may be determined under imperfect competition. In this context, these financial instruments are shown to be equivalent to a particular interest rate swap, where the principal depends on the past history of market rates. Solutions are obtained which are independent of any particular model for the evolution of the term structure of interest rates. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

The purpose of this paper is to provide an arbitrage-free approach to the valuation and hedging of demand deposits and credit card loans in markets where deposit or loan rates differ from market rates on comparable (equal risk) financial securities.

The hedging of the demand deposit liability, in particular, is a relatively un-studied issue of significant practical importance. ¹ The asset and liability management of financial institutions accepting demand deposits (e.g. commercial banks or savings and loan associations) depends crucially on an accurate understanding of the risk characteristics of demand deposits. ² Indeed, the “health” of the banking system appears related to the ability of banks to match their assets and their liabilities. For credit card portfolios, the increasing size of this financial sector justifies critical analysis (see Ausubel, 1994).

Both demand deposits and credit card loans are difficult to value because they pay/charge rates which differ from market rates on comparable (equal risk) financial securities. This difference has been attributed to markets with imperfect competition, possibly due to either market frictions (search/switching costs), institutional realities (regulatory barriers), or adverse selection problems under asymmetric information (see Ausubel, 1991; Calem and Mester, 1995; Hutchison and Pennacchi, 1996).

This paper uses the interest rate derivatives technology of Heath et al. (1992) to value both demand deposits and credit card loans. In contrast to Hutchison and Pennacchi’s (1996) equilibrium-based approach, this paper uses the arbitrage-free pricing methodology.

To justify the differences between market rates and the rates paid/charged on demand deposits/credit card loans, a “market segmentation” argument is employed. The “market segmentation” argument is that banks alone, and not individual investors, can issue demand deposits and credit card loans. In this model, both individual investors and banks can trade in frictionless and com-

² We use the familiar term “demand deposits” to mean the more broadly defined “non-maturity deposits” or “ryudosei yokin” (“liquid deposits” in Japanese). Non-maturity deposit categories include all bank deposits which have no stated maturity and where individual depositors have the right to add or subtract balances without restriction. The rate on such an account may be a fixed or floating rate. By contrast, time deposits or certificates of deposits have an explicit maturity, an explicit principal amount set at the time of the initial deposit, and a change in the principal amount only with the payment of a penalty for early withdrawal.
petitive Treasury security markets. ³ Under this structure, it is shown that demand deposits and credit card loans are equivalent to an exotic interest rate swap, where the principal depends on the past history of market rates. This interest rate swap analogy provides the necessary insights to both price and hedge these financial instruments. Solutions are obtained which are independent of any particular model for the evolution of the term structure of interest rates.


The Office of Thrift Supervision (1994) measures the interest rate risk of demand deposit balances by computing their “duration”. Duration is the percentage change in the present value determined by shifting the initial spot interest rate by 100 basis points, and then recomputing the deterministic present value. Mixing deterministic and stochastic interest rate analysis in this ad hoc manner only generates nonsensical results. Unfortunately, thrift institutions are required to provide these misspecified duration measures to the Office of Thrift Supervision. O’Brien et al. (1994) compute present values and interest rate sensitivities in two ways: one, as done by the Office of Thrift Supervision, and two, as a discounted expected value using stochastic demand deposit balances, demand deposit rates and interest rates. In this later case, the expectation represents a present value only if investors are risk-neutral. Expectations are computed using a Monte Carlo simulation under this risk-neutrality assumption. Selvaggio (1996) also computes present values in a stochastic demand deposit and interest rate environment using Monte Carlo simulations and no arbitrage arguments. Hutchison and Pennacchi (1996) compute present values using an equilibrium-based model in an economy where interest rates follow a square root, mean-reverting process.

The contribution of our paper is to provide an arbitrage-free procedure for computing present values in a stochastic interest rate environment using the Heath et al. (1992) methodology. An outline of the paper is as follows. Section 2 presents the discrete-time model. It includes the market segmentation argument, the demand deposit valuation model, and the credit card loan valuation model. Section 3 provides the continuous-time model structure and

³ This assumption can be easily modified to restrict individuals from shorting Treasury securities. This modification to the subsequent theory does not change any of the major results.
applies it to obtain analytic solutions for demand deposits. Section 4 concludes the paper. All detailed proofs are contained in an appendix.

2. Discrete time

For simplicity, we first consider a discrete time economy with trading dates \( t \in \{0, 1, 2, \ldots, \tau\} \). Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{Q})\) be a filtered probability space satisfying the usual conditions where \( \mathcal{F}_0 \) contains only trivial events and \( \mathcal{F}_\tau = \mathcal{F} \). The basic structure, with some modifications, will be that used in Jarrow (1996), which is a discrete time approximation to Heath et al. (1992).\(^4\) The discrete time analysis facilitates understanding and computer implementation.

2.1. The model

To account for differential rates on demand deposits and credit card loans with no default risk versus Treasury securities, we invoke a “market segmentation” hypothesis. The market segmentation hypothesis is that there are two types of traders: (i) banks/financial institutions and (ii) individuals. The partitioning between these two types of traders is based on their ability to issue demand deposits and credit card loans. We assume that there are significant regulatory restrictions and entry or mobility barriers associated with the demand deposit and credit card loan markets. For example, mobility barriers may be due to informational or locational advantages given consumer search/switching costs. Entry barriers may be due to the fixed costs of purchasing computer equipment and technology, data feeds, capital requirements, and the accumulation of expertise. By exogenous specification, a limited number of banks have access to the demand deposit and credit card loan markets while individual investors do not. This is the market segmentation we employ.

Both banks and individuals have access to competitive and frictionless Treasury security markets. This implies, of course, that short sales of Treasury securities are unrestricted to banks and individuals.\(^5\)

\(^4\) The extension to the continuous time analogue is straightforward, and discussed in Section 3. The restriction to a finite horizon \( \tau \) is for simplicity, and letting \( \tau \to \infty \) for the results obtained below will give a good approximation to an infinite horizon economy.

\(^5\) This implicitly assumes that there is no risk of default for banks or individual investors. Default risk is easily included along the lines of Jarrow and Turnbull (1995). One modification of this structure is to restrict the short sales of Treasuries by individual investors. This modification is discussed in a subsequent footnote.
2.2. Treasury markets

Traded are zero-coupon bonds of all maturities, and (for convenience) a money market account. A money market account is different from a demand deposit as the money market account’s return is based on the shortest maturity Treasury security.

The time $t$ price of a zero-coupon bond paying a sure dollar at time $T$ is denoted $P(t,T)$. The spot rate of interest at time $t$, denoted $r(t)$, is defined by

$$r(t) = [1/P(t,t+1)] - 1.$$  

The money market account’s value is obtained via rolling over the shortest maturity zero-coupon bond and is denoted by

$$B(t) = B(t-1)(1 + r(t-1)), \quad \text{where } B(0) = 1.$$  

We assume that there exists a unique equivalent probability measure $\tilde{Q}$ such that the conditional expectation satisfies

$$P(t,T) = \tilde{E}_t(P(t+1,T))/(1+r(t)) = \tilde{E}_t(1/B(T))B(t),$$  

for all $0 \leq t < T$.

This implies that there are no arbitrage opportunities in the Treasury securities market and that the Treasury securities market is complete. \(^7\) This is with respect to trading by both banks and individuals. \(^8\) For a sufficient set of conditions guaranteeing this hypothesis, see Jarrow (1996).

2.3. Demand deposit markets

Only a limited number of banks can issue demand deposits with rate $i(t)$ per period. The rate $i(t)$ includes any bank servicing costs and is less any fees received. Demand deposits are floating rate instruments paying the interest $i(t)$ less net servicing costs, denoted $\bar{i}(t)$, every period. Banks cannot buy demand deposits; they can simply stop issuing them. Individuals can hold demand deposits, and receive $\bar{i}(t)$ every period. Individuals cannot issue (short) demand deposits.

We assume that there are no arbitrage opportunities available for individuals, but due to the market segmentation hypothesis, we allow arbitrage opportunities for a limited number of banks. These considerations imply that ($\bar{i}(t)$ or $i(t)$) and $r(t)$ need not be equal. We now explore these issues.

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\(^6\) The notation $\tilde{E}_t(\cdot) \equiv \tilde{E}(\cdot | \mathcal{F}_t)$.

\(^7\) For an elaboration of these issues, see Heath et al. (1992) or Jarrow (1996).

\(^8\) If individuals cannot short Treasuries, then there may be arbitrage opportunities for banks that individuals cannot exploit.
First, we study the relation between $\tilde{i}(t)$ and $r(t)$. Consider the inequality: $\tilde{i}(t) > r(t)$ for some $t$. This inequality implies an arbitrage for individuals, i.e. they can buy the demand deposits earning $\tilde{i}(t)$ and sell Treasuries paying $r(t)$. Thus, no arbitrage by individuals implies $9$

$$\tilde{i}(t) \leq r(t) \quad \text{for all } t.$$  

(2a)

In contrast, the inequality

$$\tilde{i}(t) < r(t) \quad \text{for some } t$$  

(2b)

cannot be arbitraged by individuals, as they cannot issue (short) demand deposits.

Next, we study the relation between $i(t)$ and $r(t)$. Consider the inequality: $i(t) > r(t)$ for some $t$. Banks cannot arbitrage this inequality at time $t$ as they cannot buy demand deposits, although they could stop issuing them. In contrast, if

$$i(t) < r(t) \quad \text{for some } t,$$

then a limited number of banks do have an arbitrage opportunity. Indeed, those that have access to this market can issue demand deposits paying $i(t)$ and buy Treasuries earning $r(t)$. As there are entry and mobility barriers to these markets, the arbitrage opportunities are not necessarily competed away.

Let $D(t)$ be the exogenously given dollar volume of demand deposits available to a particular bank at time $t$. This process will be estimated and the estimation procedure will reflect local market characteristics, like different consumer sensitivities to changes in market rates. This process, although exogenous in our model, could be endogenously derived in an equilibrium model involving imperfect competition among banks. $10$ It restricts the rents available to a particular bank due to the above inequality.

In summary, no arbitrage opportunities for individuals, and market segmentation considerations in the demand deposits market imply that:

$$\tilde{i}(t) \leq r(t) \quad \text{with strict inequality possible}$$  

(3a)

and

$$i(t) \leq r(t) \quad \text{for all } t \text{ with } i(t) < r(t) \text{ possible.}$$  

(3b)

If the net servicing costs associated with demand deposits are small, i.e., $i(t) - \tilde{i}(t) \approx 0$, then based on expression (3a), one would expect to observe

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$9$ Of course, when examining empirical data, transactions costs in Treasury securities markets and counterparty risk could cause the violation of expression (2a). If individuals cannot short Treasuries, then expression (2a) will not hold. As shown subsequently, the violation of expression (2a) will not change any of our major results.

$10$ For one such model, see Hutchison and Pennacchi (1996).
\( i(t) \leq r(t) \) for most \( t \). \(^{11}\) It is important to emphasize that expression (3) is included to show that the stochastic processes for \( \tilde{i}(t) \) or \( i(t) \) and \( r(t) \) can be different in an equilibrium with market segmentation in the issuing of demand deposits. The minimal condition necessary for the subsequent analysis not to trivialize is that \( i(t) < r(t) \) for some \( t \) with positive probability. \(^{12}\)

2.4. Credit card loan markets

Analogous to demand deposit liabilities, only a limited number of banks can issue credit card loans. Credit card loans are floating rate instruments receiving the interest \( c(t) \) plus net servicing costs, denoted \( \bar{c}(t) \), every period. Banks cannot short credit card loans; they can simply stop issuing them. Individuals can borrow (short) credit card loans, and pay \( \bar{c}(t) \) every period. Individuals cannot issue (buy) credit card loans.

As credit card loan assets are analogous to demand deposit liabilities, but of differing sign, analogous arguments to those given for demand deposits show that:

\[
\bar{c}(t) \geq r(t) \quad \text{with strict inequality possible} \quad (4a)
\]

and

\[
c(t) \geq r(t) \quad \text{for all } t \text{ with } c(t) > r(t) \text{ possible.} \quad (4b)
\]

When \( c(t) > r(t) \), credit card loan portfolios are viewed as arbitrage opportunities by banks, and we let \( L(t) \) be the exogenously given dollar volume of credit card loans available to a particular bank at time \( t \). This process, although exogenous in our model, could be endogenously derived in an equilibrium model involving imperfect competition. \(^{13}\)

Expression (4) is not necessary for the subsequent analysis. The minimal condition necessary for the remaining analysis not to trivialize is that \( c(t) > r(t) \) for some \( t \) with positive probability.

For the present, we assume all credit card loans are default-free. This restriction is relaxed in a subsequent section.

2.5. Valuation of demand deposits

This section values demand deposits as seen by financial institutions. The demand deposit liability to a financial institution is shown to be related to an

\(^{11}\) This is supported by the empirical evidence, see (Hutchison and Pennacchi (1996), Figure 2).

\(^{12}\) If individuals cannot short Treasuries, then the first inequality in expression (3) does not hold, but the second will. This implies that all the subsequent results follow as given.

\(^{13}\) See Ausubel (1991) and Calem and Mester (1995) for equilibrium models of the credit card loan market.
exotic interest rate swap, where the principal depends on the history of market rates.

Due to the arbitrage opportunity implicit in expression (3), the bank may value each dollar in a demand deposit as more than a dollar. For simplicity, we assume that the demand deposit rate plus net servicing costs \( i(t) \) and the dollar volume of demand deposits \( D(t) \), depend only on the information set generated by the evolution of the Treasury securities. This restriction could be easily relaxed. 14 Under this assumption, using the risk-neutral valuation procedure, the time 0 net present value of the \( D(0) \) demand deposits to the bank are given in expression (5):

\[
V_D(0) = D(0) + \tilde{E}_0 \left( \sum_{t=0}^{\tau-2} \frac{D(t+1) - D(t)}{B(t+1)} \right) - \tilde{E}_0 \left( \frac{D(\tau-1)}{B(\tau)} \right) \\
- \tilde{E}_0 \left( \sum_{t=0}^{\tau-1} \frac{i(t)D(t)}{B(t+1)} \right).
\]  

(5)

The proof of expression (5) can be found in the appendix.

This net present value represents the initial deposits received \( D(0) \), plus any changes in the deposits over time, less the return of the \( \tau - 1 \) deposits at time \( \tau \), and less the present value of the aggregate costs to the deposits. 15 In economic terms, the net present value is the maximum premium above the dollar amount of deposits that a rational bank would bid to purchase the demand deposit franchise from another bank. It is analogous to the net present value computation one would make for an investment project in traditional capital budgeting problems. It represents a rent for the privilege of issuing demand deposits. Note that this value is determined independently of the use to which these funds are put within the bank.

This expression assumes that the aggregate demand deposits \( D(t) \) depend only on the evolution of the term structure of default free rates. With additional complexity, additional randomness could be included in \( D(t) \). For example, other macroeconomic/local market considerations like the income level or the unemployment rate might be included. In the estimation, the local market characteristics would be incorporated in the firm-specific parameter estimates obtained. This extension would require traded securities correlated with these

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14 This restriction is needed so that \( \tilde{E}(\bullet) \) represents the unique martingale measure generated by the term structure of interest rates and represented in Eq. (1). Otherwise, the term structure of zero-coupon bonds does not generate the unique measure needed for valuation. In this case, \( \tilde{E}(\bullet) \) in Eq. (5) would represent an extension of the measure in Eq. (1), defined over a larger state space with additional uncertainties. Uniqueness of the extended measure would require the trading of additional instruments.

15 It is easy to show that if \( i(t) = r(t) \) for all \( t \), then \( V_D(0) = 0 \). This proof follows from expression (6).
additional risks for market completeness and uniqueness of the equivalent probability measure $\tilde{Q}$ (see footnote 14). With this modification, however, the analysis is then identical to that given in expression (5).

Expression (5), although descriptive, does not indicate whether the net present value of the demand deposits is positive or negative. Algebra yields an equivalent, but more useful expression in this regard (see the appendix for a proof):

$$V_D(0) = \tilde{E}_0 \left( \sum_{t=0}^{T-1} D(t)[r(t) - i(t)]/B(t + 1) \right). \quad (6)$$

From expression (6) and condition (3), we see that the net present value of the demand deposits can be positive if Treasury rates typically exceed demand deposit rates.\(^\text{16}\)

This expression has an interesting economic interpretation. The payments on the right side are those obtainable from the trading strategy of investing $D(t)$ dollars into short term investing, receiving $r(t)$ at a cost of $i(t)$ each period. The payment of $D(t)[r(t) - i(t)]$ is received at time $t + 1$. Expression (6) is the discounted cash flows to this strategy. Note, however, that as argued earlier, the value of the demand deposits obtained is independent of this arbitrage strategy.

Expression (6) has another economic interpretation. It can also be interpreted as the value of an exotic interest rate swap lasting for $\tau$ periods, receiving floating at $r(t)$, and paying floating at $i(t)$ with an amortizing/expanding principal of $D(t)$ in period $t$. This interpretation provides intuition for subsequently discussed hedging techniques.

By definition, the present value of the demand deposit liability to the bank at time 0, denoted $C_D(0)$, is

$$C_D(0) \equiv D(0) - V_D(0). \quad (7)$$

This equals the initial demand deposits less their net present value.

To hedge the demand deposit liability, expression (7) provides the necessary insights. Hedging the demand deposit liability requires investing $D(0)$ dollars in the shortest term bond $P(0, 1)$ (rolling over at $r(t)$), and shorting the exotic interest rate swap represented by $V_D(0)$. Shorting this exotic interest rate swap provides an additional cash inflow.

\(^\text{16}\) In 1225 auctions of failed bank deposits by the Resolution Trust Corporation in the United States through 28 October 1994, the average premium paid to the RTC as a percentage of deposit balances was 2.32%, with a range from −16.87 of deposit balances to a high of 25.33%. More than 70% of deposits auctioned, on average, were time deposits and certificates of deposit which would be expected to have relatively low net present values. Six of the auctions had negative premiums, reflecting processing costs of moving deposit balances to a new organization that were greater in magnitude than the net present value of the purchased deposit franchises.
This hedge is intuitive. Indeed, if the net present value of the demand deposits were 0 (e.g., \( r(t) = i(t) \) for all \( t \)), then the only way to exactly hedge the demand deposits would be to invest all of the demand deposits in the shortest term bond (and roll it over). This is exact “maturity” matching of the assets and demand deposit liabilities of the bank. But, if the demand deposits have a positive net present value, \( V_D(0) > 0 \), a mis-matching of the asset and demand deposit liability “maturity” structures can occur. The mis-matching allowed is quantified by \( V_D(0) \), expression (6), which can be viewed as the value of an exotic interest rate swap.

Given an explicit representation of the stochastic evolution of the term structure of default-free rates, expressions (6) and (7) can be computed. This specification would also reveal how to hedge the cash flows in expression (7) using only a few Treasury securities (see Jarrow, 1996). An example will help to clarify this procedure.

**Example 1 (Deposits linear in the market rate, constant rate spread).** Let

\[
D(t) \equiv x_0 + x_1 r(t)
\]  
(8a)

and

\[
r(t) - i(t) \equiv \beta.
\]  
(8b)

This example corresponds to a demand deposit liability \( D(t) \) which is linear in the market rate and where there is a constant spread between market rates \( r(t) \) and demand deposit rates \( i(t) \).

The market conditions under which expressions (8a) and (8b) might hold are understood by substituting expression (8b) into (8a), to yield

\[
D(t) = x_0 + x_1 \beta + x_1 i(t).
\]  
(9)

Expression (9) can be interpreted as a market supply curve. 17 The net present value of the demand deposit liability under expression (8) is:

\[
V_D(0) = \bar{E}_0 \left( \sum_{t=0}^{\tau-1} [x_0 + x_1 r(t)]\beta/B(t+1) \right)
= x_0 \beta \sum_{t=0}^{\tau-1} P(0, t+1) + x_1 \beta \bar{E}_0 \left( \sum_{t=0}^{\tau-1} r(t)/B(t+1) \right).
\]  
(10)

But

\[17\] This expression also corresponds to a market with a stable deposit base of \( (x_0 + x_1 \beta) \) dollars where \( [x_1/(x_0 + x_1 \beta)] \) percent of the interest paid each period is reinvested back into the deposits. The remaining percent is withdrawn.
The proof of this last equality is contained in the appendix. Substitution of expression (11) into Eq. (10) along with repeated use of expression (1) yields:

\[ V_D(0) = x_0 \beta \sum_{t=0}^{T-1} P(0, t + 1) + x_1 \beta (1 - P(0, \tau)) , \]  

(12a)

and

\[ C_D(0) = D(0) - x_0 \beta \sum_{t=0}^{T-1} P(0, t + 1) - x_1 \beta (1 - P(0, \tau)) . \]  

(12b)

These values are computable given just the initial zero-coupon bond curve. The demand deposit liability is hedgable by investing \((D(0) - x_1 \beta - x_0 \beta P(0, 1))\) dollars in the shortest maturity bond, shorting \((x_0 \beta)\) units of the bonds with maturities 2, . . . , \(\tau - 1\), and shorting \((x_0 \beta - x_1 \beta)\) units of the \(\tau\) maturity bond. To see that this portfolio works, consider the value of this replicating portfolio at time 1. Its time 1 value is:

\[
(D(0) - x_1 \beta - x_0 \beta P(0, 1))(1 + r(0)) - x_0 \beta \sum_{t=1}^{T-1} P(1, t + 1) - x_1 \beta (-P(1, \tau))
\]

\[
= D(0)(1 + r(0)) - x_1 \beta (1 + r(0)) - x_0 \beta - D(1) + x_1 \beta + C_D(1)
\]

\[
= D(0) - D(1) + D(0)r(0) - D(0)\beta + C_D(1)
\]

\[
= D(0) - D(1) + D(0)i(0) + C_D(1).
\]  

(13)

This is exactly the interest payments required at time 1 on the initial demand deposits, plus the change in demand deposits, plus the time 1 liability of future demand deposit cash flows.

For empirical implementation, the previous example is too simple. An appropriate generalization of expressions (8a) and (8b) might be:

\[ \log D(t) = \log D(t - 1) + x_0 + x_1 t + x_2 r(t) + x_3 (r(t) - r(t - 1)) \]  

(14a)

and

\[ i(t) = i(t - 1) + \beta_0 + \beta_1 r(t) + \beta_2 (r(t) - r(t - 1)). \]  

(14b)

Expression (14a) models the percentage change in demand deposit balances across time. Demand deposit balances adjust according to the current market rate \((r(t))\), changes in market rates \((r(t) - r(t - 1))\), and a time trend \((t)\). The time trend is a proxy for other relevant macro-economic variables not included.
The parameter estimates obtained for \( (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2) \) would reflect local market characteristics like different consumer sensitivities to changing market rates, the level of local bank competition, merger activity, etc.

Expression (14b) models the change in demand deposit rates as a function of both the level of market rates and the change in market rates.

The specification of these dynamics for both the evolution of demand deposit balances and rates in expressions (14a) and (14b) is similar to those used by Hutchison and Pennacchi (1996), O’Brien et al. (1994), Selvaggio (1996), and the Office of Thrift Supervision (1994).

Expressions (14a) and (14b) are difference equations for the stochastic evolution of the demand deposit balances \( D_t \) and rates \( i_t \). The solutions to these difference equations are easily obtained via successive substitution. They are:

\[
D(t) = D(0) \exp \left\{ \alpha_0 t + \alpha_1 t(t+1)/2 + \alpha_2 \sum_{j=0}^{t-1} r(t-j) + \alpha_3 (r(t) - r(0)) \right\} \tag{15a}
\]

and

\[
i(t) = i(0) + \beta_0 t + \beta_1 \sum_{j=0}^{t-1} r(t-j) + \beta_2 (r(t) - r(0)) \tag{15b}
\]

These solutions clarify the dependence of both the demand deposit balances and rates on the average level of market rates, and the change in market rates.

Substitution of expressions (15a) and (15b) into the demand deposit valuation equation (6) yields a more realistic and complex net present value formula. Unfortunately, its evaluation for the discrete time model requires a numerical procedure along with an arbitrage-free specification for the evolution of the term structure of market rates \( r(t) \) (see Jarrow, 1996). In contrast, the continuous time formulation of this model in the next section admits an explicit analytic solution.

This completes the discrete-time modeling of the demand deposit liabilities.

2.6. Valuation of credit card loans

This section values credit card loans as seen by financial institutions. The value of credit card loan balances to a bank is shown to be equal to a type of exotic interest rate swap.

Due to the arbitrage opportunity implicit in expression (4), the bank may value each dollar in a credit card loan as worth more than a dollar. For simplicity, we assume that the credit card loan rate \( c(t) \) and the dollar volume of credit card loans \( L(t) \), depend only on the information set generated by the evolution of the Treasury securities. This restriction could be easily relaxed.
Under this assumption, using the risk-neutral valuation procedure, the net present value of the $L(0)$ credit card loans available at time 0 to the bank can be shown to be (see the appendix):

$$V_L(0) = \mathcal{E}_0 \left( \sum_{t=0}^{\tau-1} L(t)(c(t) - r(t))/B(t + 1) \right).$$

(16)

The net present value of the credit card loans is seen to be equal to the value of an exotic interest rate swap with amortizing/extending principal $L(t)$, receiving floating at $c(t)$ and paying floating at $r(t)$ per period.

As written, expression (16) ignores default risk on the side of the individual investors. A simple adjustment to expression (16) to incorporate defaults on the part of the individual investors is to assume that the defaults at time $t$ can be quantified via a dollar loss of $F(t)$ which depends only on the evolution of the zero-coupon bond price curve.\(^{18}\)

Then, expression (16) becomes:

$$V_L(0) = \mathcal{E}_0 \left( \sum_{t=0}^{\tau-1} L(t)(c(t) - r(t))/B(t + 1) \right) - \mathcal{E}_0 \left( \sum_{t=0}^{\tau-1} F(t)/B(t + 1) \right).$$

(17)

An example will clarify expression (17).

**Example 2** *(Random loan balances, a constant credit card loan rate and a constant default percentage)*.

$$L(t) = L + \gamma r(t),$$

(18a)

$$c(t) = c,$$

(18b)

and

$$F(t) = \alpha L(t) \quad \text{for } 0 \leq \alpha < 1.$$  

(18c)

This case corresponds to a random level of credit card loans outstanding, a constant credit card loan rate, and a constant default percentage on loans. The constant credit card loan rate and default percentage assumptions appear to be a good first approximation, see Ausubel (1991) and Calem and Mester (1995). Given this structure,

$$V_L(0) = \mathcal{E}_0 \left( \sum_{t=0}^{\tau-1} (L + \gamma r(t))(c - \alpha r(t))/B(t + 1) \right)$$

(19a)

\(^{18}\)This dollar loss $F(t)$ can be decomposed into different dollar losses for variously credit-scored borrowers. Default risk on the part of the bank could be included using Jarrow and Turnbull (1995).
Expression (19b) follows from expression (19a) via the use of expressions (1) and (11).

The second term in expression (19b) is positive (as \( r(t) \geq 0 \), \( (c - \alpha - r(t)) \geq 0 \), and \( B(t + 1) > 0 \)). Note that \( \gamma \) positive implies a positive correlation between spot rates and credit card loan balances. Thus, if \( \gamma \) is positive, then the credit card loan premium increases due to the correlation between interest rates and credit card loan balances.

By definition, the value of the credit card loan assets to the bank at time 0, denoted \( C_L(0) \), is

\[
C_L(0) = L(0) + V_L(0).
\]

This equals the initial credit card loans plus their net present value.

An estimate of the net present value in expression (17), normalized by the initial loan balances \( L(0) \), is available in Ausubel (1991). He reports the premiums paid for twenty-seven different interbank credit card loan sales between April 1994–April 1990. The average premium paid for a credit card loan account was 20% (the minimum paid was 3% and the maximum paid was 27%). This completes the discrete-time modeling of credit card loan balances.

3. Continuous time

To facilitate computer implementation, it is useful to derive closed-form solutions for the value of demand deposits (under the general specification of expression (14)) and credit card loans (as in Example 2). Closed-form solutions are useful because they: (i) speed up computation time, (ii) provide analytic expressions for hedges and parameter sensitivities and they (iii) provide "checks" for numerical procedures. The continuous time framework facilitates this derivation of closed-form solutions.

To illustrate the steps involved, this section sets up the continuous time model, and derives a closed-form solution for the stochastic evolution of demand deposit balances and rates as given in expressions (14a) and (14b). We only present the analysis for demand deposits as the analysis for credit card loans (as in Example 2) is identical. To obtain a closed-form solution, we restrict the evolution of the term structure of interest rates to follow a Gaussian process. In particular, we study a model first analyzed by Vasicek (1977), and
extended by Heath et al. (1992). However, an analytic solution could just as easily be obtained for any model where the distribution for the spot interest rate process is known.

3.1. The model

We consider a continuous time economy with trading horizon \([0, \tau]\). Otherwise, the setting is identical to that of the discrete time case. As in the discrete time case, the market segmentation argument is invoked to justify the rates paid/charged on demand deposits/credit card loans as in expressions (3) and (4).

Zero-coupon bond prices at time \(t\) with maturity at date \(T\) are denoted by \(P(t, T)\). To obtain the definition of the spot rate of interest, we first define forward rates. Continuously compounded forward rates, denoted by \(f(t, T)\), are implicitly defined by

\[
P(t, T) = \exp \left[ - \int_t^T f(t, v) \, dv \right].
\]  

The spot rate of interest, \(r(t)\), is defined by \(r(t) = f(t, t)\). The money market account’s value is given by

\[
B(t) = \exp \left[ \int_0^t r(v) \, dv \right].
\]

These are the analogues of the discrete time case.

To exclude arbitrage opportunities in the Treasury securities market and to guarantee that the Treasury securities market is complete, we assume that there exists a unique equivalent probability measure \(\tilde{Q}\) such that its conditional expectation satisfies:

\[
P(t, T) = \tilde{E}_t(P(s, T)/B(s))B(t) = \tilde{E}_t(1/B(T))B(t),
\]

for all \(0 \leq t \leq s \leq T\). That is, \(P(t, T)/B(t)\) is a \(\tilde{Q}\)-martingale. This completes the model structure.

3.2. Valuation of demand deposits

This section values demand deposits as seen by financial institutions. By analogy with the discrete time case, the net present value of the \(D(0)\) demand deposits at time 0 is given by

\[
V_D(0) = \tilde{E}_0 \left( \int_0^\tau \frac{D(t)[r(t) - i(t)]}{B(t)} \, dt \right).
\]
This is the continuous time analogue of expression (6). As before, this net present value is equivalent to an exotic interest rate swap paying floating at \( i(t) \) and receiving floating at \( r(t) \) on a random principal of \( D(t) \).

The demand deposit liability to the bank, denoted \( C_D(0) \), is \( C_D(0) = D(0) - V_D(0) \).

To obtain a closed-form solution for expression (24), we consider the continuous time analogue of expressions (14a) and (14b). The continuous time analogues of expressions (14a) and (14b) are:

\[
d \log D(t) = [\alpha_0 + \alpha_1 t + \alpha_2 r(t)] \, dt + \alpha_3 \, dr(t) \tag{25a}
\]

and

\[
d i(t) = [\beta_0 + \beta_1 r(t)] \, dt + \beta_2 \, dr(t). \tag{25b}
\]

As seen in expressions (25a) and (25b), if the interest rate process \( dr(t) \) is of unbounded variation (e.g. containing a Brownian motion), then so is the demand deposit balance evolution \( dD(t) \) and the demand deposit rate evolution \( di(t) \).

The solutions of these stochastic differential equations are:

\[
D(t) = D(0) \exp \left[ \alpha_0 t + \alpha_1 t^2/2 + \alpha_2 \int_0^t r(s) \, ds + \alpha_3 (r(t) - r(0)) \right] \tag{26a}
\]

and

\[
i(t) = i(0) + \beta_0 t + \beta_1 \int_0^t r(s) \, ds + \beta_2 (r(t) - r(0)). \tag{26b}
\]

Substitution of expressions (26a) and (26b) into the demand deposit valuation equation (24) yields:

\[
V_D(0) = \tilde{E}_0 \left( \int_0^\tau \left\{ D(0) \exp \left[ \alpha_0 t + \alpha_1 t^2/2 + \alpha_2 \int_0^t r(s) \, ds + \alpha_3 (r(t)
\]

\[
- r(0)) \left[ (r(t) - i(0) - \beta_0 t - \beta_1 \int_0^t r(s) \, ds - \beta_2 (r(t)
\]

\[
- r(0)) \right] / B(t) \right\} \, dt \right). \tag{27}
\]

\[19\] The general form of expression (25a) with \( \alpha_3 \equiv 0 \) was suggested to us by Alex Levin. In this case, one can rewrite expression (25a) as \( dD(t)/dt = D(t) \dot{\lambda}(t) \), where \( \dot{\lambda}(t) = \alpha_0 + \alpha_1 t + \alpha_2 r(t) \).

\[\]
This corresponds to a complex power swap where the principal changes randomly based on both the level \((r(t))\) and average of past market rates.

\[
\left( \int_0^t r(s) \, ds \right). 
\]

Simplifying expression (27) we see that it is composed of three integrals: 20

\[
V_D(0) = D(0) \exp \left[ -\alpha_3 r(0) \right] (1 - \beta_2) \int_0^t \exp \left[ \alpha_0 t \right] 
+ \alpha_1 \tau^2 / 2 \right] \tilde{E}_0 \left( r(t) \exp \left[ (\alpha_2 - 1) \int_0^t r(s) \, ds + \alpha_3 r(t) \right] \right) \, dt 
+ D(0) \exp \left[ -\alpha_3 r(0) \right] \int_0^t \exp \left[ \alpha_0 t + \alpha_1 \tau^2 / 2 \right] (-i(0) - \beta_0 t) 
+ \beta_2 r(0) \tilde{E}_0 \left( \exp \left[ (\alpha_2 - 1) \int_0^t r(s) \, ds + \alpha_3 r(t) \right] \right) \, dt 
- D(0) \exp \left[ -\alpha_3 r(0) \right] \beta_1 \int_0^t \exp \left[ \alpha_0 t + \alpha_1 \tau^2 / 2 \right] \tilde{E}_0 
\times \left( \int_0^t r(s) \, ds \exp \left[ (\alpha_2 - 1) \int_0^t r(s) \, ds + \alpha_3 r(t) \right] \right) \, dt. \quad (28)
\]

This formula is independent of any particular model for the evolution of spot interest rates. To obtain a closed-form solution, we need to evaluate the three integrals on the right side of expression (28). To do this, we need additional structure.

We consider the case of a Gaussian economy (see Heath et al., 1992) where spot rates evolve according to expression (29) under the martingale probability measure \(\tilde{Q}\):

\[
dr(t) = a \left[ \bar{r}(t) - r(t) \right] \, dt + \sigma \, d\tilde{W}(t), \quad (29)
\]

where \(a\) is a constant, \(\bar{r}(t)\) is a deterministic function of time, \(\sigma > 0\) is the spot rate’s volatility, and \(\tilde{W}(t)\) is a standard Brownian motion under \(\tilde{Q}\) initialized at \(\tilde{W}(0) = 0\).

---

20 The \((-1)\) in the exponent \((\alpha_2 - 1)\) comes from the \(B(t)\) in the denominator of expression (27).
Under the martingale probabilities $\tilde{Q}$, the spot rate is assumed to follow a mean reverting process with mean reversion parameter ($a$), a long run mean of ($r_t^\dagger$) and volatility parameter ($\sigma$).

As shown in Karoui et al. (1992), to match an arbitrary initial forward rate curve $\{f(0,T), 0 \leq T \leq \tau\}$, $\tilde{r}(t)$ must satisfy:

$$\tilde{r}(t) = f(0,t) + \frac{\partial f(0,t)}{\partial t} + \sigma^2(1 - e^{-2at})/2a]/a. \quad (30)$$

Expressions (29) and (30) yield the extended Vasicek model. It can be shown $^{21}$ (see Heath et al., 1992), that the solution to expressions (29) and (30) is:

$$r(t) = f(0,t) + \frac{\sigma^2(e^{-at} - 1)^2}{2a^2} + \int_0^t \sigma e^{-a(t-s)} d\tilde{W}(s). \quad (31)$$

Using the solution given in expression (31), the integrals on the right side of expression (28) can now be computed. The evaluation of these integrals is contained in the appendix. The net present value of the demand deposit liability $V_D(0)$ is:

$$V_D = D(0) \exp \left[ - \alpha_3 r(0) \right] \left( 1 - \beta_2 \right) \int_0^\tau \exp \left[ \alpha_0 t + \alpha_1 t^2/2 \right] M(t, \alpha_2 - 1, \alpha_3) \left[ \mu_2(t) + \sigma_2^2(t) \alpha_3 + \sigma_{12}(t)(r_2 - 1) \right] dt + D(0) \exp \left[ - \alpha_3 r(0) \right] \int_0^\tau \exp \left[ \alpha_0 t + \alpha_1 t^2/2 \right] M(t, \alpha_2 - 1, \alpha_3) \left[ \mu_2(t) + \sigma_2^2(t) \alpha_3 + \sigma_{12}(t)(r_2 - 1) \right] dt,$$

where

$$\mu_1(t) \equiv \int_0^t f(0,s) ds + \int_0^t \left[ \sigma^2(1 - \exp[ - a(t-s)])^2/a^2 \right] ds/2,$$

$$\sigma_1^2(t) \equiv \int_0^t \left[ \sigma^2(1 - \exp[ - a(t-s)])^2/a^2 \right] ds,$$

$^{21}$ The evolution of $r(t)$ under $Q$ is given by substituting $d\tilde{W}(t) = dW(t) - \phi(t) dt$, where $\phi(t)$ is the market price of risk.
\[ \mu_2(t) = f(0, t) + \sigma^2(1 - \exp[-at])^2/2a^2, \]
\[ \sigma_2^2(t) = \int_0^t \sigma^2 \exp[-2a(t-s)] \, ds, \]
\[ \sigma_{12}(t) = \sigma^2(1 - \exp[-at])^2/2a^2, \]
\[ M(t, \gamma_1, \gamma_2) = \exp[\mu_1(t)\gamma_1 + \mu_2(t)\gamma_2 + \sigma_1(t)\gamma_1^2 + 2\sigma_{12}(t)\gamma_1\gamma_2 + \sigma_2(t)\gamma_2^2]/2. \]

The cost of demand deposit liability is then \( C_D(0) = D(0) - V_D(0). \)

Given this analytic expression for \( C_D(0), \) one can easily hedge the demand deposit liability using a single zero-coupon bond, say \( P(0,T). \) This follows because every zero-coupon bond can be synthetically constructed using \( P(0,T) \) in a one-factor model. Risk management measures can now be computed using the standard techniques from the interest rate derivatives literature. The duration and convexity measures determined from these calculations are consistent with an arbitrage-free and stochastic evolution of the term structure of interest rates.

To implement this model in practice, the parameters \((a, \sigma^2)\) of the extended Vasicek model (29), the coefficients \((\alpha_0, \alpha_1, \alpha_2, \alpha_3)\) of the demand deposit balance evolution equation (25a), and the coefficients \((\beta_0, \beta_1, \beta_2)\) of the demand deposit rate evolution equation (25b) need to be estimated. The estimation of the extended Vasicek model’s parameters can follow standard procedures (see Heitmann and Trautmann, 1995). The estimation of the demand deposit balance and rate evolution parameters can be obtained via a regression analysis of expressions (14a) and (14b) as in O’Brien et al. (1994) and Selvaggio (1996). This estimation would reflect local market characteristics in the firm-specific parameters obtained. The empirical implementation of this model awaits subsequent research.

4. Conclusion

Using a segmented market argument, this paper shows how banks should price and hedge demand deposit liabilities and credit card loans. Demand deposit liabilities and credit card loans are seen to be equivalent to a special type of exotic interest rate swap with amortizing/extending principal.

The techniques in this paper provide a consistent methodology for the valuation and hedging of all of a bank’s assets and liabilities. This new asset/liability risk management tool can be used both to quantify and to control a bank’s interest rate risk exposure. In conjunction with an analysis of credit risk as in Jarrow and Turnbull (1995), it can be used by either a bank or its regulators to determine the adequacy of a bank’s capital reserves.
Acknowledgements

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Appendix A

A.1. Derivation of expressions (5) and (6)

The cash flows to the demand deposit are given in Table 1. Thus, we get expression (5):

\[ V_D(0) = \mathcal{E}_0 \left( D_0 + \sum_{t=0}^{\tau-2} \left[ \frac{D(t+1) - D(t)}{B(t+1)} \right] - \frac{D(\tau-1)}{B(\tau)} \right) \]

\[ - \mathcal{E}_0 \left( \sum_{t=0}^{\tau-1} \frac{i(t)D(t)}{B(t+1)} \right). \]

Adding and subtracting like terms gives

\[ V_D(0) = \mathcal{E}_0 \left( \frac{D(0)(1 + r(0))}{B(1)} + \sum_{t=0}^{\tau-2} \frac{D(t+1)(1 + r(t+1))}{B(t+2)} \right) \]

\[ - \sum_{t=0}^{\tau-1} \frac{D(t)}{B(t+1)} - \sum_{t=0}^{\tau-1} \frac{i(t)D(t)}{B(t+1)} \right), \]

\[ V_D(0) = \mathcal{E}_0 \left( \sum_{t=0}^{\tau-1} \frac{D(t)(1 + r(t))}{B(t+1)} - \sum_{t=0}^{\tau-1} \frac{D(t)}{B(t+1)} - \sum_{t=0}^{\tau-1} \frac{i(t)D(t)}{B(t+1)} \right), \]

\[ V_D(0) = \mathcal{E}_0 \left( \sum_{t=0}^{\tau-1} \frac{D(t)(r(t) - i(t))}{B(t+1)} \right). \]

This is expression (6). This completes the derivation.

Table 1

<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>(\tau - 1)</th>
<th>(\tau)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>+D(0)</td>
<td>−D(0)i(0)</td>
<td>−D(1)i(1)</td>
<td>...</td>
<td>−D((\tau - 2))i((\tau - 2))</td>
<td>−D((\tau - 1))i((\tau - 1))</td>
</tr>
<tr>
<td></td>
<td>−D(0)</td>
<td>D(1)</td>
<td>...</td>
<td>−D((\tau - 2))</td>
<td>−D((\tau - 1))</td>
<td>−D((\tau - 1))</td>
</tr>
<tr>
<td></td>
<td>+D(1)</td>
<td>+D(2)</td>
<td>...</td>
<td>+D((\tau - 1))</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A.2. Derivation of expression (11)

\[ \tilde{E}_0 \left( \sum_{t=0}^{T-1} r(t)/B(t+1) \right) \]

\[ = \tilde{E}_0 \left( \sum_{t=0}^{T-1} \frac{1+r(t)}{B(t+1)} \right) - \tilde{E}_0 \left( \sum_{t=0}^{T-1} \frac{1}{B(t+1)} \right) \]

\[ = \tilde{E}_0 \left( \sum_{t=0}^{T-1} \frac{1}{B(t)} \right) - \sum_{t=0}^{T-1} P(0, t+1) \]

\[ = 1 + \sum_{t=0}^{T-2} P(0, t+1) - \sum_{t=0}^{T-1} P(0, t+1) \]

\[ = 1 - P(0, \tau). \]

This derivation uses expression (1) and the fact that \( B(t) = B(t-1)(1+r(t-1)) \). This completes the derivation.

A.3. Derivation of expression (16)

To obtain expression (16), use the proof of expression (6) with

\[ D(t) \equiv -L(t) \quad \text{and} \quad i(t) \equiv c(t). \]

This completes the derivation.

A.4. Derivation of expression (32a)–(32c)

From expression (31) we can write

\[ r(t) = f(0, t) + b(0, t)^2/2 + \int_0^t \rho(s, t) \, d\tilde{W}(s) \equiv x_2, \quad \int_0^t r(s) \, ds \equiv x_1, \]

where

\[ \rho(s, t) \equiv \sigma \exp[-a(t - s)], \]

\[ b(s, t) \equiv \int_s^t \rho(s, v) \, dv = \sigma(1 - \exp[-a(t - s)])/a. \]

For computation, we need to rewrite \( \int_0^t r(s) \, ds \).

\[ \int_0^t r(s) \, ds = \int_0^t f(0, s) \, ds + \int_0^t b(0, s)^2 \, ds/2 + \int_0^t \int_0^s \rho(v, s) \, d\tilde{W}(v) \, ds. \]
It can be shown, by direct computation, that
\[ \int_0^t b(0,s)^2 / 2 \, ds = \int_0^t b(s,t)^2 / 2 \, ds. \]

Furthermore,
\[ \int_0^s \int_0^t \rho(v,s) \, d\tilde{W}(v) \, ds = \int_0^t \int_0^t \rho(v,s) \, d\tilde{W}(v) = \int_0^t b(v,t) \, d\tilde{W}(v). \]

Substitution yields
\[ \int_0^t r(s) \, ds = \int_0^t f(0,s) \, ds + \int_0^t b(s,t)^2 / 2 \, ds + \int_0^t b(s,t) \, d\tilde{W}(s). \]

So, \((x_1, x_2)\) is bivariate normal with
\[
\begin{align*}
\mu_2 &= f(0,t) + b(0,t)^2 / 2, \\
\sigma_2^2 &= \int_0^t \rho(s,t)^2 \, ds, \\
\mu_1 &= \int_0^t f(0,s) \, ds + \int_0^t b(s,t)^2 / 2 \, ds, \\
\sigma_1^2 &= \int_0^t b(s,t)^2 \, ds, \\
\sigma_{12} &= \int_0^t \rho(s,t) b(s,t) \, ds = b(0,t)^2 / 2.
\end{align*}
\]

From Hogg and Craig (1970), the moment generating function of \((x_1, x_2)\) is:
\[
M(\gamma_1, \gamma_2) \equiv \tilde{E}_0[\exp[\gamma_1 x_1 + \gamma_2 x_2]] = \exp[\mu_1 \gamma_1 + \mu_2 \gamma_2 + \sigma_1^2 \gamma_1^2 / 2 + 2 \sigma_{12} \gamma_1 \gamma_2 + \sigma_2^2 \gamma_2^2] / 2.
\]

Thus,
\[
\begin{align*}
\tilde{E}_0[x_1 \exp[\gamma_1 x_1 + \gamma_2 x_2]] &= \frac{\partial M(\gamma_1, \gamma_2)}{\partial \gamma_1} = M(\gamma_1, \gamma_2) [\mu_1 + \sigma_1 \gamma_1 + \sigma_{12} \gamma_2], \\
\tilde{E}_0[x_2 \exp[\gamma_1 x_1 + \gamma_2 x_2]] &= \frac{\partial M(\gamma_1, \gamma_2)}{\partial \gamma_2} = M(\gamma_1, \gamma_2) [\mu_2 + \sigma_2 \gamma_2 + \sigma_{12} \gamma_1],
\end{align*}
\]
For later use, we note that

\[ P(0, t) = \tilde{E}_0(e^{-\lambda t}) = M(-1, 0). \]

Thus,

\[
\tilde{E}_0 \left( r(t) \exp \left[ (\alpha_2 - 1) \int_0^t r(s) \, ds + \alpha_3 r(t) \right] \right)
= M(\alpha_2 - 1, \alpha_3) \left[ \mu_2 + \sigma_2^2 \alpha_3 + \sigma_{12}(\alpha_2 - 1) \right],
\]

\[
\tilde{E}_0 \left( \exp \left[ (\alpha_2 - 1) \int_0^t r(s) \, ds + \alpha_3 r(t) \right] \right) = M(\alpha_2 - 1, \alpha_3),
\]

\[
\tilde{E}_0 \left( \int_0^t r(s) \, ds \exp \left[ (\alpha_2 - 1) \int_0^t r(s) \, ds + \alpha_3 r(t) \right] \right)
= M(\alpha_2 - 1, \alpha_3) \left[ \mu_1 + \sigma_1^2 (\alpha_2 - 1) + \sigma_{12} \alpha_3 \right].
\]

Substitution gives the result in expression (32a)–(32c). This completes the derivation.

References


Office of Thrift Supervision, Department of the Treasury, 1994. The OTS Net Portfolio Value Model.
