Current Advances in the Modeling of Credit Risk

The theory underlying the pricing models for credit derivatives is well developed but the empirical validation of the models awaits further research.

Credit derivatives are widely believed to represent the next growth area in the OTC derivative securities markets. Estimates of for the market's size in the year 2000 range from $100 billion to $1 trillion.¹ To grow to this potential, derivatives dealers that participate—the intermediaries—need to be able to act as brokers and offset their risks. In essence, there are only two ways to do this. The traditional approach is to find both a long and a short side to every trade. Although the simplest, this approach limits market growth since it requires a significant effort to match offsetting counterparties for each trade. The second approach to offsetting intermediation risks is not to match the trade with a counterparty, but rather to offset the trade by hedging in the underlying defaultable—or credit risky—securities. This approach provides the most flexibility because the credit derivative can be tailored to the needs of only one client, not two. Unfortunately, it requires that a model decompose the credit derivative into its component parts so that each of these parts can then be offset by a reverse trade in the underlying corporate debt market. Consequently, models for credit derivatives are essential for the expansion of the credit derivatives markets. Models for credit derivatives are essential in the same way that they have been for the expansion in the equity options, FX, and interest rate derivative markets.²

Since models are so important to the growth of the credit derivative markets, this article will review the current advances in the modeling of credit derivatives. Although inherently a mathematical topic, the mathematics will be kept to a minimum. Though no knowledge of pricing technology is assumed, we will describe the current frontiers of credit derivatives research. The abstract theory under the standard—perfect market assumptions—is well developed. The standard perfect market assumptions are:

- no transaction costs
- no taxes
- competitive markets.³

The remaining open questions are not theoretical, but practical. The practical

---

¹ Robert Jarrow is a professor at the Johnson Graduate School of Management, Cornell University, and Kamakura Corporation. He was named Financial Engineer of the Year in 1997 by the International Association of Financial Engineers. This article was completed December 1997 and updated February 1998.
Risk-neutral pricing methodology.

Let a random future cash flow at time 1 be denoted by \( X_1 \) and let \( r_0 \) be the interest rate earned on a riskless investment over the time period [0,1]. Under the risk-neutral pricing methodology, the present value of this cash flow, written \( V_0 \), is computed as:

\[
V_0 = E(X_1 / (1 + r_0)) \tag{1}
\]

where \( E(\cdot) \) is an expectations operator using risk adjusted probabilities. Computing the present value in expression (1) is done as it is in standard finance textbooks—as a discounted expected value. The only difference between expression (1) and the standard textbook presentation is that the probabilities used in computing the expectation in (1) are adjusted for the risk of the cash flow \( X_1 \), and not the discount factor. The traditional textbook computation adjusts the discount factor and not the probabilities. Expression (1) is the valuation technique underlying the famous Black-Scholes option pricing formula.

What makes the risk-neutral valuation approach preferable to the traditional textbook presentation is that there is an elegant mathematical theory showing that the risk adjusted probabilities can often be computed using only current prices and volatilities, both of which are observable. For example, this is the case with the Black-Scholes formula. In contrast, in the traditional textbook approach, one must compute the risk adjusted discount factor. This risk adjusted discount factor is an unobservable that must be estimated from market data. These risk adjusted discount factors are difficult, if not impossible, to estimate with any precision. In addition, as an added benefit, the risk neutral valuation approach provides a procedure for hedging (offsetting risk), while the traditional textbook approach does not. As argued above, hedging is the key characteristic of credit derivative models needed for the expansion in the credit derivatives markets.

To understand hedging in more detail, let us imagine that the time 1 cash flow \( X_1 \) is being generated by a derivative security. A derivative security is a financial security whose value or cash flow is derived from the value or cash flow to some underlying asset. An example of a derivative security would be a call option on a stock. The call option derives its value from the fact that the stock’s price may exceed the strike price of the option at the option’s maturity date.

For our analysis, let \( S_0 \) represent the time 0 price of the asset underlying the derivative security’s cash flow \( X_1 \). Let \( S_t \) be the underlying asset’s time 1 price. Then, we can rewrite expression (1) as:

\[
V_0(S_0) = E(X_1(S_0S_t) / (1 + r_0)) \tag{2}
\]

Here we have made explicit the facts that \( X_1 \) depends on \( S_0 \) and \( S_t \). This, in turn, implies that the present value depends on the current stock price \( S_0 \).

To determine how to hedge the derivative, we need to compute the derivative’s delta. The derivative’s delta is the (partial) derivative of \( V_0 \) with respect to \( S_0 \), denoted \( dV_0 / dS_0 \).

Let \( V \) represent the value of the derivative security at time 1. In this simple case, the derivative’s value at time 1 is equal to its time 1 cash flow,
The basic model for pricing credit derivatives is similar to that for pricing equity options. 

\[ V_1 = X_1 \text{, but in more complex cases this will not always be true.}^{9} \]

Then, it can be shown\(^{10} \) that:

\[ V_1 - V_0 = \text{constant} + dV_0/dS_0 [S_1 - S_0] + \text{small error.} \tag{3} \]

This gives the decomposition of the derivative security into its component parts. It says that the change in the value of the derivative security \([V_1 - V_0]\) is equivalent to \(dV_0/dS_0\) units of the change in the value of the underlying asset. Hence, the derivative security is equivalent to \(dV_0/dS_0\) units of the underlying asset. To hedge, therefore, one just takes an offsetting position in \(dV_0/dS_0\) units of the underlying asset. As long as the underlying trades, then this hedge can be created. Markets where hedges for the derivative security can be created are called “complete.” Complete markets are the ones where the growth potential for their derivative markets is the greatest. For example, the equity, foreign currency, and interest rate markets are widely believed to be complete.

This is the basic structure of the model. The model works well under the perfect market assumptions of no transaction costs, no taxes, and competitive markets. To apply this model to credit derivatives, it needs to be expanded in two ways. One is that the model needs to allow the derivative security to have multiple cash flows at multiple dates. Second, the model needs to allow the derivative to depend on more than one underlying asset.

These extensions are easily included. We first discuss multiple cash flows. Let \(X_1, X_2, X_3, ..., X_T\) represent the multiple (random) cash flows at future dates 1, 2, ..., T. Let \(r_0, r_1, r_2, ..., r_{(T-1)}\) be the corresponding rates of interest on a riskless investment over the time periods \([0,1], [1,2], ..., [T-1,T]\), respectively. Then, the generalized present value calculation is:

\[ V_0 = E(X_1/[1+r_0]) + E(X_2/[1+r_0]) \]
\[ + ... + E(X_T/[1+r_0]) \]
\[ + [1+r_1] ... [1+r_{(T-1)}]). \tag{4} \]

The present value of a derivative security with multiple cash flows is just the sum of the present values of each of the separate cash flows. This is no more difficult to compute than expression (1).

Next, we consider the extension where the derivative security has multiple underlying assets. An example of such a derivative security is a put option on the minimum value of two different corporate bonds. A put option gives the holder the right to sell the underlying asset for a predetermined price. In this case, the underlying is the worst performing of the two corporate bonds. For multiple underlying assets, one uses an expanded version of expression (2).

Let \(S_0, S_1, P_0, P_1\) represent different underlying asset prices, then we can simply rewrite the time 1 cash flow \(X_1\) as dependent on these values, i.e., \(X_1(S_0, S_1, P_0, P_1)\). This in turn implies that the present value depends on the two underlying prices at time zero, \(V_0(S_0, P_0)\). Next, the derivative security can be decomposed into units of these two underlying assets:

\[ V_1 - V_0 = \text{constant} + dV_0/dS_0 [S_1 - S_0] + dV_0/dP_0 [P_1 - P_0] + \text{small error.} \tag{5} \]

In expression (5), the change in the value of the derivative security \([V_1 - V_0]\) is equal to \(dV_0/dS_0\) units of the change in the value of the first underlying plus \(dV_0/dP_0\) units of the second change in the value of the second underlying. Thus, the derivative security is equal to \(dV_0/dS_0\) units of the first underlying plus \(dV_0/dP_0\) units of the second. A hedge is obtained by taking these offsetting positions in both the underlying assets. Applying this extended model is no more difficult than applying the simpler expression (3).

In fact, both extensions to the basic model can apply in conjunction. In which case both expressions (4) and (5) apply simultaneously. This completes the discussion of the basic model.

**Credit Risk—the FX Analogy**

The basic credit derivative pricing problem can now be easily understood. Consider pricing a loan of X dollars made to an individual. The loan repayment date is time 1. The loan is risky because the individual may default on his promised payment. Using the risk-neutral valuation procedure, we can price and (possibly) hedge this loan. Let P represent the actual payment. Note that X represents the promised payment. P can be less than X. Then, applying expression (1), we get:

\[ V_0 = E(P/[1+g_0]). \tag{6} \]

To understand this formula, we need to represent the actual payment \(P\) in terms of the promised payment \(X\). To...
do this, we use the FX analogy of Jarrow and Turnbull.11

The FX analogy is easily explained. Consider two currencies, dollars and
promised dollars, both with their own term structure of interest rates.
Promised dollars is the hypothetical currency formulated to construct the
analogy. Each term structure is default free in its own currency. There is an
exchange rate linking the two currencies. The exchange rate of dollars per
promised dollar is usually unity, except in default, when it drops to less than
one. Thus, there exists significant exchange rate risk. Derivatives on these
two term structures or currencies are usually called FX derivatives. In our
case, these are called credit derivatives.

This FX analogy states that we can write the actual payment as:

$$ P = eX $$  \tag{7}

where $e$ is the exchange rate of promised dollars to actual dollars at
the repayment date 1. If the individual defaults, then $e < 1$. In this case
the actual payment $P$ is less than the promised payment $X$. If the individual
does not default, then $e = 1$. Here, the actual payment equals the promised
payment.

The importance of the FX analogy is that FX derivative pricing and hedging is a well-studied and well-understood field.12 This implies that the study of credit derivatives has a firm foundation upon which to build. The only significant difference between the two applications is in the modeling of the "exchange rate" process $e$.

Understanding this exchange rate process from promised dollars to
dollars is the key to credit derivatives pricing, and must, therefore, be
investigated in more detail. In this context, we can rewrite this exchange rate as:

$$ e = \delta < 1 \text{ if default occurs, with probability } \lambda \\
= 1 \text{ if no default, with probability } 1 - \lambda. $$  \tag{8}

Here, $\delta$ is called the recovery rate. Historical estimates of the recovery rate are readily available.13 $\lambda$ is the cumulative (risk - adjusted) probability of default by time 1. Although historical estimates of $\lambda$ are also available,14 these estimates must be adjusted to incorporate a risk premium.15

Given this identification, we can rewrite expression (6) in a very
revealing form:

$$ V_0 = E(X\delta/[1+r_0] | \text{ default})\lambda + \\
E(X/[1+r_0] | \text{ no default})(1 - \lambda). $$  \tag{9}

This is the final valuation formula in its simplest form. The value of the risky loan is equal to the discounted payoff in default times the probability of default plus the discounted payoff if no default occurs times the probability of no default. These
quantities are easily computed.

Generalizations involve multiple time periods and multiple cash flows.

These extensions are as given in expressions (4) and (5) above.

For applications, the remaining debate revolves around the models used to determine the default probabilities - $\lambda$. For most applications, the recovery rate is taken as an estimated constant, so there is less debate on this quantity.

MODEL CLASSIFICATION

The different models can be classified as to how the default probabilities are modeled. Jarrow and Turnbull16 contains a model where the default probabilities are non-random constants. This is the simplest formulation and it leads to simple analytic formulas for various credit derivatives. The limitation of this formulation is that it cannot price credit derivatives whose payoffs depend on the credit rating of the underlying. The reason is that the model does not include credit rating as an important variable. In addition, it implies that the credit spread to Treasuries is a constant across time. This implication of the model is inconsistent with the evidence.

Credit ratings have been included in a model of the default probabilities by Jarrow, Lando, and Turnbull.17 This model has received significant attention in the professional community, because it uses a simple Markov transition probability matrix of credit ratings in the valuation. A Markov transition probability matrix of credit ratings is a table that has credit ratings, say [AAA, AA, A, ...., C, default] as both the row and column labels. The rows indicate the current credit rating status of a firm (under consideration). The columns represent the future credit rating status of the firm after some time period has passed,
Most commonly a year. The table entries in a row for a particular column provide the probability of moving from the current credit rating status (the row) to the future credit rating status (the column) in a year. Historical credit rating probability matrices are available. The simplest form of the model implemented, which has all the entries in the table constant across time, again implies a constant credit rating spread to Treasuries across time.

To eliminate the implication of a constant credit spread to Treasuries across time, one needs to let the default probabilities (or the Markov probability transition matrix) be random. The easiest way to accommodate this is to let the default probability depend on the current state of the economy at any point in time. The most general formulation for modeling the default probability allows there to be numerous macro- and micro-variables that influence its level. For example, the macro-variables could include the level of interest rates, a measure of general economic activity like GNP, or a market stock index. The micro-variables might include the firm’s stock price. Then, as these macro- and micro-variables change, the default probability changes. For example, if the economy goes into a depression, both GNP and the stock market index would fall. Since the default probability depends on their levels, the probability of default would increase.

These general formulations for the default probabilities can be found in Jarrow and Turnbull,16 Lando,19 Madan and Udal,20 or Duffie and Singleton.21 There are a plethora of models to choose from in this regard. The differences in the models correspond (mostly) to subtle technical differences in the modeling of the default probability. The theoretical models are well developed. These models are quite robust, their only limitations are contained in:

1. The underlying perfect market assumptions—no transaction costs, no taxes, and competitive markets.
2. The data available for estimation and implementation.
3. Their computational complexity.

We will discuss each of these in turn.

**Perfect Market Assumptions**

The perfect market assumptions—no transaction costs, no taxes, and competitive markets—are the same assumptions underlying the traditional models used to price equity, FX, and interest rate derivatives. These models have been employed with reasonable success in these various markets, despite the fact that the assumptions are not formally true. Nonetheless, they are reasonable approximations to reality. The reasons are that for professional traders, transaction costs are low and profits are usually all short-term capital gains, so that taxes do not differentially affect the analysis. In addition, the volume of trading in these markets is large enough so that liquidity is readily available, i.e., the competitive markets assumption is reasonable.

For credit derivatives and the underlying corporate debt market, the same arguments should apply, with perhaps one exception. The volume of trading in the corporate debt market is not as great as it is in the equity, FX, or interest rate (swaps) markets. This implies that liquidity risk may be an important factor to consider in the pricing of credit derivatives. The above-discussed models do not explicitly incorporate this noncompetitive market feature.22 This omission may imply increased model error or model risk.

The importance of this omission and the inclusion of liquidity risk into credit derivative models are topics of current research.23

**Empirical Implementation**

Given the analogy between FX derivatives and credit derivatives, the empirical methodologies applied to the pricing and hedging of FX derivatives apply to credit derivatives as well. The problems faced in both areas are due to the complications caused by multiple term structures and multiple factors. The existence of multiple term structures is self-evident given the existence of two different currencies (dollars and promised dollars). The existence of multiple factors needs some explanation. A “factor” is an economic force generating randomness in prices. Examples of factors would be inflation, production, aggregate savings, or consumption. The more term structures that exist, the more “factors” or economic forces that are

---

16 Jarrow and Turnbull, supra note 16.
22 Duffie and Singleton (1997) consider liquidity costs implicitly in their valuation equations.
23 For a paper in this direction see Cherian, Jacquier, Jarrow, and Ma, “Pricing the Convenience Yield of Treasuries: Theory and Evidence,” working paper, Cornell University.
Liquidity risk becomes important in pricing credit derivatives.

needed to capture the risks inherent in the related markets.

For valuation, the relevant parameters are obtained from an estimated covariance matrix of changes in the value of the default free and credit risky bonds, across the different maturities. This covariance matrix contains the term structure of the bonds’ volatilities and correlations. In addition, for pricing credit derivatives, the (risk-adjusted) probability of default (λ) and the recovery rate (δ) need to be estimated.

There are two methods for estimating these quantities—historic estimation and implicit estimation. “Historic estimation” uses past observations of the bond prices to estimate the relevant statistics. “Implicit estimation” (sometimes called market calibration) uses only current market prices. Given the market price for a risky bond, the model is inverted to find those parameters such that the model price for the risky bond equals the market price.

In the standard FX derivatives pricing problem, both methods are appropriate. For pricing credit derivatives, historic estimation is more problematic. The reason is that given a particular (nonbankrupt) firm, it is impossible to estimate the default probability and recovery rate directly, given that the firm has not defaulted. These historic default probability estimates are a necessary first step, needed prior to the second step, which is the risk adjustment made for their use in valuation.

It is possible, although difficult, to estimate the default probability and the recovery rate from the average returns on the firm. This is possible because the risky bond’s expected returns reflect the probability of default. However, given the difficulty in estimating expected returns using market prices, this approach does not appear promising. The difficulties in estimating expected returns result from modeling non-stationarities (and the large standard errors of the estimates).

An alternative procedure, but still using historic prices (and default experiences), is to estimate the default probabilities and the recovery rate using a “representative” firm. A “representative” firm is a firm “like” the firm under consideration, as measured by various attributes—same asset size, credit rating, industry, debt structure, etc. Then, one uses the estimated default probabilities and recovery rate from the “representative” firm for the firm under consideration. This is the procedure followed by Jarrow, Lando, and Turnbull. Although promising, it is not always possible to find a suitable “representative” firm. After all, how many IBMs are there?

Given the above difficulties with using historic price (and default) data, implicit estimation becomes the preferred alternative. Implicit estimation of volatilities, for example, is quite commonplace in the pricing of equity, FX, and interest rate derivatives. These procedures use a set of market prices and compute (numerically on a computer) those volatilities that equate the model’s prices to the market prices. Implicit volatilities can be interpreted as the “market’s beliefs.” Implicit estimates, therefore, have the added benefit of being forward looking, while historic estimates (by construction) are not. Similar procedures can be applied to credit derivatives.

Surprisingly, the empirical evidence with respect to which models best fit the data and why they do is limited. This is mainly due to the fact that not enough time has passed to generate a significant study of the new pricing models. But, it is also partly due to the fact that given the illiquidity in the corporate bond market, time series prices of corporate bonds are not readily available. This makes scientific study of the different models difficult. The empirical validation of the various models awaits subsequent research.

Computation

The last issue in using credit derivatives pricing models concerns their computation. Using the FX analogy as a reference, the computational complexity of the credit derivatives pricing model can be seen to be equivalent to the computational complexity of the FX derivatives pricing model. Unfortunately, the computational complexity of the FX derivatives pricing model is quite large because of multiple factors and multiple term structures. The credit derivatives pricing models have the identical problems.

Since the computational techniques are the same as those used for pricing FX derivatives, a detailed discussion of these issues can be found in other sources. Here, we just provide an overview of the basic techniques available. The basic techniques can all be understood by recognizing

---

24 Jarrow, Lando, and Turnbull, supra note 17.
25 See Jarrow and Turnbull, supra note 4.
27 Id.
that computing credit derivative prices is equivalent to calculating an expectation (see expression (1)). Expectations can be computed using lattices, trees (nonrecombinating), techniques for solving partial differential equations—implicit and explicit difference procedures, and Monte Carlo simulation. Each of these procedures can be used to generate a distribution for future cash flows, from which an expectation can be inferred. Each of these procedures also have their benefits and costs. No procedure dominates the others. For low dimensional problems—lattices, trees, and partial differential equation techniques—are preferred. For high dimensional problems, Monte Carlo becomes the preferred procedure. This is true because the approximation error for lattices, trees, and partial differential equation techniques grows with the size of the dimension. In contrast, the approximation error for Monte Carlo simulation is independent of the problem's dimension. The dimension of the problem is (roughly) equivalent to the number of factors used in the model. Credit derivative pricing problems, like FX derivative pricing problems, are sometimes high dimensional problems.

**CONCLUSION**

This article has summarized the advances in the modeling of credit derivatives. The theory underlying the models is well developed, but the implementation of the models is less so. Of the perfect market assumptions underlying the model structure—no transaction costs, no taxes, and competitive (liquid) markets—the illiquidity of the corporate bond market has the biggest potential for generating model error. In implementation, estimation of the model's parameters is crucial. Estimation of the credit derivative model parameters is different from the traditional derivatives pricing model estimation. This is because many of the needed parameters correspond to default, and they cannot be directly estimated before default occurs. Direct estimation of these parameters is, therefore, impossible. "Representative" firm procedures are used instead. Implicit estimation (or market calibration) of the required parameters plays an important role. Due to the newness of the models and the sparsity of accurate price data, the empirical validation of the various models awaits subsequent research.