Mopping up Liquidity

Robert Jarrow and Ajay Subramanian present
a quantitative treatment of liquidity
considerations for value-at-risk

State-of-the-art techniques for calculating value-at-risk, such as JP Morgan’s RiskMetrics or Hull (1997), rigorously include both market and credit risk; i.e., they are based on sound economic theories. However, this is not true for the quantification of liquidity risk. Liquidity risk is included in standard VAR computations only in an ad hoc fashion. The ad hoc adjustment increases the time horizon over which VAR is calculated to account for the time necessary to liquidate a large position. Although it is sensible to account for this, the adjustment is not based on an economic model and could consequently omit important considerations, leading to an underestimation of true loss levels.

The purpose of this article is to provide a rigorous method for quantifying liquidity risk and incorporating it into VAR (as detailed in Jarrow & Subramanian, 1997). Our method is based on sound economic theories and, fortunately, is both intuitive and simple to use. Most importantly, this modified VAR calculation confirms that current procedures systematically underestimate the potential losses due to liquidation. Such procedures exclude three quantities that tend to increase the loss level: a liquidity discount, the volatility of the liquidity discount and the volatility of the time horizon to liquidation.

We study the problem faced by a firm (or trader) that must liquidate an investment portfolio over some fixed horizon, i.e., how to determine the optimal liquidation policy which, in turn, determines the liquidation value of the portfolio. What makes this problem difficult, and different from standard theory, is that large sale quantities can have a negative impact on the price received and can also increase the time taken to execute an order. Standard theories such as the Black-Scholes formula assume that the trader can liquidate an investment portfolio, no matter how large, at the market price, and can do so without execution lags.

Let’s first take a simple version of the trader’s liquidation problem. Consider a trader who at time 0 has a portfolio of S shares of a risky security that needs to be liquidated by time T. The trader’s sales decrease the price received and there is an execution lag in filling his orders. We will consider no other explicit transaction costs (out-of-pocket commissions). For discounting purposes, a money market account with return r per unit time is available. The trader is assumed to be risk-neutral (this is one of the conditions that can be easily relaxed).

The risky asset will be called a “stock”. The “market price” for the stock is defined as the quote for one round lot, a unit sale (or purchase). Let p(t) denote the stock’s market price at time t. We assume that between trades, the stock’s price follows a geometric Brownian motion, i.e:

\[ dp(t) = p(t) \alpha dt + \sigma dW(t) \]  

where \( \alpha \) and \( \sigma \) are constants, and \( W(t) \) is a standard Brownian motion. The phrase “between trades” describes the situation when the trader is not in the market.

When the trader sells \( S \) shares at time \( t \), given a market price of \( p(t) \), the price he receives, per share, is:

\[ c(s)p(t) \]

where:

\[ 0 \leq c(s) \leq 1 \]

The coefficient \( c(s) \) represents a quantity discount. It is assumed to be non-increasing in \( S \), that is:

\[ c(s_1) \leq c(s_2) \quad \text{for} \quad s_1 \geq s_2 \]

This implies that as the number of shares sold increases, so does the quantity discount (or it may remain constant). The quantity discount \( c(s) \) is also random and (for simplicity) independent of the market price process \( p(t) \). It is random because the size of the discount can be unknown prior to the trade.

In addition, given that the trader’s sell order is placed at time \( t \), we will let it be executed only at time \( t + \Delta(s) \) where \( \Delta(s) \) is an execution lag. The execution lag is assumed to be non-decreasing in \( S \), that is:

\[ \Delta(s_1) \geq \Delta(s_2) \quad \text{for} \quad s_1 \geq s_2 \]

The larger the sales, the longer the time to execute, when everything else is held constant. The execution lag is random and (for simplicity) independent of the market price process \( p(t) \) and the quantity discount \( c(s) \).
It is random because prior to the trade, the exact time for its completion is unknown.

To ensure that liquidation has a cost, we also impose the condition that:
\[
\alpha(s) \exp \left( \alpha - r \right) \Delta(s) \leq 1 \quad \text{for all } s
\]  
(5)

This states that the combined impact of the quantity discount from selling $s$ shares and the appreciation on the stock prior to execution (discounted to the present) is less than or equal to one. That is, there is a loss in value to selling $s$ shares. The effect of the shares sold on the market price is cumulative. This is captured by imposing the condition that after a sale is executed, the new market price begins at a magnitude determined by the quantity discount, i.e.,
\[
p(t^*) = c(s)p(t)
\]  
(6)

where $t^*$ means an instant after time $t$.

This completes our description of the market structure. We must now consider the trader's liquidation problem, given the market structure previously defined. As already stated, the trader wants to liquidate $S$ shares over the horizon $0$ to $T$. He can sell the shares however he wishes, either as a block or slowly in smaller quantities.

Formally, the trader sells shares using a trading strategy, defined as a collection of dates and shares sold, $(t_i, s_i)$ for $i = 1, \ldots, n$ such that:
\[
0 \leq t_1 + \Delta(s_1) \leq t_2 + \Delta(s_2) \leq \cdots \leq t_n \leq T \quad \text{and}
\]
\[
0 \leq s_i \quad \text{for all } i \quad \text{where } s_1 + s_2 + \cdots + s_n = S
\]  
(7)

The trader is allowed to sell the $S$ shares in a block or individually, depending on the market conditions. But all shares must be brought to the market by the last trading time $T$. The last trade can take place at $T$; if so, it will be executed at time $T + \Delta(s_i)$. The times at which sell orders are placed must increase; and the previous sale must be executed before a new sale is offered.

The trader's liquidation problem can now be stated. It is to choose a trading strategy $(t_i, s_i)$ for $i = 1, \ldots, n$ to maximise:
\[
E_0 \left\{ \sum_{i=1}^{n} c(s_i)p(t_i, \Delta(s_i)) \exp \left[ -\left[ t_i + \Delta(s_i) \right] \right] \right\}
\]  
(8)

where:
\[
p(t + \Delta(s_i)) = p(t) \exp \left[ \frac{\alpha - \sigma^2}{2} \Delta(s_i) + \sigma \left( W(t_i) + \Delta(s_i) \right) - W(t_i) \right]
\]

This represents the discounted proceeds from the sale of the $S$ shares by time $T$. To see this, note that the proceeds received from the sale of $s_i$ shares initiated at time $t_i$ is $s_i c(s_i)p(t_i, \Delta(s_i))$. This is the number of shares multiplied by the price received per share at the execution time (the quantity discount is included). The trader’s goal is, of course, to maximise his proceeds from the sales.

To understand the solution to the trader’s liquidation problem, we first solve it when there is no liquidity risk (Arrow & Leland, 1983). The liquidation problem without liquidity risk is to choose a trading strategy $(t_i, s_i)$ for $i = 1, \ldots, n$ to maximise:
\[
E_0 \left\{ \sum_{i=1}^{n} s_i p(t_i) \exp \left[ -\alpha - \sigma^2 / 2 \right] \right\}
\]  
(9)

where:
\[
p(t_i) = p(0) \exp \left[ \frac{\alpha - \sigma^2}{2} t_i + \sigma (W(t_i) - W(0)) \right]
\]

The difference between this maximisation problem and the one in expression (8) is that the condition in (9) contains no quantity impact on the price and there is no execution lag in selling $s_i$ shares.

Let $u^*(p, S)$ represent the maximum discounted proceeds received from the sale of $S$ shares when the current market price is $p$. This value is determined by solving for the optimal liquidation policy in expression (9). It can be shown that the optimal liquidation strategy, given no liquidity risk, is a block sale. More formally, the solution to the liquidation problem can be described as follows:

If $[\alpha - \sigma^2 / 2] t + \sigma (W(t) - W(0)) \leq 0$, then immediate liquidation is optimal, with proceeds:
\[
u^*(p, S) = S p
\]

If $[\alpha - \sigma^2 / 2] t + \sigma (W(t) - W(0)) > 0$, then terminal date liquidation is optimal, with proceeds:
\[
u^* (p, S) = S p \exp \left[ \frac{\alpha - \sigma^2}{2} T + \sigma (W(T) - W(0)) \right]
\]

If the appreciation on the stock after discounting is non-positive, then it pays to liquidate immediately. In this case, the proceeds are just the market price multiplied by the number of shares sold, $S$. Alternatively, if the appreciation on the stock after discounting is positive, then it pays to wait to liquidate. Here, the proceeds from the sale are the discounted value of selling all $S$ shares at time $T$.

This result shows that when there is no liquidity risk, marking to market $S p$ always provides the proper liquidation value of a portfolio. The market price multiplied by the number of shares sold, $S$. Alternatively, if the appreciation on the stock after discounting is positive, then it pays to wait to liquidate. Here, the proceeds from the sale are the discounted value of selling all $S$ shares at time $T$.

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Market risk

it with the above result. In general, without additional structure, little can be said about the solution to the trader's liquidation problem. It could be either a block sale of the $S$ shares or slow, steady liquidation. Surprisingly, the difference between these two alternatives can be characterised by a single condition, called the "economies of scale in trading" condition:

$$sc(s) \exp \left[ (\alpha - \gamma) \Delta(s) \right] + (S - s) c(S - s) s \exp \left[ (\alpha - \gamma) \Delta(s - s) \right]$$

$$< Sc(s) \exp \left[ (\alpha - \gamma) \Delta(s) \right] \quad \text{for all} \quad 0 \leq s \leq S$$

This condition compares the proceeds received from two types of liquidation strategies. The left side represents a sale of $S$ shares, immediately followed by a sale of $(S-s)$ shares. As the quantity impact is cumulative, the second term on the left side includes the product of both quantity discounts; the right side represents a trade of $S$ shares alone. The economies of scale in trading condition states that two trades are more costly than one.

It can be shown that this condition characterises the trade-off between a block sale and slow liquidation. The result is that a block sale (either at time $T$ or $T$) is optimal, if and only if, the economies of scale in trading conditions hold. This is a useful result. In principle, you can examine the current market structure at the time of liquidation to see if it holds. So whether or not the economies of scale in trading conditions hold is an empirical question. If it does hold, then the proceeds from the liquidation of the $S$ shares can be easily quantified.

Let $u(p,S)$ represent the maximal discounted proceeds to the trader facing liquidation risk. The maximum revenue from the sale is determined from solving for the optimal liquidation policy of expression (8).

The solution to the trader's liquidation problem, under the economies of scale in trading condition, is:

- If $[\alpha - \gamma] \leq 0$, then immediate liquidation is optimal with proceeds:
  $$u(p,S) = Sc(s) \exp \left[ (\alpha - \gamma) \Delta(s) \right]$$

- If $[\alpha - \gamma] > 0$, then terminal date liquidation is optimal, with proceeds:
  $$u(p,S) = Sc(s) \exp \left[ (\alpha - \gamma) (T + \Delta(s)) \right]$$

The trading strategy is identical to that without liquidity risk. If the rate of appreciation on the stock, less the discount rate, is non-positive, it pays to liquidate immediately. Alternatively, if the appreciation rate on the stock less the discount rate is positive, it pays to wait to liquidate. But, in contrast to the situation without liquidity risk, the maximum proceeds received from liquidation differ due to the quantity discount and the execution lag. By condition (2), the discounted expected proceeds from liquidation, given liquidity risk, are always less.

Given the above solution, we can now determine the liquidation value of the portfolio. One measure of the liquidation value would be the discounted expected proceeds from the sale. This would be acceptable for the case where the price appreciation on the stock is less than the discount rate, but not if it exceeds the discount rate. Here, as for the case without liquidity risk, this valuation method would explicitly book the trader's trading profits (from the price appreciation) before they occur. This has obvious incentive problems associated with it.

Instead, we provide a valuation procedure that avoids this problem. Our valuation method is a straightforward generalisation of the marking-to-market approach used in the case without liquidity risk. We can effec-

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uly mark to market and account for the liquidation costs, not the trading profits, in the following way. The idea is to determine a hypothetical price in a market without liquidity risk that would provide the same proceeds given under the liquidity risk solution. Since for a trader facing no liquidity risk, marking to market provides the proper valuation method, we can use that here as well, but with the hypothetical price.

We define the per-share liquidation value of the stock to be that market price \( p^* \) such that a trader facing no liquidity risk would receive the same proceeds as a trader facing liquidity risk and the current market price \( p \). i.e., the per-share liquidation value is \( p^* \) such that:

\[
    u^*(p^*, S) = u(p, S)
\]  

(10)

This per-share liquidation value \( p^* \) gives an equivalent market price that implicitly incorporates liquidity risk and ignores potential trading profits. Hence, the liquidation value of our portfolio can be obtained by marking-to-market using \( p^* \) and not \( p \). i.e., the liquidation value of our portfolio is \( p^* S \).

Under the economies of scale in trading condition, it can be shown (Jarrow & Subramanian, 1997) that the per-share liquidation value is:

\[
    p^* = \rho c(s) \exp(\alpha - r\Delta(s))
\]  

(11)

This is comforting, as it provides us with the same liquidation value that would be obtained by using the discounted expected proceeds from sale, when the appreciation on the stock is less than the discount rate. In this case, we know that there are no trading profits to be gained from holding the stock and only liquidation costs to a sale. Our approach to valuation, however, also gives this same value when the appreciation on the stock exceeds the discount rate — which is the desired result.

Given liquidity risk, we see that marking to market \( S_p \) always overvalues the portfolio. The liquidation value of the portfolio \( p^* S \) is seen to be less than \( S_p \) by a discount factor that includes the quantity discount multiplied by the stock's appreciation rate (see condition (5)).

The above formulation was based on the condition that the economies of scale in trading condition holds. This will not always be the case. Nonetheless, we can still use the above to provide us with a conservative estimate of the liquidation value of the portfolio. When the economies of scale in trading condition is not satisfied, a block trade is not optimal, by the above theory. In this case, the expected discounted proceeds from liquidation will always be at least as large as those received from a block trade. Hence, the liquidation value determined above will provide a lower bound for the liquidation value under slow liquidation. Consequently, we have shown that valuing the portfolio under condition (11) always provides a conservative estimate of the liquidation value of the portfolio.

Let's now look at our quantification of liquidity risk for the computation of VAR. For comparison, we first compute the standard VAR measure for the trader's portfolio. Let \( \delta \) be the horizon over which the change in the portfolio's value is considered, usually selected to be between one and 10 days. Notice that the horizon is independent of the shares sold. We will calculate the 95% confidence interval. Since under condition (1), the stock price is lognormally distributed, we need to consider a two-standard-deviation movement in value. Under this structure, the standard VAR measure is easily computed. It is:

\[
    \text{Standard VAR} = \rho S \left[ \mathbb{E} \left[ \log(p(\delta)/p) \right] - 2 \text{std}\left[ \log(p(\delta)/p) \right] \right] 
\]

(12)

where \( \rho = \rho(0) \) and \( \text{std}(\ast) \) represents the standard deviation. A simple calculation under (1) yields:

\[
    \text{Standard VAR} = \rho S \left[ \alpha - \sigma^2 / 2 \right] \delta - 2 \sigma \delta^{1/2}
\]

(13)

This represents the loss in the dollar value of the portfolio due to a two-standard-deviation move below the mean.

Using the conservative estimate of the liquidation value as given by expression (11), we can compute the liquidity-adjusted VAR (LA - VAR). It is:

\[
    \text{LA - VAR} = \rho S \left[ \mathbb{E} \left[ \log(p(\Delta S)c(\delta)/p) \right] - 2 \text{std}\left[ \log(p(\Delta S)c(\delta)/p) \right] \right] 
\]

(14)

Using expression (8), a non-trivial calculation yields:

\[
    \text{LA - VAR} = \rho S \left[ \alpha - \sigma^2 / 2 \right] \mathbb{E} \left[ \Delta S \right] + \mathbb{E} \left[ \log c(\delta) \right] 
\]

\[
    -2 \left[ \sigma \mathbb{E} \left[ (\Delta S)^{1/2} \right] + \alpha - \sigma^2 / 2 \right] \text{std}\left[ \Delta S \right] + \text{std}\left[ \log c(\delta) \right] 
\]

(15)

The dollar loss in the value of the portfolio including liquidity risk is greater than that implied by the standard VAR measure. It differs from the standard calculation in three ways.

First, the liquidity horizon \( \delta \) is replaced by the expected execution lag in selling the \( S \) shares, \( E(\Delta S) \). This may differ due to the size of the shares in the portfolio. Second, the initial quantity discount on the shares sold needs to be included. This is the term \( E(\log c(\delta)) \). It is negative because \( c(\delta) \leq 1 \). Third, the volatility of the changes in value needs to be increased to include the volatility of the execution time, \( \alpha - \sigma^2 / 2 \text{std}(\Delta S) \), as well as the volatility of the quantity discount, \( \text{std}(\log c(\delta)) \).

This liquidity adjusted VAR measure is easy to calculate. It requires an estimate of the mean and standard deviation of the market price's movement (\( \alpha, \sigma \)), an estimate of the mean and standard deviation of the quantity discount (\( E(\log c(\delta)), \text{std}(\log c(\delta)) \)), and an estimate of the mean and standard deviation of the execution time for a block of \( S \) shares (\( E(\Delta S), \text{std}(\Delta S) \)). In principle, these should be easy to estimate.

This is certainly the case for the mean and standard deviation of the market price, which can be obtained using standard techniques, in stable markets, where sales and purchases are roughly balanced. Computing the remaining parameters in the liquidity-adjusted VAR measure, however, is more problematic. This calculation requires knowledge of the quantity discount (\( c(\delta) \)) and the execution lag variable \( \Delta S \). To our knowledge, there is no readily available data source that can be used for estimating these quantities, which implies that firms (or traders) need to collect time-series data on the shares traded, prices received and times to execution. From this data, the estimates of the necessary parameters can be obtained. In the short run, however, you can use subjective estimates of these parameters based on trading experience.

Alternatively, you could calculate the standard deviation of the market price, conditional upon a serious market decline. This conditional standard deviation may be a reasonable proxy for the sum of the standard deviations of the market price and the liquidity discount. The intuition is that in a market crash, sales are dominating purchases. So the prices observed are due to the joint movements of the market price and the quantity discount combined. The above analysis is simple so that the economic reasoning can easily be understood. In practice, however, you can include multiple assets, stochastic volatilities with jumps, and risk aversion. These generalizations complicate the computation, but the logic is identical (Jarrow & Subramanian, 1997).

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