The HJM Model: Its Past, Present, and Future

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I. INTRODUCTION

When I was thinking about what to discuss in this address, my mind wandered across many topics. I first thought about discussing martingale probability measures and the topologies of asset pricing. I quickly discarded this topic for obvious reasons (and one not so obvious). The not so obvious one is that I was asked to avoid using any equations. I next thought about preaching on the evils of “value at risk,” but learned that Steve Ross had talked about this in the past. One doesn’t want to imitate a master. So, I rejected that topic. By then I was desperate. To illustrate how much so, I even thought about a banking topic—asset/liability management. I rejected this as well because after lunch, there is always the danger of putting everyone to sleep.

Then, an inspiration hit. I remembered when I was a graduate student in the late 70s at MIT, how much I enjoyed listening to Bob Merton and Fischer Black talk about the historical development of the Black–Scholes formula.1 Who did what, and when? What were the stumbling blocks in the development, in particular, solving the Black–Scholes partial differential equation. To this day, I still like recounting those stories to my students.

So, I thought I would share my insights and recollections on the development process of the Heath–Jarrow–Morton term structure model2 with you. In the process, I will take advantage of the historic progression to discuss the salient issues regarding its derivation and implementation. In addition, I will also discuss its extensions to foreign currency and credit derivatives. Finally, I will share my predictions with you on the future of research in this area, and on the future of the field of financial engineering itself.

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II. THE PAST

I guess it all started, in some sense, way back when—in my graduate program at MIT. I did my thesis on the term structure of interest rates and the expectations hypothesis. My thesis was partly an empirical investigation showing that the expectations hypothesis does not hold. This is a topic directly related to the HJM model.

The Motivation

About seven years later, my colleague in the OR&IE department at Cornell—David Heath—and a student of ours—Andrew Morton—were writing some software for David's friend (and consulting client) which was to implement the discrete time Ho and Lee model. To understand the Ho and Lee model in more detail, they wanted to study its continuous time limit—which is, of course, what the discrete time model is approximating. David asked me if I would like to join them in investigating this problem. I did.

To explain our strategy, let me go back 10 years to the state of the art of pricing interest rate derivatives. At that time, the interest rate derivatives pricing technology suffered from a number of limitations. These limitations were two.

One, all existing term structure models required estimates of the market price of risk (i.e., the expected excess return to a risk factor per unit of risk). This is equivalent to estimating the expected return on a bond—a difficult, if not impossible task. To show just how difficult a task this is, the finance profession is still unsure how to estimate expected returns for equities, and it has been estimating asset pricing models for nearly 35 years. This dependence on the market price of risk was unlike the Black–Scholes model, which only requires an estimate of the volatility, and this made the existing term structure models difficult to use.

Two, all the existing term structure models could not exactly match an arbitrary initial zero–coupon bond price curve (zero curve). This meant that there were arbitrage opportunities across zeros as implied by the existing models, prior to pricing derivatives. At the time, this included the Vasicek (1977) and Brennan and Schwartz (1979) models. This same criticism was also partially true for the Cox, Ingersoll, Ross (1985) model. Although similar in structure to the others, it differed only because they outlined in their paper how to overcome this limitation by inverting the model's parameters to match the initial zero curve. Unfortunately, this inversion is not always possible in their model, and when possible, it is computationally intensive.

Besides these limitations, David, Andrew, and I were also aware that the Brownian bridge process used by Ball and Torous (1983) was inconsistent with the absence of arbitrage. We realized that the key issue in the specification of a term structure model, when one takes as given the stochastic process for the entire
zero curve, is an arbitrage free evolution. This was a simple, but powerful insight. The technique we wanted to use to characterize the arbitrage free evolution was the then recently developed, martingale measure technology.

So, we set out on our investigation.

The Mathematics
The first mathematical problem we faced was how to specify (uniqueness and existence of) a stochastic differential equation for the zero-coupon bond prices for which \( P(t, T) > 0 \) and \( P(T, T) = 1 \) (where \( P(t, T) \) is the time \( t \) price for a sure dollar delivered at time \( T \)). This was the issue that the Brownian bridge process was trying, but failed, to overcome.

The solution to this boundary problem was easy. It was to use either the forward rates or the bond’s yield as the exogenously given process. The relation of yields or forward rates to the bond’s price (an exponential of an integral across time) guaranteed that both conditions for the zeros are satisfied simultaneously.

How to decide between forward rates and yields? Looking at the deterministic problem convinced us that forward rates would generate the simplest solutions. It can be easily shown (in fact it is very intuitive) that in a deterministic economy, no arbitrage implies that the realized spot rate at time \( T \) must be equal to forward rate \( f(t, T) \) from any earlier time \( t \), i.e., \( f(t, T) = f(t_2, T) = r(T) \) for all \( t, t_2 \leq T \) (where \( f(t, T) \) is the time \( t \) forward rate for date \( T \) and \( r(t) \) is the time \( t \) spot rate). This gives a constant—fixed—evolution. The no arbitrage relation for yields is not as simple. So, forward rates it was.

Finally, we wanted non-negative forward rates, or \( P(t, T) \leq 1 \). Why? In an economy with money (currency), negative rates on zeros would imply an arbitrage opportunity—sell zeros and store currency. Since the economy we live in trades currency, non-negative forward rates are desired.

Drawing an analogy to the Black–Scholes model, we wanted to first study forward rates which were lognormally distributed. So, the next step was to find conditions on lognormally distributed (instantaneous) forward rates such that there exists an equivalent martingale probability measure. This turned out to be a frustrating process. We were convinced at the start that such a measure must exist. But, we could not find an existence proof. This process lasted for about six months. I remember one time I had a proof which was valid, as long as a particular “boundedness” type condition was satisfied. David then showed, to my disappointment at the time, that the hypothesis was in fact violated by the lognormal process. (By the way, this proof provided the blueprint for the basic existence theorem's proof for no arbitrage found in the original *Econometrica* paper.)

This was the proverbial “straw that broke the camel’s back.” Seeing this, we changed our conjecture, and then looked for a nonexistence proof. We discovered one the next day. It was easy to show that since the drift exploded in finite time,
the stochastic forward rate process must as well. This implies, of course, that lognormally distributed (instantaneous) forward rates are inconsistent with arbitrage-free bond prices! This was a surprise!

Having spent six months studying the problem, once we could prove the nonexistence of a martingale measure for the lognormal case, we understood the mathematical problem in its entirety. We had the general mathematical solution.

The mathematical solution was simple. No arbitrage in the evolution of the forward rate curve could be guaranteed (necessary and sufficient) by ensuring that the drifts (in the risk-neutral transformed economy) were a specific function of the volatilities. This drift adjustment is now sometimes called "the convexity adjustment" in the Eurodollar markets when used to construct Eurodollar spot zero curves from Eurodollar futures market prices.

This drift adjustment led to an unexpected complication. In studying the model, we became aware that the form of this drift adjustment causes the most common processes with non-negative rates to be path dependent (or, to explode). That is, the price of zeros depend on the path of forward rates from which they are attained. In a tree, this means that an up node followed by a down node leads to different bond prices than a down node followed by an up node. This gives a bushy tree as opposed to a lattice. These bushy trees grow exponentially in the number of steps, i.e., for large step sizes, bushy trees become too time intensive to compute.

Computation

Therefore, we were not done. We still had to convince ourselves, as well as the academic and professional communities, that one could compute with the HJM model for realistic stochastic processes in reasonable times. David and Andy, whose expertise is also in computer science, went at the computer programming aspect. In parallel, I started writing up our research, trying to get it published, and looked at obvious extensions (the foreign currency extension, commodities pricing, and credit risk). More on these extensions later.

David and Andy came up with an efficient code for computing with a bushy tree. The code used "standard," but clever, computer tricks to get accurate answers with minimal memory requirements and with only a few steps of a bushy tree. In fact, the technique was so accurate, that for European options, 10 steps had better accuracy than a 100-step binomial tree!

This accuracy is best understood by realizing that because of martingale pricing, derivative valuation is equivalent to computing an expectation. In a lattice, after 100 steps, there are 101 outcomes or nodes. In a 10-step bushy tree, there are $2^{10} = 1024$ nodes. The 10-step bushy tree gives a much finer grid for computing expected values. Hence, its increased accuracy. David and Andy had shown that computing with the HJM model was feasible for most interest rate derivatives. The bushy tree approach only starts having difficulties with long
dated options of the American type, when there are numerous decision nodes (≥ 15). In this case, there are four alternatives: (i) to use bigger, faster computers (possibly in parallel), (ii) to use special cases of the model where lattices arise, (iii) to use special cases of the model where the the evolution is strong Markov and partial differential equation methods apply, or (iv) to use Monte Carlo simulation. For the unrestricted evolutions and smaller computers, Monte Carlo procedures appear the most promising of these four alternatives.

Extensions
While David and Andy were working on computation, I studied extensions of HJM to related problems. With my student, Kaushik Amin, I applied HJM to equities and foreign currencies. It was a straightforward extension, with one twist. In multiple term structures, the arbitrage free evolution requires the link between the various term structures, the spot exchange rate, also to have a restricted drift. The restriction is easy to impose, involving both the domestic and foreign spot rate of interests.

Another student, Robin Brenner, under my direction carefully looked at the Vasicek model and showed how to embed it within the HJM framework. (The CIR model embedding was done in the original paper.) This demonstrated that the existing models, properly interpreted, could be seen as special cases of the HJM model. This implied that the HJM model was a unifying framework for understanding all the existing term structure models. This was an important observation.

Shortly thereafter, Stuart Turnbull and I studied credit risk. We discovered that the solution to pricing credit derivatives was to use an FX analogy. What is the FX analogy? It’s easy to explain. Consider the risky debt of XYZ company, say, a zero coupon bond. XYZ’s debt can be thought of as default free in a hypothetical currency, its promised dollars. The promised dollars gives the foreign currency term structure. At the date of payment, however, these promised dollars may not be worth an actual dollar. There is an exchange rate, which is one if XYZ is not in default, or less than one otherwise. Combined with the term structure of government bonds, we get the FX analogy. With this analogy, it is easy to see that the mathematics is the same as a foreign currency derivatives pricing problem. Hence, the mathematical problem is solved. As an aside, practical implementation of this technology is an exciting and “hot” area of current research.

Concurrent with our continuing research (as described above), other researchers began studying special cases of HJM, where computation could be done either analytically or using lattices. Gaussian models were heavily studied. This class of models results in lattices, but have negative forward rates possible. Nonetheless, these are a useful special case. The extended Vasicek model lies in this class.
A lognormal process for discrete rates model (in contrast to continuously compounded rates) was recently developed. Unfortunately, it only gives analytic solutions for the particular discrete rate selected. Derivatives based on other discrete rates and/or the continuously compounded rates must be computed numerically. (It gives the Black formula for one Eurodollar rate, but not for others.) Again, a useful special case.

Why do I say a "special" case? My personal feeling with respect to these examples is that for consistent pricing across a wide range of derivatives, these special cases are of more short-term than long-term interest. Why? Because, in applied contexts, it is essential to let the evolution specify the appropriate process, and not computing and/or analytic efficiency. Hedging is the key concern in risk management (and pricing). To hedge well, one must understand how prices actually move, not how we want them to! In the short term, however, because of computing constraints, these special cases have their place.

III. THE PRESENT

Moving to the present, as the OTC derivatives markets have expanded since 1987 and computer technology has improved, the HJM model is more and more the natural choice for pricing interest rate sensitive securities. Why? Because, it is an all-purpose model. It has three characteristics that enables it to be used in almost any situation.

One, it can price arbitrary interest rate derivatives consistently, using an arbitrage free term structure which does not explicitly depend on the market price of risk; only the volatilities of the forward rate process need to be estimated. This is analogous to Black–Scholes.

Two, it can work with any given (finite dimensional) term structure evolution. It need not even have continuous sample paths. It can incorporate multiple factors and empirically based evolutions. The data describe the process used for valuation and not analytic convenience and/or computer efficiency. This is important, of course, for accurate hedging.

Three, it is what I call a "lego building block" technology. It can be easily built upon to add multiple term structures (foreign currencies, credit risk), other commodities, even stochastic volatilities. Hence, it won't become obsolete as risk management practices expand and eventually necessitate the consistent pricing across a firm's various trading desks. Why is consistent pricing essential? One reason is that different models at different trading desks within the same firm can lead to third parties arbitraging across the firm's trading desks! A scary thought. A second reason is that for incentives, rewards, and risk management one wants consistency across groups. This promotes internal equity, less gaming the system, and allows better management control, since decisions will have the same impact across areas (from a risk perspective).
Let me digress for a moment to discuss two related asides. One is with respect to the volatility structure of the forward rate curve, and the second is with respect to curve fitting.

First, consider the observed volatility structure of forward rates. Fortunately, the volatility structure (through principal components estimation) appears to be reasonably stable across time. The evidence, roughly translated for Treasury curves, is that there appear to be three important and stable factors across time. The first is a parallel shift of the forward rate curve, with the short end being slightly more volatile. This factor explains about 75–85 percent in the variation of changes in forward rates across time. The second factor is a shift of the slope of the forward rate curve. The short and long end can have different signs for changes in rates. The second factor brings the explanatory power up to between 85–95 percent. The third factor is a bending of the forward rate curve. The long and short end can move differently from the middle segment. These three factors combined explain between 95–99 percent of the variation. Surprisingly, these same three factor forms also appear in Eurodollar curves, copper futures price curves, and oil futures price curves. An open question, in my mind, is why these same three factor forms appear in so many commodity price evolutions.

Second, consider curve fitting (calibration). There is a dangerous and recent trend in derivatives research (especially equity options) to focus on curve fitting. What do I mean by this? Choosing some (too) simple volatility structure and implicitly fitting its parameters to match an initial curve of cap (call option) prices. This is done to calibrate the model’s parameters to the market. It provides an implicit estimate, often thought to be a better estimator than a historic estimate (obtained using historic data). The danger is that curve fitting may give the false impression of accurate hedging. After all, if you can match the market prices for a term structure of caps, won’t it also hedge caps well? The answer is no.

Why? The reason is that if there is a two–factor model operating, then using a more general 1 factor model will still not create a perfect hedge. My own view is that we should be concentrating more on hedging than fitting an initial cap curve. Again, the way out of this “mess” is to let the actual evolution of the term structure drive the valuation model, because then the historic volatilities will better match the implicit or calibrated volatilities. In this case, the two approaches coincide. Many researchers are aware of this danger, and avoid it, but there are also many who are not.

**IV. THE FUTURE**

This brings me to my speculations on the future and ends my reminiscing about the HJM model (except for the war stories regarding publication which I save for conversations over a drink sometime). In this regard, I have two forecasts. One is about a relatively unexplored, but important research area in financial
engineering that I believe represents the next major trend in research focus. The other is about the future of the field itself.

For the next major trend in research focus I have two words to share—"liquidity risk." I define liquidity risk as a quantity impact of trades on the price. Alternatively stated, demand curves for financial securities are downward sloping. The more you buy, the larger the price paid. The more you sell, the lower the price received. This quantity impact can be state and trader dependent. It can depend on the information the trade reveals, or the absence of readily available supplies or demands. I believe that the evidence is overwhelming that it exists. Quantifying liquidity risk is an important missing component in our understanding of the pricing and hedging of derivatives. We need to understand it better. I illustrate its importance for pricing interest rate derivatives with two examples.

Liquidity risk is important for computing value at risk (VAR). I have written a research paper which demonstrates that the current recommended adjustment to VAR to account for liquidity risk—increase the horizon of the interval over which VAR is computed—is incomplete. One needs to include at least two additional modifications: an initial adjustment to incorporate the price discount needed to move a large position, and an increase in the standard deviation of the portfolio's return to account for the volatility of the price discount itself. For reasonable structures, simple analytic formulas are available.

The second example of liquidity risk's importance is in the pricing of Treasuries and Treasury derivatives. Illiquidity is important for constructing yield curves; we have known this for quite some time because of the off-the-run and on-the-run classifications. But, these illiquidity are also important for pricing as well, because they can generate convenience yields in Treasuries, analogous to convenience yields for the storage of crude oil. This is due to the repo markets and the Treasury auction mechanism.

Convenience yields change prices. Indeed, it's easy to see using the standard—forward/price parity theorem—that Treasury forward prices would change. Then, it is an easy step to believe that Treasury futures prices would change as well. If Treasury futures prices change, so would Treasury futures option prices, and so forth.

Besides the pricing of Treasuries and Treasury derivatives, these convenience yields are also important for understanding the relation between Treasuries and Eurodollar deposits. Eurodollar deposits do not have similar convenience yields, as they are not subject to the same auction mechanism as are Treasuries. The absence of convenience yields in Eurodollar deposits drives a wedge between the two term structures, partially explaining the TED spread.

These two examples are just the proverbial "tip of the iceberg." With illiquidity, the theory for pricing and hedging derivatives is completely modified. Why? Because with illiquidity, the notion of arbitrage changes as there are now differences between real and paper wealth. Because with illiquidity, market
manipulation becomes a real consideration, e.g., front running a large order. Because with illiquidities, option prices are affected by a large trader’s anticipated future position in the underlying, e.g., immediately before an option’s expiration. Because with illiquidities, asset pricing models and the notion of “risk” need to be augmented, e.g., there can be another beta reflecting a large trader’s position. We know only a little about risk management with illiquidities, and I’d like to know more.

Now for my forecast concerning the future of financial engineering. When I graduated from MIT in 1979, the conventional wisdom at the time was that option pricing theory was a “dead” research area. Given that the Black–Scholes formula was discovered, it was believed that there was nothing else to do in the area of derivatives. Recent graduates inform me that the same conventional wisdom applies today with respect to the HJM model. This is nonsense! It was nonsense in 1979 and it is nonsense now. The field has only just passed another critical stage, as it did with the Black–Scholes model. There are a plethora of open and unanswered questions (e.g., illiquidity risk). In fact, financial engineering is perhaps the most vibrant and active of all research areas. I expect this to continue.

Now for my prediction. It is widely believed that financial engineering is the field where economists have had the most direct impact on practice. I agree with this belief. Because of this impact, in the not too distant future, I predict that we will see finance departments in business schools transforming to departments of financial engineering. In fact, this is already happening at some business schools today. In contrast to basic corporate finance, financial engineering will be the mainstay of our teaching and students’ demand. The core course in finance curriculums (and textbooks) will correspondingly change their emphasis to derivatives. Let’s wait and see. We really don’t have any other choice. Or do we?

NOTES

1. See Black and Scholes (1973) and Merton (1973).
6. This assertion was proven in Heath, Jarrow, Morton (1992).
8. See Morton (1988) for this proof.
15. See Jarrow and Turnbull (1997).
17. See Jarrow (1994).

REFERENCES


