Options Markets, Self-Fulfilling Prophecies, and Implied Volatilities

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Abstract. This paper answers the following often asked question in option pricing theory: if the underlying asset's price does not satisfy a lognormal distribution, can market prices satisfy the Black-Scholes formula just because market participants believe it should?

In complete markets, if the underlying asset's objective distribution is not lognormal, then the answer is no. But, in an incomplete market, if the underlying asset's objective distribution is not lognormal and all traders believe it is, then the answer is yes! The Black-Scholes formula can be a self-fulfilling prophecy.

The proof of this second assertion consists of generating an economy where self-confirming beliefs sustain the Black-Scholes formula as an equilibrium. An asymmetric information model is provided, where the underlying asset's price has stochastic volatility and drift. This model is distinct from the existing pricing models in the literature, and it provides new empirical implications concerning Black-Scholes implied volatilities and the bid/ask spread. Similar to stochastic volatility models, this model is consistent with the implied volatility "smile" pattern in strike prices. In addition, it is consistent with implied volatilities being biased predictors of future volatilities.

Keywords: Black-Scholes model, options, implied volatility, self-fulfilling prophecy

1. Introduction

After studying the Black-Scholes model, one of the first questions a student asks is: if the underlying asset's price does not satisfy a lognormal distribution, can market prices satisfy the Black-Scholes formula just because market participants believe it should? Appropriately translated to an economist, can the Black-Scholes formula be a self-fulfilling prophecy? Unfortunately, standard option pricing theory only provides a partial answer to this question. The partial answer is that in a complete market, if beliefs are lognormal, but the objective distribution is otherwise, yet agreeing on zero probability events, then the Black-Scholes formula will not hold in any equilibrium. The reason is that in a complete market, synthetic replication of the option depends only on the zero probability events of the objective distribution. Exact synthetic replication generates arbitrage opportunities for the knowledgeable trader, and arbitrage opportunities are inconsistent with equilibrium. But, for the above argument, the hypothesis of a complete market is crucial. The answer to this question is unknown in an incomplete market, where synthetic replication is impossible.

The primary purpose of this paper is to answer this self-fulfilling prophecy question in an incomplete market where exact synthetic replication is impossible. Surprisingly, we show that in a partial equilibrium setup, the answer to this question is yes! The Black-Scholes formula can be a self-fulfilling prophecy, even if the objective distribution for the
underlying asset's price is not lognormal. The demonstration of such an equilibrium is a major contribution of this paper.3

Although there are other option pricing models which generate the Black-Scholes formula in an incomplete market setting (see Rubinstein, 1976; and Brennan, 1979), these papers focus on the alternative issue of discrete versus continuous trading.4 These models have incomplete markets because trading can only take place discretely. Prices are determined via a Walrasian competitive equilibrium, where traders have both symmetric information and identical preferences (up to a risk aversion parameter). In contrast, ours is the first model to generate the Black-Scholes formula as a self-fulfilling prophecy in a market where incompleteness is due to asymmetric information. The asymmetric information concerns the underlying asset’s volatility and drift, both of which are stochastic in our model.

Furthermore, our equilibrium notion is that of a Bayesian Nash equilibrium, which has become common in the market microstructure literature (see Glosten and Milgrom, 1985; and Easley and O’Hara, 1987). It is also a rational expectations or self-fulfilling equilibrium in the sense of Azaridis (1981) because equilibrium prices confirm investors’ conjectures, based on their subjective beliefs. Unfortunately, as our model is essentially static (involving two periods that are not repeated), convergence of the subjective to the objective distribution via learning is not formally addressed herein. This line of inquiry awaits subsequent research. There are other models of option pricing in the market microstructure setting (see Biais and Hillion, 1990; John, Koticha, and Subrahmanyam, 1991; Back, 1993; and Easley and O’Hara, 1994), but these papers study different issues, and result in equilibrium option prices which are not Black-Scholes.

If our model is to provide a plausible explanation of market prices, then it needs to be consistent with the observed “smiles” across strike prices in Black-Scholes implied volatilities5 (see Canina and Figlewski, 1993). A second contribution of our paper is the demonstration that the equilibrium option prices generated by our Black-Scholes self-fulfilling equilibrium are consistent with the “smile” effect. This is not surprising, however, as our model for the evolution of the underlying asset’s price has stochastic volatility, and stochastic volatility option models are known to generate such “smiles” (see Hull and White, 1987; Wiggins, 1987; Stein and Stein, 1991; Bates, 1996). In contrast to these simpler stochastic volatility option pricing models, which do not consider bid/ask spreads or order flows, our approach has additional empirical implications. First, our equilibrium prices can be shown to be consistent with the observed over-reactions effect in implied volatilities (Stein, 1987; Canina and Figlewski, 1993). Second, our model generates testable implications regarding the bid/ask spread and the option’s volume, depending upon which traders dominate the order flow (i.e., traders with information on the volatility or the drift). These later implications have been the subject of empirical testing in Cherian and Vila (1997).

An outline of this paper is as follows. Section 2 describes the structure of the economy. Section 3 defines, and proves the existence of a self-fulfilling Black-Scholes equilibrium in a two-period setting. Weak-form and strong-form implied volatility equilibria are defined in Section 4. The existence of a weak-form equilibrium is proved, as well as the non-existence of a strong-form equilibrium. Section 5 specializes the analysis to an economy with athe-money options. This allows us to characterize the equilibrium in implied volatilities.
and to study its properties. Section 6 provides a discussion of the empirical implications of the model. Section 7 concludes the paper. All proofs of the lemmas and propositions are contained in the appendix, unless otherwise specified.

2. The Economy

This section constructs and motivates the model used in this paper to study whether the Black-Scholes formula can be a self-fulfilling prophecy. As we are seeking to prove the existence of such an equilibrium, we purposely keep the model simple in order to facilitate intuition and understanding. Nonetheless, the model structure is reasonably robust, and it can be generalized so that its refinements provide a reasonable approximation to reality. The relevant generalizations are noted at the appropriate points in the text.

We construct a partial equilibrium model of a call option market, where the option cannot be synthetically replicated given trading in an underlying stock and riskless bond. As such, in this incomplete market setting (as argued in the introduction), arbitrage arguments alone cannot justify the satisfaction of the Black-Scholes formula. Consequently, we will invoke a Bayesian Nash equilibrium in a market microstructure setup similar to Glosten and Milgrom (1995) and Easley and O'Hara (1987) to determine the equilibrium option prices.

Our market is incomplete due to asymmetric information concerning the underlying stock's stochastic volatility and stochastic drift. This is distinct from the alternative derivations of the Black-Scholes formula in incomplete markets by Rubinstein (1976) and Brennan (1979), where the incompleteness is due to discrete trading alone. For simplicity, we also invoke a discrete trading model.

There are two time periods with two trading dates, denoted by \( t = 0 \) and 1. Trading are a stock, a riskless bond, and a European type call option on the stock. The call option matures at time 2 with a strike price of \( K \).

There are two types of market participants: (i) a risk-neutral, competitive market maker, and (ii) a continuum, indexed by the interval \([0, 1]\), of variously informed traders. These traders are subdivided into three types: (a) the fraction \( 0 < \alpha < 1 \) of volatility or \( \sigma \)-traders, (b) the fraction \( 0 < \beta < 1 \) of directional or \( \mu \)-traders, and (c) the fraction \( 0 < 1 - \alpha - \beta < 1 \) of liquidity traders or hedgers. The \( \sigma \)- and \( \mu \)-traders are risk-neutral, the liquidity traders are risk averse. Risk aversion for liquidity traders is consistent with their hedging motive. The assumption of risk-neutrality for the market maker, the \( \sigma \)-, and the \( \mu \)-traders can be relaxed.

The volatility or \( \sigma \)-traders have information relating to the volatility of the stock, the directional or \( \mu \)-traders have information relating to the mean return of the stock, and the hedgers are uninformed. The hedgers trade regardless of the actions of the informed traders. A precise description of each trader's beliefs and information structures will be given below. These three types of traders provide a reasonable approximation to the types of option traders present in actual markets.

We model the call option market taking as given both the initial stock price at time 0, denoted by \( S \), and the continuously compounded riskless rate per period, denoted by \( r \). We interpret the initial stock price \( S \) as the last recorded transaction price in the equity market. This price is known to all participants in the option market. This partial equilibrium set-up
is motivated by the standard setting of traditional option pricing models. It provides a useful simplification for our analysis, necessary (we believe) to obtain closed-form solutions. The relaxation of this partial equilibrium set-up is a fruitful area for subsequent modeling.5

All market participants enter the economy at time 0 with identical prior beliefs over the possible outcomes for the stock’s price at times 1 and 2. These beliefs are illustrated in Figure 1. Analogous to time 0, the evolution of the stock’s price and riskless rate through time are taken as given. We will describe the stock price tree moving from left to right.

At time 0, the initial stock price $S$ is observed. There are three possible stock prices at time 1, $S(1 + u)$, $S$, $S(1 + d)$ where $1 + u > e^r > 1 + d$. However, there are four possible states of nature. These are delineated by a third parameter, the volatility (high $\bar{\sigma}$ or low $\sigma$ where $\bar{\sigma} > \sigma$). This volatility corresponds to the stock return’s volatility over the next time period (1 to 2). The four states of nature are (i) $S(1 + u)$, $\bar{\sigma}$; (ii) $S$, $\bar{\sigma}$; (iii) $S(1 + d)$, $\bar{\sigma}$; and (iv) $S$, $\sigma$. States (i)–(iii) correspond to a high volatility market at time 1, with changing prices (high, flat, low). State (iv) corresponds to a low volatility and a flat (unchanged) stock market. These four states are known to all the market participants at time 1. These four states introduce the stochastic drift and stochastic volatility to the stock price process. The number of states is purposefully kept small in order to facilitate understanding. As seen below, expanding the number of states merely complicates the derivation, but it will not change any of the qualitative conclusions drawn.

Figure 1. The stock price tree with prior beliefs where $W(\sigma)$ is a random variable whose distribution depends on $\sigma$ with a variance increasing in $\sigma^2$ and a mean satisfying $E(e^{W(\sigma)}) = e^r$. 

The prior beliefs for the likelihoods of these four states are given on the branches of the tree. The probability of a high volatility occurring is 1/2. Given a high volatility, the high, flat, and low stock price outcomes are equally likely. This equally likely restriction is without loss of generality as the magnitudes of an up jump (u) and down jump (d) are unrestricted. Recall that for risk-neutral traders, only expected values matter. The probability of a low volatility occurring is 1/2 > 0. The fact that the high and low volatilities are equally likely is also without loss of generality as \( \sigma \) and \( \sigma \) are arbitrary.

Finally, given the stock price at time 1 \( S_1 \) the stock price at time 2 \( S_2 \) satisfies \( S_2 = S_1 \exp(W(\sigma)) \). The stock price’s distribution at time 2 is left unspecified and determined by the prior beliefs over the random variable \( W(\sigma) \), denoted by \( \text{Prob}(W(\sigma)) \). The notation makes explicit the fact that the random variable \( W(\sigma) \) depends on the volatility \( \sigma \) observed. In particular, the variance of the prior beliefs over \( W(\sigma) \) is assumed to be increasing in \( \sigma^2 \). This condition just formalizes the interpretation that \( \sigma \) is the stock’s volatility.

It is important to emphasize that these beliefs can differ from the objective distribution generating the randomness in \( W \). These beliefs are left unspecified because their equilibrium determination is essential in the notion of a self-fulfilling equilibrium a la Azariadis (1981). This will be discussed further in Section 3 below.

A number of remarks need to be made concerning this tree. First, as there are three stock price outcomes at time 1, the call option market is incomplete with trading in the stock and bond over the first time period. Secondly, as there are a continuum of stock prices possible at time 2 (and only discrete trading), the call option market is incomplete in the second time period as well. Arbitrage arguments alone, therefore, will not generate the Black-Scholes formula.

The evolution of the stock price process as in Figure 1, although simplified to facilitate the analysis, can be motivated as a randomly distributed stock price with information events during the option’s life corresponding to volatility shocks \( \overline{\sigma}, \sigma \) and mean stock return shocks \( \mu, \mu \). As the subsequent analysis will document, the beliefs structure and evolution of the stock price process is easily modified.

To ensure that the stock price process is fairly priced to both the market maker and the uninformed hedger, before trading occurs in the option’s market, we impose the following time 0 restriction:

\[
S = e^{-r} E(S_1).
\]  

The expected return on the stock, given no information except the last transaction price, equals the risk-free return. Simple algebra yields an equivalent form of this restriction, i.e.,

\[
(u + d)/6 = e^r - 1.
\]  

Of course, the informed \( \sigma \)- and \( \mu \)-traders, given their information, will see trading opportunities available in the stock market under restriction (1). However, as seen below, they will also see trading opportunities available in the options markets. Given the partial equilibrium nature of our model, we concentrate only on their trading behavior in the call option market.
Finally, we assume that the beliefs structure of Figure 1 satisfies the time 1 fair pricing condition,

\[ S_1 = e^{-r} E(S_2 \mid S_1, \sigma) \]  

where \((S_1, \sigma) \in \{(S(1 + u), \overline{\sigma}), (S, \overline{\sigma}), (S(1 + d), \overline{\sigma}), (S, \sigma)\}\).

This restriction is needed since the state of nature \((S_1, \sigma)\) is known to all the market participants at time 1.

We now turn to the information structures and trading mechanism. Trading takes place at times 0 and 1. The trading mechanism is a competitive, risk-neutral, market maker quoting bid/ask prices at each date. The traders, knowing the bid/ask prices, submit market orders to buy or sell one call option at each of these times. Only one trade occurs at each trading date.

As constructed, our model does not include quantity flow as a decision variable for the informed traders nor as an information source for the market maker. This is done, in part, to keep the analysis simple. More complicated order flows could be included.

As the model is constructed, all of the strategic trading and related information revelation occurs at time 0. At time 1, the state of nature \((S_1, \sigma) \in \{(S(1 + u), \overline{\sigma}), (S, \overline{\sigma}), (S(1 + d), \overline{\sigma}), (S, \sigma)\}\) is observed by all market participants. As such, the option market at time 1 trivializes to the case where all the market participants agree on the call option’s price as the discounted, expected payoff to the option. This reconfirms the necessity of the stock market restriction condition (3) above. More will be said about this in the next section.

The time 0 information structures and action choices for the various market participants are detailed in Figure 2. We describe this figure from left to right. This figure characterizes our model as an extensive form game with incomplete information.

The game has four stages at time 0.

(STAGE 1). First, the market maker quotes bid (\(B\)) and ask (\(A\)) option prices, given he observes the last transacted stock price \(S\) and the order submitted (sell or buy).

(STAGE 2). Second, nature chooses, according to the probabilities provided, the states of nature \((S_1, \sigma) \in \{(S(1 + u), \overline{\sigma}), (S, \overline{\sigma}), (S(1 + d), \overline{\sigma}), (S, \sigma)\}\). The \(\sigma\)-trader observes \(\overline{\sigma}\) or \(\overline{\sigma}\). She does not know \((S(1 + u), S, S(1 + d))\) if the volatility is high (\(\overline{\sigma}\)). If the volatility is low (\(\sigma\)), she knows that \(S\) occurs.

The \(\mu\)-trader observes \(S(1 + u), S, \) or \(S(1 + d)\). In the case of \(S\), she does not know either \(\overline{\sigma}\) or \(\sigma\). Otherwise, she knows \(\overline{\sigma}\) has occurred.

This information structure is consistent with \(\sigma\)-traders knowing the volatilities, but not the direction of stock price movements; and, with \(\mu\)-traders knowing the direction of stock price movements, but not the volatilities.

(STAGE 3). Third, nature chooses who trades, \(\sigma\)-traders (with probability \(\alpha\)), \(\mu\)-traders (with probability \(\beta\)), and hedgers (with probability \(1 - \alpha - \beta > 0\)). This stage can be interpreted as the informed traders only becoming informed with the indicated probabilities.

(STAGE 4). Fourth, at their decision nodes, the \(\sigma\)- and \(\mu\)-traders see the bid/ask prices and decide to either buy, make no trade, or sell. The hedgers buy with probability \(1/2\) and sell with probability \(1/2\). They trade independently of the actions of the informed traders.

Finally, time 1 occurs, the state \((S_1, \sigma)\) is revealed, and we return to Figure 1. This completes the set-up of the model.
Figure 2. The information structures and action choices of the various market participants at time 0. Squares connect the information partition of the $\sigma$-trader. Circles connect the information partition of the $\mu$-trader.
3. Self-Fulfilling Black-Scholes Equilibrium

This section defines and characterizes a self-fulfilling equilibrium for the call option market. The equilibrium notion employed is the standard one from the market microstructure literature modified to include beliefs as in Azariah (1981). Details are provided below. We demonstrate the existence of the Black-Scholes formula as a self-fulfilling equilibrium. The equilibrium analysis is divided into two time steps. First, we discuss the equilibrium in call option prices, given a set of prior beliefs over $W(\sigma)$. Then, we discuss an equilibrium in both beliefs and prices.

a. Equilibrium at Time 1

First, at time 1, equilibrium call prices are determined strictly via prior beliefs, as the market maker is risk neutral and no differential information remains. He values the call option at time 1 as its discounted expected value at time 2. The priors in Figure 1 give the time 1 equilibrium call prices under each state as:

\[ C(1, \sigma) = E(\max[S_2 - K, 0] \mid S(1), \sigma)e^{-r} \]

\[ C(1, \sigma) = E(\max[S_2 - K, 0] \mid S, \sigma)e^{-r} \]

\[ C(1, +d, \sigma) = E(\max[S_2 - K, 0] \mid S(1 + d), \sigma)e^{-r} \]

\[ C(1, \sigma) = E(\max[S_2 - K, 0] \mid S, \sigma)e^{-r} \]

(4a)

Call prices at time 1 are given conditioned on the different initial stock prices and volatilities. We assume that the prior distribution over $W(\sigma)$ is such that $C(1, \sigma)$ is (i) increasing in both $S_1$ and $\sigma$, and (ii) convex in $S_1$. From standard option pricing theory, it is known that numerous distributions over $W(\sigma)$ will satisfy these restrictions.

For simplicity, we also add restrictions on the parameters $d, \sigma$, and $\sigma$ such that $C(1, \sigma) > C(1, +d, \sigma)$. The complimentary case could be handled in an analogous manner.

These equilibrium call prices are the starting point for the subsequent equilibrium analysis. Since the exact distribution over $W(\sigma)$ is left unspecified at this stage, there is a myriad of candidate equilibria possible based on condition (4). To tie the current analysis to the self-fulfilling literature, the set of beliefs over $W(\sigma)$, which may be driven completely by extraneous factors, is what will determine the equilibrium outcomes. These beliefs need not correspond to the objective distribution generating the randomness in $W(\sigma)$. Hence, an equilibrium will be parameterized by both beliefs and prices. The determination of a self-fulfilling equilibrium is one in which prices and beliefs are formed simultaneously.
b. Equilibrium at Time 0

Given the equilibrium call prices at time 1, we can proceed to the equilibrium analysis at time 0. We first study the optimality conditions of the various informed traders. The three strategic traders are the $\sigma$-traders, $\mu$-traders, and the market maker. The hedgers are non-strategic and trade regardless of the actions of the informed traders.

1. $\sigma$-Traders

At time 0, the $\sigma$-traders know the volatility $\overline{\sigma}$ or $\sigma$. Consequently, their market valuation for the call option is:

given $\overline{\sigma}$,

$$E(C_1(S_1, \sigma) \mid \overline{\sigma})e^{-r} = \left[ \frac{1}{3} C_1(S(1 + u), \overline{\sigma}) + \frac{1}{3} C_1(S, \overline{\sigma}) + \frac{1}{3} C_1(S(1 + d), \overline{\sigma}) \right] e^{-r}$$  \hspace{1cm} (5a)

given $\sigma$,

$$E(C_1(S_1, \sigma) \mid \sigma)e^{-r} = C_1(S, \sigma)e^{-r}$$  \hspace{1cm} (5b)

Condition (5a) is seen to be a weighted average of the time 1 call prices, where the weights correspond to the conditional likelihoods of the various stock prices $S(1 + u)$, $S$, $S(1 + d)$. Condition (5b) is the call’s value for the state $(S, \sigma)$. It is shown in the appendix that the value in expression (5a) exceeds the value in expression (5b). This implies that volatility traders have higher values when $\overline{\sigma}$ occurs than when $\sigma$ occurs.

The $\sigma$-trader observes the bid/ask prices, and the optimality conditions are:

buy if and only if $A < E(C_1(S_1, \sigma) \mid \sigma)e^{-r}$ \hspace{1cm} (6a)

no trade if and only if $B \leq E(C_1(S_1, \sigma) \mid \sigma)e^{-r} \leq A$ \hspace{1cm} (6b)

sell if and only if $E(C_1(S_1, \sigma) \mid \sigma)e^{-r} < B$. \hspace{1cm} (6c)

where $\sigma \in [\overline{\sigma}, \sigma]$.

The $\sigma$-trader buys if her valuation exceeds the ask price, sells if her valuation is less than the bid price, and doesn’t trade otherwise.

2. $\mu$-Traders

At time 0, the $\mu$-traders know the direction of stock prices, $S(1 + u)$, $S$, or $S(1 + d)$. Consequently, their market valuation for the call option is:

given $S(1 + u)$,

$$E(C_1(S_1, \sigma) \mid S(1 + u))e^{-r} = C_1(S(1 + u), \overline{\sigma})e^{-r}$$  \hspace{1cm} (7a)

given $S$,

$$E(C_1(S_1, \sigma) \mid S)e^{-r} = (1/4)C_1(S, \overline{\sigma})e^{-r} + (3/4)C_1(S, \sigma)e^{-r}$$  \hspace{1cm} (7b)
given $S(1 + d)$,

$$E(C_1(S_1, \sigma) \mid S(1 + d)) e^{-r} = C_1(S(1 + d), \sigma) e^{-r} \tag{7c}$$

Conditions (7a) and (7c) are the discounted time 1 call values. Condition (7b) is a weighted average of time 1 call prices, where the weights correspond to the $\mu$-traders conditional probabilities of the various volatilities $\sigma$ and $\bar{\sigma}$.

By construction, the value in expression (7a) exceeds the values in expressions (7b) and (7c). In addition, because $C_1(S, \sigma) > C_1(S(1 + d), \bar{\sigma})$ expression (7b) exceeds (7c). This implies that the information on stock price movements is more valuable to the $\mu$-trader than is the volatility knowledge. The complimentary case can be handled in a similar manner.

The $\mu$-trader observes the bid/ask prices, and the optimality conditions are:

- buy if and only if $A < E(C_1(S_1, \sigma) \mid S_1) e^{-r}$
- no trade if and only if $B \leq E(C_1(S_1, \sigma) \mid S_1) e^{-r} \leq A$
- sell if and only if $E(C_1(S_1, \sigma) \mid S_1) e^{-r} < B$ \tag{8a}

where $S_1 \in \{S(1 + u), S, S(1 + d)\}$.

The $\mu$-trader buys if her valuation exceeds the ask price, sells if her valuation is less than the bid price, and doesn’t trade otherwise.

3. The Market Maker

The market maker determines his bid/ask prices knowing the game tree structure (as in Figure 2). The market maker conditions his beliefs on a buy, no trade, or sale. Conditioned on this trade, and his conjecture over who is trading (in equilibrium), he determines his posterior probabilities for the states $(S_1, \sigma)$. These posterior probabilities are denoted $\text{Prob}(S_1, \sigma \mid \text{BUY})$, $\text{Prob}(S_1, \sigma \mid \text{NO TRADE})$, and $\text{Prob}(S_1, \sigma \mid \text{SELL})$.

There are six possible conjectures over who buys/sells at the various states. These cases are given in Figure 3. For the $\sigma$-trader, the only conjecture is that he buys if $\sigma$ is high and sells if $\sigma$ is low. For the $\mu$-trader, however, there are six possible cases. These correspond to all rational combinations of BUY, NO TRADE, SELL for the three different information sets she observes $(S(1 + u), S, S(1 + d))$. Recall that (7b) is greater than (7c). Note that consistent with the information structure, the decision for the $\mu$-trader is the same at the nodes $(S, \sigma)$ and $(S, \bar{\sigma})$.

Given each of these conjectures, the market maker’s posterior probabilities can be determined. These posterior probabilities are contained in the appendix.

Given that he is a competitive, risk-neutral market maker, he sets bid and ask prices according to the following rule:

- $A = E(C_1(S_1, \sigma) \mid \text{BUY}) e^{-r}$ \tag{9a}

and

- $B = E(C_1(S_1, \sigma) \mid \text{SELL}) e^{-r}$ \tag{9b}

Both the bid and ask prices are seen to be weighted averages of time 1 call prices.
Given the optimality conditions, we can now define our time 0 self-fulfilling equilibrium in both beliefs and prices.

Definition 1 (A Self-Fulfilling Call Market Equilibrium). A self-fulfilling call market equilibrium is defined to be a set of prior beliefs $\text{Prob}(W(\sigma))$, a set of conjectured trades by the market maker (from Figure 3) and a pair of ask/bid prices $(A, B)$ satisfying condition (9) such that under these ask/bid prices, his prior beliefs and conjectured trades are verified with conditions (6) and (8) holding for the $\sigma$- and $\mu$-trader, respectively.

Given predetermined beliefs, this is the standard Bayesian Nash equilibrium for a market-maker economy. It is a Nash equilibrium as the market participants would not want to deviate from their actions, given the equilibrium selections of the other market participants. The twist in this definition is that the beliefs are also determined as part of the equilibrium, and seen to be self-confirming. These beliefs are self-fulfilling in the sense of Azariadis (1981), modified to a market maker economy. The equilibrium is also a rational expectations equilibrium in that investors' price conjectures and beliefs are confirmed in the equilibrium. As our model is essentially static (involving two periods that are not repeated), convergence of the subjective beliefs to the objective distribution via learning cannot be formally addressed herein. This important line of inquiry awaits subsequent research.

Proposition 1 characterizes all possible call market equilibrium in this economy for a given set of beliefs.

Proposition 1 (Characterization of the Call Market Equilibrium for a Given Set of Beliefs) For a given set of beliefs over $W(\sigma)$, there is at most one call market equilibrium in this economy. An equilibrium exists in
case 3 if and only if
\begin{align*}
(i) 
4\beta C_1(S(1 + u), \sigma) &= 9(1 - \alpha - \beta)(E(C_1(S_1, \sigma) | \sigma) - C_1(S, \sigma)) \\
&+ 2\beta(C_1(S, \sigma) + C_1(S(1 + d), \sigma)) \\
(ii) 
2\beta C_1(S, \sigma) &= 3(1 - \alpha - \beta)(E(C_1(S_1, \sigma) | \sigma) - C_1(S, \sigma)) \\
&+ 2\beta(C_1(S, \sigma) - C_1(S, \sigma)) + 2\beta C_1(S(1 + d), \sigma) \\
(iii) 
(4\alpha + 2)E(C_1(S, \sigma) | S) &= (1 - \alpha - \beta)(C_1(S(1 + u), \sigma) + C_1(S(1 + d), \sigma)) \\
&+ 2\beta C_1(S(1 + d), \sigma) + 6\alpha C_1(S, \sigma)
\end{align*}
or,

case 5 if and only if
\begin{align*}
(i) 
4\beta C_1(S(1 + u), \sigma) &= 9(1 - \alpha - \beta)(E(C_1(S_1, \sigma) | \sigma) - C_1(S, \sigma)) \\
&+ 2\beta(C_1(S, \sigma) + C_1(S(1 + d), \sigma)) \\
(ii) 
2\beta C_1(S, \sigma) &= 3(1 - \alpha - \beta)(E(C_1(S_1, \sigma) | \sigma) - C_1(S, \sigma)) \\
&+ 2\beta C_1(S(1 + d), \sigma) \\
(iii) 
(4\alpha + 2)E(C_1(S, \sigma) | S) &\geq (1 - \alpha - \beta)(C_1(S(1 + u), \sigma) + C_1(S(1 + d), \sigma)) \\
&+ 2\beta C_1(S(1 + d), \sigma) + 6\alpha C_1(S, \sigma)
\end{align*}

The proof of this proposition is contained in the appendix. It follows by a careful examination of the inequalities in the definition of a self-fulfilling call market equilibrium for a given set of prior beliefs. Cases 1 and 2 are excluded because the \(\mu\)-trader wants to sell when he observes \(S(1 + d)\). Cases 4 and 6 are excluded because the \(\mu\)-trader doesn’t want to buy when he observes \(S\). The only remaining equilibria possible, therefore, are cases 3 or 5. These cases are mutually exclusive due to condition (iii); either the \(\mu\)-trader doesn’t trade when he observes \(S\) (case 5), or he sells when he observes \(S\) (case 3).

The \(\mu\)-trader’s decision to not trade (case 5) or sell (case 3) when he observes \(S\) can be characterized via the level of hedgers in the economy \((1 - \alpha - \beta)\).

**Corollary (Level of Hedgers)** For a fixed ratio \((\alpha/\beta)\), there exists a level of hedgers \(\gamma^* \in (0, 1)\) relative to \(\mu\)-traders \(\beta^* \in (0, 1)\) for this economy such that\(^{11}\)

if \((1 - \alpha - \beta)/\beta \leq \gamma^*/\beta^*\), then the only equilibrium is case 5,

and

if \((1 - \alpha - \beta)/\beta > \gamma^*/\beta^*\), then the only equilibrium is case 3.

The difference between the two equilibriums can be explained by the bid/ask prices \((A, B)\). When the level of hedgers trading in the economy is low relative to the \(\mu\)-traders \((1 - \alpha - \beta)/\beta \leq \gamma^*/\beta^*\), adverse selection is high, so the market maker chooses large
values for $A$ and low values for $B$. This makes the optimal strategy for the $\mu$-traders not to trade in flat markets (when they observe $S$). In contrast, with a high level of hedgers trading in the economy relative to $\mu$-traders $(1 - \alpha - \beta)/\beta > \gamma^*/\beta^*$, adverse selection is less important, so the $\mu$-traders always see profit opportunities. Of the two economies, we believe that a low level of hedgers trading in the economy relative to $\mu$-traders is a more plausible representation of actual call option markets.

We now give one of the key results in our paper.

Proposition 2 (Black-Scholes as a Self-Fulfilling Call Market Equilibrium) There exists a self-fulfilling call market equilibrium with prior beliefs over $W(\sigma)$ being normally distributed. In this case,

\[
C_1(S_1, \sigma) = BS(S_1, \sigma) \quad \text{where}
\]

\[
BS(S_1', \sigma) = S_1 N(h(S_1, \sigma)) - K e^{-r} N(h(S_1, \sigma) - \sigma),
\]

\[
h(S_1, \sigma) = \log(S_1/K e^{-r})/\sigma + (1/2)\sigma,
\]

and

\[
N(h) = \int_{-\infty}^{h} e^{-x^2/2}dx/\sqrt{2\pi}
\]

The proof of this proposition is trivial.\(^\text{12}\) It consists of showing that $C_1(S_1, \sigma) = BS(S_1, \sigma)$ and that $BS(S_1, \sigma)$ satisfies the conditions specified for $C_1(S_1, \sigma)$ following expression (4), i.e., (i) it is increasing in both $S_1$ and $\sigma$, and (ii) it is convex in $S_1$. The expectation calculation is straightforward and the remaining properties are both well-known implications of the Black-Scholes formula.

This proposition shows that in incomplete markets, the Black-Scholes formula is a self-fulfilling equilibrium in the sense of Azariadis (1981). The proof of this proposition shows the existence of a self-fulfilling call market equilibrium (the Black-Scholes formula), but it does not prove uniqueness. Since beliefs over $W(\sigma)$ determine the option pricing formula, $C_1(S_1, \sigma)$, the set of beliefs consisting of a normally distributed $W(\sigma)$ can be equivalently viewed as traders entering option markets with Black-Scholes priors. Hence, the self-fulfilling equilibrium in this case is the Black-Scholes formula. In fact, any set of beliefs generating time 1 call values which are both (i) increasing in $S_1, \sigma$ and (ii) convex in $S_1$ will work. Non-uniqueness of the equilibrium outcome is a standard property of similar models with sunspot equilibrium in incomplete markets, see Cass and Shell (1983). Our justification for using the Black-Scholes formula is that it is the standard model used in practice.

Example (Illustration of a Self-Fulfilling Black-Scholes Equilibrium). To illustrate the set of Black-Scholes self-fulfilling equilibrium, we consider the following parameter values:

\[
\bar{\sigma} = .364, \quad \bar{\alpha} = .336,
\]

\[
u = 0.35, \quad S = 100, \quad \text{and} \quad e^r = e^{.05}.
\]
We study nine cases. The nine cases are three subcases each for in-the-money ($S = 1.25K e^{-r}$), at-the-money ($S = K e^{-r}$), and out-of-the-money ($S = .75 K e^{-r}$) options corresponding to (i) $\sigma$-traders dominating the market ($\alpha = 2\beta$), (ii) $\sigma$-traders and $\mu$-traders equally likely ($\alpha = \beta$), and (iii) $\mu$-traders dominating the market ($\alpha = \beta/2$).

Under the above parameters, we get the call market equilibria of cases 3 or 5. Table 1 shows the critical values for $\gamma^*/\beta^*$ such that for values of $(1 - \alpha - \beta)/\beta \leq \gamma^*/\beta^*$, the equilibrium is case 5, and for values of $(1 - \alpha - \beta)/\beta > \gamma^*/\beta^*$, the equilibrium is case 3.

In all of the equilibria in proposition 2, the bid and ask prices are seen to be conditional expectations of the Black-Scholes formula as in expression (9). This fact is the only implication of a call market equilibrium used in the next section.

4. Implied Volatility Equilibria

This section studies implied volatilities for the Black-Scholes self-fulfilling equilibrium of Section 3. It demonstrates that the Black-Scholes self-fulfilling equilibrium prices are consistent with a "smile" effect across strike prices in Black-Scholes implied volatilities. We take as given a call market equilibrium pair of bid/ask prices $(A, B)$ as given in propositions 1 and 2 under normally distributed priors for $W(\sigma)$. The following results depend only on the fact that the call option prices in expression (9) are an expectation of Black-Scholes values. It is important to emphasize that in our incomplete market due to asymmetric information, the market micro-structure model of the previous sections is essential to obtain this characterization of the equilibrium option price. This insight, and the ability to obtain a bid-ask spread, distinguishes our model from the other random volatility models in the literature (e.g., Hull and White, 1991; and Wiggins, 1987).

Definition 2 (Weak Form Implied Volatility Equilibrium). A weak-form implied volatility equilibrium is a Black-Scholes self-fulfilling equilibrium $(A, B)$ such that

$$A = BS(S, a) \text{ and } B = BS(S, b) \text{ for some } (a, b) \in \mathbb{R}_+ \times \mathbb{R}_+.$$
equates to the market clearing option price (associated with the Black-Scholes self-fulfilling equilibrium price). We next show that a weak-form implied volatility equilibrium always exists.

**Proposition 3 (Existence of a Weak-Form Implied Volatility Equilibrium)** Any Black-Scholes self-fulfilling equilibrium has a weak-form implied volatility equilibrium.

Proposition 3 shows that a weak-form implied volatility equilibrium always exists. As seen in the appendix, the proof of Proposition 3 uses only the fact that the Black-Scholes formula is monotonically increasing in \( \sigma \).

If the Black-Scholes formula applied exactly as in the "arbitrage-free" derivation, then the implied volatilities for the same stock should not depend on the strike price of the option. Hence, we end this section by defining a strong-form implied volatility equilibrium, and then showing its non-existence.

**Definition 3 (Strong-Form Implied Volatility Equilibrium).** A strong-form implied volatility equilibrium is a Black-Scholes self-fulfilling equilibrium \((A, B)\) where the ask and bid implied volatilities, \(a\) and \(b\), for a fixed expiration date, \(T\), and fixed interest rate, \(r\), are independent of the strike price, \(K\).

The following proposition shows that a strong form implied volatility equilibrium never exists, i.e., the implied volatility must be a function of the strike price, \(K\).

**Proposition 4 (Nonexistence of a Strong-Form Implied Volatility)** The bid and ask implied volatilities are nonconstant functions of \(K\).

The proof of this proposition is contained in the appendix. In fact, the proof implies that for the parameters used in the example, the slope of the implied volatility with respect to the strike price is negative for deep-in-the-money options. This slope is illustrated in Figures 4–6. A symmetric argument holds for the bid price \(B\). This generates a "U-shape" in the implied volatilities as a function of \(K\).

This proposition is consistent with the empirical evidence regarding usage of the Black-Scholes formula. For example, Canina and Figlewski (1993) have documented "U-shaped" implied volatilities when plotted against the strike price of the option. They find that deep-in-the-money call options have downward-sloping implied volatilities, while deep-out-of-the-money call options have constant or slightly upward-sloping implied volatilities. In fact, traders call this phenomenon a "half smile."

As demonstrated previously in the stochastic volatility option pricing literature (Hull and White, 1987; Wiggins, 1987; Stein and Stein, 1991; Bates, 1996), a smile effect will occur in Black-Scholes implied volatilities whenever the call's price can be written as an expectation over Black-Scholes values (as in expression (9)). Our proposition shows that we would also expect implied volatilities to depend on an option's strike price as a result of asymmetric information and adverse selection in the trading of options. Although this implication of our model can be generated in the simpler stochastic volatility option pricing
models, other implications regarding bid/ask spreads and options' trading volume cannot. These are discussed in the next section.

5. Towards a Theory of Implied Volatilities

The next question to be studied is the relationship which exists between the implied volatilities, from a weak-form implied volatility equilibrium, and the unconditional estimate of the volatility. Via propositions 1 and 2, we restrict our analysis to markets where hedgers represent a small percent of the trading volume relative to $\mu$-traders, so that the Black-Scholes self-fulfilling equilibrium is given by case $S$ in Figure 3. The analysis for the other market structure would follow in a similar fashion.

In order to study an information-based model of implied volatilities, we would like to remove some of the deterministic or parametric biases the Black-Scholes formula introduces into the implied volatilities due to non-linearities in $\sigma$. In this spirit, it has been shown by Feinstein (1989) that at-the-money ($S = Ke^{\mu\tau}$) Black-Scholes options prices are approximately linear in volatilities. We use this insight in the following proposition.
Figure 5. Graph of ask volatility versus strike price for $\alpha = \beta = 0.025$ where $\bar{\sigma} = 0.365, \sigma = 0.336$, $u = 0.35, r = .05$ and $S = 100$.

Proposition 5 (Implied Volatilities as Biased Forecasts of True Volatility) For at-the-money options,

\begin{align}
a &= E[\sigma \mid \text{BUY}] e^{-r} + \eta(\text{BUY}) e^{-r} + o(u^2) + o(d^2) \\
b &= E[\sigma \mid \text{SELL}] e^{-r} + \eta(\text{SELL}) e^{-r} + o(u^2) + o(d^2)
\end{align}

(10a)

where $\eta(\text{BUY}) > 0$, $\eta(\text{BUY}) > \eta(\text{SELL})$, and

\begin{align}
\eta(\text{BUY}) &= \left\{ \left( \frac{\sqrt{\pi}}{\sqrt{2}} + \left( \frac{u + \sigma^2/2}{\sigma} \right) \right) u P(S(1 + u), \bar{\sigma} \mid \text{BUY}) \right. \\
&\left. + \left( \frac{\sqrt{\pi}}{\sqrt{2}} + \left( \frac{d + \sigma^2/2}{\sigma} \right) \right) d P(S(1 + d), \bar{\sigma} \mid \text{BUY}) \right\}
\end{align}

\begin{align}
\eta(\text{SELL}) &= \left\{ \left( \frac{\sqrt{\pi}}{\sqrt{2}} + \left( \frac{u + \sigma^2/2}{\sigma} \right) \right) u P(S(1 + u), \bar{\sigma} \mid \text{SELL}) \right. \\
&\left. + \left( \frac{\sqrt{\pi}}{\sqrt{2}} + \left( \frac{d + \sigma^2/2}{\sigma} \right) \right) d P(S(1 + d), \bar{\sigma} \mid \text{SELL}) \right\}
\end{align}

and $a$ and $b$ are the ask and bid implied volatilities which set the option ask and bid prices equal to the Black-Scholes prices, respectively.
Figure 6. Graph of ask volatility versus strike price for $\alpha = \beta/2$, $1 - \alpha - \beta = .025$ where $\bar{\alpha} = .365$, $\bar{\beta} = .336$, $u = 0.35$, $r = .05$ and $S = 100$.

The proof of this proposition is contained in the appendix.

**Corollary (At-the-Money Calls)** For at-the-money options, if $\sigma r \approx 0$, then

$$a \approx E(\sigma \mid \text{BUY}) + \eta(\text{BUY})e^{-r} \tag{11a}$$

$$b \approx E(\sigma \mid \text{SELL}) + \eta(\text{SELL})e^{-r} \tag{11b}$$

where

$\eta(\text{BUY}) > 0$ and

$$\eta(\text{SELL}) = \left[6(e^r - 1) \left(\sqrt{\pi}/2 + \bar{\alpha}/2\right) + u^2 + d^2\right](1 - \alpha - \beta)/2(-2\beta + 3)$$

$$+ \beta d(1 + d)/(-2\beta + 3).$$

If $(1 - \alpha - \beta) \approx 0$, then $\eta(\text{SELL}) < 0$.

This corollary has the immediate implication that an ask implied volatility for an at-the-money option is an upward biased estimate of the market maker's expected volatility. This
upward bias reflects the non-linearities present when taking an expectation of the Black-Scholes formula over the time 1 call prices with $S_1 \in \{S(1+d), S(1+u)\}$. These time 1 options are either in- or out-of-the-money. Furthermore, if $(1 - \alpha - \beta) \approx 0$, then the bid implied volatility is a downward biased estimate of the market maker’s expected volatility. This is the case, for example, with the parameter estimates used in the example following proposition 2.

Define the unconditional expectation and variance of volatility as $\sigma^{**} = (\bar{\sigma} + \overline{\sigma})/2$ and $\Phi^2 = (\bar{\sigma} - \overline{\sigma})^2/4$, respectively. Calculating the appropriate conditional probabilities in a Bayesian manner, the following result is obtained.

**Lemma 1** The market maker’s expected volatilities are given by

$$E(\sigma \mid BUY) = \sigma^{**} + \frac{\Phi^2}{(\bar{\sigma} - \overline{\sigma})} \left[ \frac{2(3\alpha + \beta)}{-2\beta + 3} \right]$$

(12a)

$$E(\sigma \mid SELL) = \sigma^{**} + \frac{\Phi^2}{(\bar{\sigma} - \overline{\sigma})} \left[ \frac{2(3\alpha + \beta)}{-2\beta + 3} \right]$$

(12b)

and

$$E(\sigma \mid NO \ TRADE) = \sigma^{**} + \frac{\Phi^2}{(\bar{\sigma} - \overline{\sigma})}$$

(12c)

The following inequalities follow for different levels of informed trading $\alpha$ and $\beta > 0$.

if $\alpha = 0$, then $E(\sigma \mid BUY) = E(\sigma \mid SELL) = \sigma^{**} + \frac{\Phi^2}{(\bar{\sigma} - \overline{\sigma})} \left( \frac{2\beta}{-2\beta + 3} \right) > \sigma^{**}$

(13a)

if $0 < \alpha < \beta/3$, then $E(\sigma \mid BUY) > E(\sigma \mid SELL) > \sigma^{**}$

(13b)

if $\alpha \geq \beta/3$, then $E(\sigma \mid BUY) \geq \sigma^{**} \geq E(\sigma \mid SELL)$

(13c)

Our assumed state space structure results in a downward revision in the expectation of future volatility given no trade, see equation (12c). This is not surprising as the probability of volatility being low ($\sigma$) is three times that of volatility being high ($\overline{\sigma}$), given that no trade has taken place. The result is consistent with models of market microstructure which make explicit the role of time in price adjustment (see O’Hara, 1995, Chapter 6.3 for the references). In our case, the absence of trades provides information about volatility to the market maker, thus causing him to revise his price (and consequently, implied volatility).

This lemma implies the following proposition.

**Proposition 6 (Implied Volatilities and Unconditional Expectations)** For at-the-money options, if $\sigma r \approx 0$, then

$$\sigma \approx \sigma^{**} + \frac{\Phi^2}{(\bar{\sigma} - \overline{\sigma})} \left( \frac{2(3\alpha - \beta)}{-2\beta + 3} \right) + \eta(BUY)e^{-r}$$

(14a)
\[ b \approx \sigma^{**} + \frac{\phi^2}{(\sigma - \sigma)} \left( \frac{2(3\alpha - \beta)}{2\beta + 3} \right) + \eta(\text{SELL})e^{\tau} \tag{14b} \]

\[ a - b \approx \frac{\phi^2}{(\sigma - \sigma)} \left[ \frac{12\alpha}{3 - 2\beta} \right] + [\eta(\text{BUY}) - \eta(\text{SELL})]e^{\tau}. \tag{14c} \]

Thus,

\[ \text{if} \quad 0 \leq \alpha/\beta < \frac{1}{3}, \text{ then } a > \sigma^{**}, a > b, \text{ and } b \leq \sigma^{**} \tag{14d} \]

\[ \text{if} \quad \alpha/\beta > \frac{1}{3}, \text{ then } a > \sigma^{**}, a > b, \text{ and } b \leq \sigma^{**} \tag{14e} \]

\[ \text{if} \quad \alpha/\beta \geq \frac{1}{3} \text{ and } 1 - \alpha - \beta \approx 0, \text{ then } a > \sigma^{**} > b \tag{14f} \]

The proof of this proposition follows from direct substitution of the lemma into proposition 5.

This proposition implies that there exists a non-empty set of parameters for which the ask volatility strictly exceeds the bid volatility which strictly exceeds the unconditional estimate of volatility. Indeed, if there are no informed volatility traders present (\( \alpha = 0 \)) or if the \( \mu \)-traders dominate the order flow (\( \alpha < \beta/3 \)), then the bid volatility can exceed the unconditional estimate of the volatility \( \sigma^{**} \). This result concurs with the intuition that the dominant presence of informed \( \mu \)-traders skews volatilities upwards and it could be a possible explanation for the "overreactions" result of Stein (1987).

In contrast, when \( \sigma \)-traders dominate the order flow (\( \alpha \geq \beta/3 \)) in markets where hedgers represent a small fraction of the traders (\( 1 - \alpha - \beta \approx 0 \)), then the ask volatility (a) exceeds the unconditional estimate of the volatility (\( \sigma^{**} \)) which exceeds the bid volatility (b). This situation corresponds to the traditional ordering of bid/ask and unconditional prices found in the market microstructure literature.

It is important to note that the previous results generalize to an economy where all the market participants (\( \sigma - \) and \( \mu \)-traders, market maker) are risk averse. Under this extension, all propositions hold, but without the simple closed-form expressions available under risk neutrality. This generalization involves transforming the expectations operator \( E(\bullet) \) at time 1, in the preceding analysis, to an equivalent martingale measure, \( E^{*}(\bullet) \). This follows standard techniques. The equivalent martingale measure implicitly incorporates risk aversion. At time 0, the expectations operator \( E(\bullet) \) is replaced by \( E(u(\bullet)) \) for the \( \mu \)- and \( \sigma \)-traders, where \( u(\bullet) \) is a trader dependent, increasing, concave utility function of time 1 wealth. The first-order conditions for optimality (expressions (6a)–(6c) and (8a)–(8c)) extend in a straightforward fashion. For the market maker, \( E(\bullet) \) is replaced by an equivalent martingale measure \( E^{*}(\bullet) \) explicitly incorporating his risk aversion (in expression (9)). This equivalent probability measure \( E^{*}(\bullet) \) is obtained from the first-order conditions for the market maker’s optimal inventory decision, conditioning on a buy or a sell.
6. Empirical Implications

We introduce this section by contrasting our model with extant stochastic volatility models like Hull and White (1987), Stein and Stein (1991), and Wiggins (1987). The latter models provide testable implications on the cross-sectional, i.e., across strikes, as well as the time-series properties of implied volatilities. More specifically, these models predict that:

(i) Stochastic volatility has the effect of increasing model option prices above that of Black-Scholes under certain parametric assumptions (example, for a mean-reverting volatility process, this result obtains when the long-run average level of volatility is set equal to the Black-Scholes volatility). The Black-Scholes implied volatility obtained from the stochastic volatility model option price will be correspondingly higher than the Black-Scholes volatility.

(ii) Stochastic volatility tends to be more important for in-the-money and out-of-the-money options due to leptokurtosis and skewness effects implicit in such models. "U-shaped" implied volatilities result with the at-the-money options having the lowest implied volatilities.

Our model, with some added structure, includes as a subclass, the stochastic volatility models mentioned above. Hence, the results outlined above are also implications of our model. In addition, our assumption of incompleteness due to the asymmetry of information results in additional testable hypotheses. These hypotheses follow from proposition 6. The first two hypotheses concern the bid-ask spread as quantified in expression (14c).

In our model, the informed $\sigma$ - and $\mu$-traders only transact when a signal is received. Otherwise, they are inactive. In addition, when one is informed, the other isn't. Hedgers trade randomly. Only one type of trader transacts at each instant. For the empirical implementation, we say that the $\mu$-traders dominate the market when: (i) they are informed/active and the $\sigma$-traders are not, and (ii) $1 - \alpha - \beta \approx 0$, so that hedgers are only a small part of the market. In this case, the larger the number of $\mu$-traders (larger $\beta$) in the market, by expression (14c), the larger will be the bid-ask spread. The relation, however, is non-linear. We can proxy the number ($\beta$) of $\mu$-traders in this circumstance by the volume of trading. This implies the following hypothesis.

$H1$. When $\mu$-traders dominate the market, then an increasing, but non-linear relationship exists between the cleared contract volume in options and the bid-ask spread.

Similarly, when $\sigma$-traders dominate the market, we get a second testable hypothesis:

$H2$. When volatility traders dominate the market, then bid-ask spread is directly and positively proportional to the cleared contract volume in options.

Proposition 6 also implies a theoretical relationship between bid and ask implied volatilities and the true underlying volatility, $\sigma^*$. These relationships depend on the ratio of the number of $\sigma$-traders ($\alpha$) to the number of $\mu$-traders ($\beta$). When $\alpha/\beta > \frac{1}{3}$ and $1 - \alpha - \beta \approx 0$
we say that the $\sigma$-traders dominate the order flow. Here, they dominate the market (at a point in time), and they are likely to again at the next trading instant. In this case, we get:

$H3$. If volatility traders dominate the order flow, then the bid and ask implied volatilities, $a$ and $b$, respectively, straddle the unconditional estimate of the true underlying volatility, $\sigma^{**}$. Mathematically, $a > \sigma^{**} > b$.

Similarly, if $\alpha / \beta \leq \frac{1}{3}$ and $1 - \alpha - \beta \approx 0$, we say the $\mu$-traders dominate the order flow. In this case, we get:

$H4$. If $\mu$-traders dominate the order flow, then the bid implied volatility, $b$, can exceed the unconditional estimate of the true underlying volatility, $\sigma^{**}$. Mathematically, $a > b > \sigma^{**}$.

The reason why hypotheses $H1$ and $H2$ are different from the standard market microstructure results (i.e., that volume and the bid-ask spread are inversely related) is that our hypotheses are conditional on the presence of information traders, albeit of different types. Whether it is directional or volatility information traders dominating the order flow, the effect is to widen the bid-ask implied volatility spread, since the market maker believes the overall information risk of trading with an informed trader is higher. We view this as a contribution of our paper.

The empirical hypotheses $H1$–$H4$ hinge on correctly identifying the two types of informed traders. In recent work, Cherian and Vila (1997) provide identification tests which aid in distinguishing between volatility and directional trades. They provide results which support hypotheses $H3$–$H4$. Even more recent empirical work by Cherian and Weng (1997) finds that the bid-ask implied volatility spread widens after observing both directional and volatility trading. They also find that the volume of directional and volatility trading are positively correlated to bid-ask implied volatility spreads, although the correlation is not significant at the 5% level for the volatility trading sample.

7. Conclusion

This paper prices call options in a partial equilibrium model where there is adverse selection. We show that the Black-Scholes formula can be supported in equilibrium as a self-fulfilling prophecy even when the option is not spanned by the underlying security and bonds. We show that the level of hedges in the market influences the equilibrium which results.

We also study how information on the future volatility is incorporated into the market's conditional estimate of the future volatility. In order to remove some of the biases in implied volatilities that the Black-Scholes formula generates due to non-linearities, we specialize the analysis to at-the-money options. This allows us to characterize the equilibrium in implied volatilities and to study the outcomes of such an equilibrium, especially the ones which predict systematic biases in implied volatilities. Our general result is that volatility risk is priced in equilibrium.

This result is consistent with the predictions of Canina and Figlewski (1993) and Lamoureux and Lastrapes (1993). In our model, the types of traders in the market determines the
position and size of the bid-ask spread. In this respect, the dominant presence of \( \mu \)-traders in particular markets would tend to bias the implied volatilities in such markets upwards relative to markets which lack \( \mu \)-trading. This could be a possible explanation for the recent empirical evidence concerning an “overreactions” effect in options’ implied volatilities (see Stein, 1987).

In Section 6, we show that if \( \mu \)-traders dominate the order flow \( \alpha < \beta/2 \), then the bid implied volatility can exceed the unconditional estimate of volatility. In this situation, historical volatility would be a better predictor of future volatility than is implied volatility (see Canina and Figlewski, 1993). We also show, by way of example, how equilibrium prices in options can produce “U-shaped” implied volatilities when plotted against the strike price of the option. This has been documented in numerous studies, including the ones by Canina and Figlewski (1993) and Rubinstein (1985). As our results are from an economic equilibrium, there are no arbitrage opportunities in implied volatilities associated with “half smiles” in our model.

Finally, an extension of the current model is taken up in Cherian (1996). In that model, traders are given a choice between two different maturity zero-gamma call options. The main result is that the informed volatility traders’ optimal strategies are found to be a function of the pattern followed by the true volatility process.

Appendix (Mathematical Proofs)

Proof that (5a) > (5b)

Note that \( C_1(S, \sigma) \) is strictly convex in \( S \). Thus,

\[
C_1(S(1 + (u + d)/2), \sigma) < [C_1(S(1 + u), \sigma) + C_1(S(1 + d), \sigma)]/2
\]

But

\[
(u + d)/2 = 3(\epsilon' - 1) > 0 \quad \text{so}
\]

\[
C_1(S, \sigma) < C_1(S(1 + (u + d)/2), \sigma), \quad \text{giving}
\]

\[
2C_1(S, \sigma) < C_1(S(1 + u), \sigma) + C_1(S(1 + d), \sigma) \quad \text{or}
\]

\[
C_1(S, \sigma) < [C_1(S(1 + u), \sigma) + C_1(S, \sigma) + C_1(S, \sigma) + C_1(S(1 + d), \sigma)]/3.
\]

This completes the proof as

\[
C_1(S, \sigma) < C_1(S, \sigma).
\]

Posterior Probabilities of the Market Maker

Case 1

\[
P(S(1 + u), \sigma \mid \text{BUY}) = (1/6)(\alpha + \beta + 1)/(\beta + 1)
\]
\[\begin{align*}
P(S, \bar{\sigma} \mid \text{BUY}) &= (1/6)(\alpha + \beta + 1)/(\beta + 1) \\
P(S + d, \bar{\sigma} \mid \text{BUY}) &= (1/6)(\alpha + \beta + 1)/(\beta + 1) \\
P(S, \sigma \mid \text{BUY}) &= (1/2)(-\alpha + \beta + 1)/(\beta + 1) \\
P(S(1 + u), \bar{\sigma} \mid \text{SELL}) &= (1/6)(1 - \alpha - \beta)/(1 - \beta) \\
P(S, \bar{\sigma} \mid \text{SELL}) &= (1/6)(1 - \alpha - \beta)/(1 - \beta) \\
P(S(1 + d), \bar{\sigma} \mid \text{SELL}) &= (1/6)(1 - \alpha - \beta)/(1 - \beta) \\
P(S, \sigma \mid \text{SELL}) &= (1/2)(\alpha - \beta + 1)/(1 - \beta)
\end{align*}\]

**Case 2**

\[\begin{align*}
P(S(1 + u), \bar{\sigma} \mid \text{BUY}) &= (1/2)(\alpha + \beta + 1)/(3 - 2\beta) \\
P(S, \bar{\sigma} \mid \text{BUY}) &= (1/2)(\alpha - \beta + 1)/(3 - 2\beta) \\
P(S(1 + d), \bar{\sigma} \mid \text{BUY}) &= (1/2)(\alpha - \beta + 1)/(3 - 2\beta) \\
P(S, \sigma \mid \text{BUY}) &= (3/2)(1 - \alpha - \beta)/(3 - 2\beta) \\
P(S(1 + u), \bar{\sigma} \mid \text{SELL}) &= (1/6)(1 - \alpha - \beta)/(1 - \beta) \\
P(S, \bar{\sigma} \mid \text{SELL}) &= (1/6)(1 - \alpha - \beta)/(1 - \beta) \\
P(S(1 + d), \bar{\sigma} \mid \text{SELL}) &= (1/6)(1 - \alpha - \beta)/(1 - \beta) \\
P(S, \sigma \mid \text{SELL}) &= (1/2)(\alpha - \beta + 1)/(1 - \beta)
\end{align*}\]

\[\begin{align*}
P(S(1 + u), \bar{\sigma} \mid \text{NO TRADE}) &= 0 \\
P(S, \bar{\sigma} \mid \text{NO TRADE}) &= 1/5 \\
P(S(1 + d), \bar{\sigma} \mid \text{NO TRADE}) &= 1/5 \\
P(S, \sigma \mid \text{NO TRADE}) &= 3/5
\end{align*}\]

**Case 3**

\[\begin{align*}
P(S(1 + u), \bar{\sigma} \mid \text{BUY}) &= (1/2)(\alpha + \beta + 1)/(-2\beta + 3) \\
P(S, \bar{\sigma} \mid \text{BUY}) &= (1/2)(\alpha - \beta + 1)/(-2\beta + 3) \\
P(S(1 + d), \bar{\sigma} \mid \text{BUY}) &= (1/2)(\alpha - \beta + 1)/(-2\beta + 3) \\
P(S, \sigma \mid \text{BUY}) &= (3/2)(1 - \alpha - \beta)/(-2\beta + 3) \\
P(S(1 + u), \bar{\sigma} \mid \text{SELL}) &= (1/2)(1 - \alpha - \beta)/(2\beta + 3) \\
P(S, \bar{\sigma} \mid \text{SELL}) &= (1/2)(-\alpha + \beta + 1)/(2\beta + 3) \\
P(S(1 + d), \bar{\sigma} \mid \text{SELL}) &= (1/2)(-\alpha + \beta + 1)/(2\beta + 3) \\
P(S, \sigma \mid \text{SELL}) &= (3/2)(\alpha + \beta + 1)/(2\beta + 3)
\end{align*}\]
Case 4

\[
P(S(1 + u), \sigma | \text{BUY}) = \frac{1}{2}(\alpha + \beta + 1)/(2\beta + 3)
\]
\[
P(S, \sigma | \text{BUY}) = \frac{1}{2}(\alpha + \beta + 1)/(2\beta + 3)
\]
\[
P(S(1 + d), \sigma | \text{BUY}) = \frac{1}{2}(\alpha - \beta + 1)/(2\beta + 3)
\]
\[
P(S, \sigma | \text{BUY}) = \frac{3}{2}(-\alpha + \beta + 1)/(2\beta + 3)
\]

\[
P(S(1 + u), \sigma | \text{SELL}) = \frac{1}{6}(1 - \alpha - \beta)/(1 - \beta)
\]
\[
P(S, \sigma | \text{SELL}) = \frac{1}{6}(1 - \alpha - \beta)/(1 - \beta)
\]
\[
P(S(1 + d), \sigma | \text{SELL}) = \frac{1}{6}(1 - \alpha - \beta)/(1 - \beta)
\]
\[
P(S, \sigma | \text{SELL}) = \frac{1}{2}(\alpha - \beta + 1)/(1 - \beta)
\]

\[
P(S(1 + u), \sigma | \text{NO TRADE}) = 0
\]
\[
P(S, \sigma | \text{NO TRADE}) = 0
\]
\[
P(S(1 + d), \sigma | \text{NO TRADE}) = 1
\]
\[
P(S, \sigma | \text{NO TRADE}) = 0
\]

Case 5

\[
P(S(1 + u), \sigma | \text{BUY}) = \frac{1}{2}(\alpha + \beta + 1)/(-2\beta + 3)
\]
\[
P(S, \sigma | \text{BUY}) = \frac{1}{2}(\alpha - \beta + 1)/(-2\beta + 3)
\]
\[
P(S(1 + d), \sigma | \text{BUY}) = \frac{1}{2}(\alpha - \beta + 1)/(-2\beta + 3)
\]
\[
P(S, \sigma | \text{BUY}) = \frac{3}{2}(1 - \alpha - \beta)/(-2\beta + 3)
\]

\[
P(S(1 + u), \sigma | \text{SELL}) = \frac{1}{2}(1 - \alpha - \beta)/(-2\beta + 3)
\]
\[
P(S, \sigma | \text{SELL}) = \frac{1}{2}(1 - \alpha - \beta)/(-2\beta + 3)
\]
\[
P(S(1 + d), \sigma | \text{SELL}) = \frac{1}{2}(-\alpha + \beta + 1)/(-2\beta + 3)
\]
\[
P(S, \sigma | \text{SELL}) = \frac{3}{2}(\alpha - \beta + 1)/(-2\beta + 3)
\]

\[
P(S(1 + u), \sigma | \text{NO TRADE}) = 0
\]
\[
P(S, \sigma | \text{NO TRADE}) = 1/4
\]
\[
P(S(1 + d), \sigma | \text{NO TRADE}) = 0
\]
\[
P(S, \sigma | \text{NO TRADE}) = 3/4
\]
Case 6:

\[
P(S(1 + u), \overline{\sigma} | \text{BUY}) = (1/2)(\alpha + \beta + 1)/(2\beta + 3)
\]

\[
P(S, \overline{\sigma} | \text{BUY}) = (1/2)(\alpha + \beta + 1)/(2\beta + 3)
\]

\[
P(S(1 + d), \overline{\sigma} | \text{BUY}) = (1/2)(\alpha - \beta + 1)/(2\beta + 3)
\]

\[
P(S, \sigma | \text{BUY}) = (3/2)(-\alpha + \beta + 1)/(2\beta + 3)
\]

\[
P(S(1 + u), \overline{\sigma} | \text{SELL}) = (1/2)(1 - \alpha - \beta)/(-2\beta + 3)
\]

\[
P(S, \overline{\sigma} | \text{SELL}) = (1/2)(1 - \alpha - \beta)/(-2\beta + 3)
\]

\[
P(S(1 + d), \overline{\sigma} | \text{SELL}) = (1/2)(-\alpha + \beta + 1)/(-2\beta + 3)
\]

\[
P(S, \sigma | \text{SELL}) = (3/2)(\alpha - \beta + 1)/(-2\beta + 3)
\]

**Proof of Proposition 1:** To facilitate the algebra, define \(C^{s, \sigma} \approx C_1(S(1 + x), \sigma)e^{-r}\).

**Case 1**

This case is rejected because given \(S(1 + d)\) is observed by the \(\mu\)-trader, \(A \leq C^{d, \overline{\sigma}}\) contradicted. Indeed, algebra shows

\[
A = \frac{\alpha + \beta + 1}{6(\beta + 1)} C^{u, \overline{\sigma}} + \frac{4(-\alpha + \beta + 1)}{6(\beta + 1)} \left( \frac{C^{o, \overline{\sigma}}}{4} + \frac{3C^{o, \sigma}}{4} \right) + \frac{2\alpha}{2(\beta + 1)} C^{o, \overline{\sigma}} + \frac{(\alpha + \beta + 1)}{6(\beta + 1)} C^{d, \overline{\sigma}} > C^{d, \overline{\sigma}}
\]

**Case 2**

This case is rejected because given \(S(1 + d)\) is observed by the \(\mu\)-trader, \(B \leq C^{d, \overline{\sigma}}\) is contradicted. Indeed, algebra shows \(B \leq C^{d, \overline{\sigma}}\) if and only if

\[
(1 - \alpha - \beta)(C^{o, \overline{\sigma}} - C^{d, \overline{\sigma}}) + 4(1 - \alpha - \beta) \left( \frac{C^{o, \overline{\sigma}}}{4} + \frac{3C^{o, \sigma}}{4} - C^{d, \overline{\sigma}} \right)
\]

\[
+ 6\alpha(C^{u, \overline{\sigma}} - C^{d, \overline{\sigma}}) \leq 0
\]

which is impossible since the left side of this inequality is positive.

**Case 3**

This case has a possible equilibrium. Given the \(\sigma\)-trader sees \(\overline{\sigma}\), he buys if \(A < [C^{u, \overline{\sigma}} + C^{o, \overline{\sigma}} + C^{d, \overline{\sigma}}]/3\). This holds if and only if

\[
4\beta C^{u, \overline{\sigma}} < 3(1 - \alpha - \beta)(C^{u, \overline{\sigma}} + C^{o, \overline{\sigma}} + C^{d, \overline{\sigma}} - 3C^{o, \sigma}) + 2\beta(C^{o, \overline{\sigma}} + C^{d, \overline{\sigma}}).
\]
This is condition (i).

Given the \( \sigma \)-trader sees \( \sigma \), he sells if \( C^\sigma < B \). This holds if and only if

\[
2\beta C^\sigma < (1 - \alpha - \beta)(C^{u,\overline{\sigma}} + C^{o,\overline{\sigma}} + C^{d,\overline{\sigma}} - 3C^\sigma) + 2\beta(C^{o,\overline{\sigma}} - C^\sigma)
+ 2\beta C^{d,\overline{\sigma}}
\]

This is condition (ii).

Given the \( \mu \)-trader sees \( S(1 + u) \), he buys if \( A < C^{u,\overline{\sigma}} \). This always holds as \( C^{u,\overline{\sigma}} \) exceeds \( C^{o,\overline{\sigma}}, C^{d,\overline{\sigma}} \) and \( C^\sigma \).

Given the \( \mu \)-trader sees \( S \) or \( S(1 + d) \), and that \( C^{d,\overline{\sigma}} < [C^{o,\overline{\sigma}} + 3C^\sigma]/4 \), he sells in both cases if \( [C^{o,\overline{\sigma}} + 3C^\sigma]/4 < B \) This holds if and only if

\[
(4\alpha + 2)(C^{o,\overline{\sigma}} + 3C^\sigma)/4 < 2\beta C^{d,\overline{\sigma}} + (1 - \alpha - \beta)(C^{u,\overline{\sigma}} + C^{d,\overline{\sigma}}) + 6\alpha C^\sigma
\]

This is condition (iii).

Case 4 and Case 6

Both these cases are rejected because given the \( \mu \)-trader sees \( S, A < [C^{o,\overline{\sigma}} + 3C^\sigma]/4 \) is contradicted. (Both cases 3 and 6 have identical ask prices.) Indeed, algebra shows \( A < [C^{o,\overline{\sigma}} + 3C^\sigma]/4 \) if and only if

\[
4\beta(C^{u,\overline{\sigma}} - C^{d,\overline{\sigma}}) + (4\alpha + 2)(C^{u,\overline{\sigma}} + C^{o,\overline{\sigma}} + C^{d,\overline{\sigma}} + 3C^\sigma)
+ 2(C^{o,\overline{\sigma}} + C^{d,\overline{\sigma}} + 2C^\sigma) < 0.
\]

This is impossible because the left side is positive. Note the last term is positive because \( C^\sigma \) is convex in \( \overline{\sigma} \) (see the proof that (5a) > (5b)).

Case 5

This case has a possible equilibrium. Given the \( \sigma \)-trader sees \( \overline{\sigma} \), he buys if \( A < [C^{u,\overline{\sigma}} + C^{o,\overline{\sigma}} + C^{d,\overline{\sigma}}]/3 \). This holds if and only if

\[
4\beta C^{u,\overline{\sigma}} < 3(1 - \alpha - \beta)(C^{u,\overline{\sigma}} + C^{o,\overline{\sigma}} + C^{d,\overline{\sigma}} - 3C^\sigma) + 2\beta(C^{o,\overline{\sigma}} + C^{d,\overline{\sigma}})
\]

This is condition (i).

Given the \( \sigma \)-trader sees \( \sigma \), he sells if \( C^{o,\overline{\sigma}} < B \). This holds if and only if

\[
2\beta C^{o,\overline{\sigma}} < (1 - \alpha - \beta)(C^{u,\overline{\sigma}} + C^{o,\overline{\sigma}} + C^{d,\overline{\sigma}} + 3C^\sigma + 2\beta C^{d,\overline{\sigma}}
\]

This is condition (ii).

Given the \( \mu \)-trader sees \( S(1 + u) \), he buys if \( A < C^{u,\overline{\sigma}} \). This always holds as \( C^{u,\overline{\sigma}} \) exceeds \( C^{o,\overline{\sigma}}, C^{d,\overline{\sigma}} \) and \( C^\sigma \).

Given the \( \mu \)-trader sees \( S \), he does not trade if \( B \leq [C^{o,\overline{\sigma}} + 3C^\sigma]/4 \leq A \). We consider each inequality separately. First, \( [C^{o,\overline{\sigma}} + 3C^\sigma]/4 \leq A \) if and only if

\[
0 \leq (12\alpha + 6)((C^{u,\overline{\sigma}} + C^{o,\overline{\sigma}} + C^{d,\overline{\sigma}})/3 - C^\sigma)
+ 4\beta(C^{o,\overline{\sigma}} - C^{d,\overline{\sigma}}) + 2(C^{u,\overline{\sigma}} + C^{d,\overline{\sigma}} - 2C^\sigma)
\]
This last inequality is true due to the fact that (5a) > (5b) and $C^{x, \bar{r}}$ is convex in $x$ (see the proof of (5a) > (5b)). Second, $B \leq [C^{\alpha, \bar{r}} + 3C^{\alpha, \bar{r}}]/4$ if and only if
\[ (1 - \alpha - \beta)(C^{u, \bar{r}} + C^{d, \bar{r}}) + 2\beta C^{d, \bar{r}} + 6\alpha C^{\alpha, \bar{r}} \leq (2 + 4\alpha)(C^{\alpha, \bar{r}} + 3C^{\alpha, \bar{r}})/4. \]
This is condition (iii).
Finally, given the $\mu$-trader sees $S(1 + d)$, he sells if $C^{d, \bar{r}} \leq B$. This is true as $C^{d, \bar{r}} < C^{\alpha, \bar{r}} < C^{\alpha, \bar{r}}$
This completes the proof of proposition 1.

**Proof of Corollary:** Given a fixed rate $(\alpha/\beta)$, case 3 equilibrium occurs if
\[
\left(\frac{4\alpha}{\beta} + \frac{2}{\beta}\right) E(C_{1}(S, \sigma) \mid S) - \left(\frac{1}{\beta} - \frac{\alpha}{\beta} - 1\right) (C_{1}(S(1 + u), \bar{\sigma}) + C_{1}(S(1 + d), \bar{\sigma})) - 2C_{1}(S(1 + d), \bar{\sigma}) - 6(\alpha/\beta)C_{1}(S, \bar{\sigma}) < 0
\]
By the convexity of $C(x, \bar{\sigma})$ in $x$ (see the proof that (5a) > (5b)), $2E(C_{1}(S, \sigma) \mid S) < C_{1}(S(1 + u), \bar{\sigma}) + C_{1}(S(1 + d), \bar{\sigma})$. Thus, the expression on the left side is continuous and decreasing in $(1/\beta) \in [1 + (\alpha/\beta), +\infty)$. The left side is positive if $1/\beta = \frac{\alpha}{\beta} + 1$. Thus, there exists a $1/\beta^{*} > 1 + (\alpha/\beta)$ such that for $1/\beta > 1/\beta^{*}$, the left side is less than zero. Define $\alpha^{*} \in [0, 1]$ such that $\alpha^{*}/\beta^{*} = \alpha/\beta$, and define $\gamma^{*} = 1 - \alpha^{*} - \beta^{*}$. Then case 3 equilibrium if $\frac{1}{\beta} > \frac{1}{\beta^{*}}$ if and only if
\[
\frac{1}{\beta} - \frac{\alpha}{\beta} - 1 > \frac{1}{\beta^{*}} - \frac{\alpha^{*}}{\beta^{*}} - 1 = \gamma^{*}/\beta^{*}.
\]
This completes the proof.

**Proof of Proposition 3:** Since the Black-Scholes price is strictly increasing in volatility, a higher estimate of volatility implies a higher Black-Scholes price. Let
\[ f(\sigma) = A - BS(S, \sigma) = 0. \]
It is easy to see that $f: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$ is continuous in $\sigma$. Realizing that $A$ is determined from the equilibrium in Proposition 1 gives the existence of $\sigma^{*} > 0$ such that
\[ f(0) \leq 0 \leq f(\sigma^{*}) \]
By the Intermediate Value Theorem, there exists an implied volatility, $a \in [0, \sigma^{*}]$ such that $f(a) = 0$. A symmetric argument holds on the bid side.

**Proof of Proposition 4:** Following the proof of Proposition 1, the ask price, $A$, is given by
\[ A = e^{-r} (\eta BS(S, \bar{\sigma}, K) \mid BUY, \bar{\sigma}) + (1 - \eta) BS(S, \bar{\sigma}, K) \]
for $\eta = P(\bar{\sigma} \mid BUY) \in (0, 1)$.
where the expectation is over \( S_t \in [S(1 + a), S, S(1 + d)] \). By Proposition 3, we have

\[
e^{-r}[\eta E(BS(S_1, \bar{\sigma}, K) \mid \text{BUY}, \bar{\sigma}) + (1 - \eta)BS(S, \bar{\sigma}, K)] - BS(S, a, K)
\]

where \( a \) is the ask implied volatility. Define

\[
G(a, K) \equiv e^{-r}[\eta E(BS(S, \bar{\sigma}, K) \mid \text{BUY}, \bar{\sigma}) + (1 - \eta)BS(S, \bar{\sigma}, K)] - BS(S, a, K) = 0.
\]

Since

\[
\frac{\partial G}{\partial a} = \frac{\partial}{\partial a} BS(S, a, K) < 0,
\]

we have by the Implicit Function Theorem, the existence of a unique function \( g: \mathbb{R}_+ \to \mathbb{R}_+ \) such that

\[
G(g(K), K) = 0.
\]

We want to show that,

\[
\frac{\partial g}{\partial K} = -\frac{\partial G/\partial K}{\partial G/\partial a} \neq 0.
\]

This will prove our result as it shows that the implied volatility \( a = g(K) \), is a non-constant function of \( K \). Note that

\[
\frac{\partial G}{\partial K} = e^{-r}\left\{ \eta E\left( \frac{\partial}{\partial K} BS(S_1, \bar{\sigma}, K) \mid \text{BUY}, \bar{\sigma} \right) + (1 - \eta)\frac{\partial}{\partial K} BS(S, \bar{\sigma}, K) \right\}
\]

\[
-\frac{\partial}{\partial K} BS(S, a, K)
\]

\[
e^{-r}(N(h(S, a) - a) - \eta E(N(h(S, \bar{\sigma}) - \bar{\sigma}) \mid \text{BUY}, \bar{\sigma})e^{-r}
\]

\[-(1 - \eta)N(h(S, \bar{\sigma}) - \bar{\sigma})e^{-r}\}
\]

(11)

where

\[
h(S, \sigma) = \frac{\log(S/K e^{-r})}{\sigma} + \frac{1}{2}\sigma.
\]

In general, \( \partial G/\partial K \) is non-zero, hence, \( \partial g/\partial K \) will be non-zero. This completes the proof. \( \blacksquare \)

**Proof of Proposition 5:** Let \( S = Ke^{-r} \).

Note that \( N(x) \approx N(0) + N'(0)x + o(x^2) = 1/2 + x/\sqrt{2\pi} + o(x^2) \) and \( \log(1 + x) = x - x^2/2 + o(x^2) \) where \( \lim_{x \to 0} o(x^2)/x^2 = 0 \).

Thus, using these approximations gives:

\[
BS(S, \sigma) \approx S\sigma/\sqrt{2\pi}
\]

\[
BS(S(1 + x), \sigma) \approx S\sigma/\sqrt{2\pi} + Sx \left( 1/2 + \left( \frac{x + \sigma^2/\sqrt{2\pi}}{\sigma} \right) \right)
\]
Using
\[ BS(S, a) = e^{-r}[BS(S(1 + u), \sigma) P(S(1 + u), \sigma | BUY) + BS(S, \sigma) P(S, \sigma | BUY) + BS(S(1 + d), \sigma) P(S(1 + d), \sigma | BUY) + BS(S, \sigma) P(S, \sigma | BUY)]. \]

Substitution and simplification yields
\[ a \approx E(\sigma | BUY) e^{-r} + \left\{ \left( \frac{\sqrt{\pi}}{\sqrt{2}} + \frac{(u + \sigma^2/2)}{\sigma} \right) u P(S(1 + u), \sigma | BUY) + \left( \frac{\sqrt{\pi}}{\sqrt{2}} + \frac{(d + \sigma^2/2)}{\sigma} \right) d P(S(1 + d), \sigma | BUY) \right\} e^{-r}. \]

A similar analysis gives
\[ b \approx E(\sigma | SELL) e^{-r} + \left\{ \left( \frac{\sqrt{\pi}}{\sqrt{2}} + \frac{(u + \sigma^2/2)}{\sigma} \right) u P(S(1 + u), \sigma | SELL) + \left( \frac{\sqrt{\pi}}{\sqrt{2}} + \frac{(d + \sigma^2/2)}{\sigma} \right) d P(S(1 + d), \sigma | SELL) \right\} e^{-r}. \]

So
\[ \eta(BUY) = \left( \frac{\sqrt{\pi}}{\sqrt{2}} + \frac{\sigma}{2} \right) (u P(S(1 + u), \sigma | BUY) + d P(S(1 + d), \sigma | BUY)) \frac{u^2}{\sigma} P(S(1 + u), \sigma | BUY) + \frac{d^2}{\sigma} P(S(1 + d), \sigma | BUY) \]
\[ \eta(SELL) = \left( \frac{\sqrt{\pi}}{\sqrt{2}} + \frac{\sigma}{2} \right) (u P(S(1 + u), \sigma | SELL) + d P(S(1 + d), \sigma | SELL)) \frac{u^2}{\sigma} P(S(1 + u), \sigma | SELL) + \frac{d^2}{\sigma} P(S(1 + d), \sigma | SELL). \]

Using the market maker’s probabilities we get
\[ u P(S(1 + u), \sigma | BUY) + d P(S(1 + d), \sigma | BUY) = [(u + d)(1 + \alpha) + \beta(u - d)]/2(-2\beta + 3) > 0 \]
giving \( \eta(BUY) > 0 \) and
\[ \eta(SELL) = \left[ 6(e^r - 1) \left( \frac{\sqrt{\pi}}{\sqrt{2}} + \frac{\sigma}{2} \right) + u^2 + d^2 \right] (1 - \alpha - \beta)/2(-2\beta + 3) \]
\[ + \beta d(1 + d)/(-2\beta + 3). \]

The first term is positive, the second term is negative as \( (1 + d) > 0 \) and \( d < 0 \). This completes the proof.
\textbf{Proof of the Lemma:} The implied volatility on the ask side is given by

\[ E(\sigma \mid \text{Buy}) = \sigma \Pr(\sigma = \sigma \mid \text{Buy}) + \sigma \Pr(\sigma = \sigma \mid \text{Buy}) \]

where by Bayes’ Rule

\[ \Pr(\sigma = \sigma \mid \text{Buy}) = \frac{\Pr(\text{Buy} \mid \sigma = \sigma)\Pr(\sigma = \sigma)}{\Pr(\text{Buy} \mid \sigma = \sigma)\Pr(\sigma = \sigma) + \Pr(\text{Buy} \mid \sigma = \sigma)\Pr(\sigma = \sigma)} \]

\[ = \frac{\alpha + \frac{1}{2}(1 - \alpha - \beta)}{\alpha + \frac{1}{2} \beta + \frac{1}{2} (1 - \alpha - \beta) (1/2)} \]

Plugging the above into the ask equation, and after some tedious algebra, the desired result obtains.

A symmetric argument holds on the bid side. The difference lies in the fact that the market maker’s conditioning set is over the market order to sell. Hence, the requisite conditional probability is

\[ \Pr(\sigma = \sigma \mid \text{Sell}) = \frac{\Pr(\text{Sell} \mid \sigma = \sigma)\Pr(\sigma = \sigma)}{\Pr(\text{Sell} \mid \sigma = \sigma)\Pr(\sigma = \sigma) + \Pr(\text{Sell} \mid \sigma = \sigma)\Pr(\sigma = \sigma)} \]

\[ = \frac{\frac{1}{2} \beta + \frac{1}{2} (1 - \alpha - \beta) (1/2)}{\frac{1}{2} \beta + \frac{1}{2} (1 - \alpha - \beta) (1/2) + \frac{1}{2} (1 - \alpha - \beta) (1/2)} \]

\[ \blacksquare \]

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\textbf{Notes}

1. See Azariadis (1981) for a treatise on self-fulfilling prophecies in economic situations, where prices move because they are expected to and not because fundamentals dictate. Azariadis’ research is in the spirit of the \textit{sunset equilibrium} literature due to Shell (1977) and Cass and Shell (1983), who formally model the interdependence of “rational” beliefs leading to self-justifying equilibria.

2. Recall that in a complete market, all that is required for a unique price is that all traders agree on the zero probability events. They need not agree on the objective distribution. This is most easily seen in the binomial model where the option’s price does not depend on the objective distribution, except through the null events.

3. Though only call options are covered in this paper, a corresponding analysis applies to put options.
4. One can interpret these models as providing an answer to this self-fulfilling prophecy question if the traders' beliefs in these models (lognormal) differ from the objective distribution, which is left unspecified.
5. An implied volatility is the volatility which equates the market price of the option to the Black-Scholes value.
6. For a paper which takes this approach, see Back (1993).
7. By algebra, this condition is equivalent to $E(\exp(W(\sigma))) = e^\sigma$. This restriction is given on Figure 1.
8. Similar to the traditional market microstructure models, hedgers (noise traders) are necessary for the existence of an equilibrium in our model.
9. As a stark illustration, it is possible that the objective distribution over $W(\sigma)$ is that of a nonrandom constant. In this case, the uncertainty over $W(\sigma)$ is extraneous and entirely attributable to beliefs.
10. Indeed, this is most easily seen by referring to expressions (11a) and (11b) in Azariadis (1981, p. 386), and making the following identifications. Due to risk neutrality $U(y_{t+1}) = y_{t+1}$, $G(y_t) = y_t$. The probabilities $q_1$ and $q_2$ are our prior probabilities over $W$, and $y_t$ corresponds to our call value at time $t$. The only remaining difference is that our model is non-Markov in $y_t$.
11. When $\alpha = \beta = 0$, the only possible equilibrium is case 5. Here, no equilibrium exists as conditions (i) and (ii) are violated. This shows that hedgers (or noise traders) are necessary in our model to obtain an equilibrium.
12. This is most easily seen if one defines $W(\sigma) = r - \sigma^2/2 + \sigma \phi$ where $\phi$ is normal with mean zero and variance 1.

References

Feinstein, Steven. (1989). "The Black-Scholes Formula is Nearly Linear in $\sigma$ for At-the-Money Options; Therefore Implied Volatilities from At-the-Money Options are Virtually Unbiased," working paper, Federal Reserve Bank of Atlanta.


