THE SECOND FUNDAMENTAL THEOREM OF ASSET PRICING

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This paper presents a resolution of the paradox proposed by the example of an economy with complete markets and a multiplicity of martingale measures constructed by Artzner and Heath (1995). The resolution lies in noting that completeness is with respect to a topology on the space of cash flows and is connected with uniqueness of the price functional in the topological dual space. Uniqueness may be lost outside the dual and this is what occurs in the counterexample of Artzner and Heath.

KEY WORDS: state price density, market completeness

The second fundamental theorem of asset pricing relates market completeness, or the ability to hedge arbitrary claims, to the uniqueness of martingale measures. The result was first established for finite state, finite time economies, (see Taqqu and Willenger 1987). Harrison and Pliska (1983) extended the result to continuous trading in finitely many assets with continuous semimartingale price processes. The results were extended by Delbaen (1992) to the case of infinitely many securities trading in continuous time, with price processes that are bounded, continuous semimartingales. The notion that the second fundamental theorem may not be true in general was generated by an example in Artzner and Heath (1995). They provide an economy with a complete market but an infinity of martingale measures.

The purpose of this paper is to study the second fundamental theorem in a one-period setting with infinitely many securities (with bounded jumps in prices). This setting is an intermediate step to the consideration of continuous trading. This setting is sufficient, however, to clarify and resolve the paradox presented by the Artzner and Heath (1995) example. The key result of this paper is that, given the proper notion of completeness, the second fundamental theorem of asset pricing holds.

The proper notion of completeness relates to the topology used to define it. The space we consider is one of bounded cash flows defined on a probability space. The probability represents the statistical measure. On this space of cash flows, there are many different topologies that may be used to define the notion of completeness. There is the weak
topology induced by the bounded cash flows. There are the weak topologies generated by the \( q \) sumizable elements for \( q \geq 1 \). Furthermore, considering the bounded cash flows as a subspace of the space of \( q \) sumizable elements for \( q \geq 1 \), there are also the various norm topologies.

The weak and norm topologies provide very different notions for approximating cash flows. Given a choice of topology, the space of cash flows has a well-defined dual space. Our key result is that completeness in a particular topology is always equivalent to uniqueness of the signed state price density in the dual space associated with its topology. The fact that we have only signed state price densities, which could be negative, implies that arbitrage opportunities may exist in these economies. This is reasonable, as the notion of completeness is logically separate from the notion of no arbitrage.

Therefore, a secondary contribution of our paper, as noted above, is separating the notion of completeness from the notion of no arbitrage. Completeness refers to the ability to construct "arbitrary random cash flows." The two concepts are logically distinct—that is, one can hold without the other (or neither, or both). In fact, in practice, one often looks for arbitrage opportunities in complete markets. Given this, it is also important that the notion of completeness be independent of a martingale measure (which may not exist). Our definition meets this condition, as it only depends on the statistical measure. This is in contrast to the definitions of completeness found in either Harrison and Pliska (1983) or Delbaen (1992).

For economies with no arbitrage, we show that the second fundamental theorem holds in a more restricted form. For weak topologies, completeness in a topology is equivalent to uniqueness of a positive state price density in its dual space, but only if certain conditions are satisfied. One condition requires the space of traded assets to include all call options. When this condition is not satisfied, we provide a counterexample showing that one may have uniqueness of a positive state price density without completeness. For the norm topologies, similar results hold, but only when \( q \) is less than \( \infty \). For the essential supremum norm, no results for positive state price densities could be proven.

The resolution of the Artzner and Heath (1995) counterexample lies in identifying the appropriate dualities. Their cash flow space has the norm topology with \( q = 1 \). The martingale measures they exhibit are not in the dual space for this topology, but instead they are in the space of summable elements with the norm topology of \( q = 1 \). Using our results, we can understand their example more fully. For their example, given the non-uniqueness of the measure, we expect non-completeness in the dual topology—the essential supremum norm topology on the space of bounded cash flows. This is in fact the case. Hence, in the essential supremum norm topology the counterexample is just an example of non-uniqueness and non-completeness. Furthermore, for their example, given the completeness in the norm topology of \( q = 1 \), we expect uniqueness of the martingale measure in the dual space of essentially bounded functions. This is also true. Thus, in the norm topology of \( q = 1 \), their example illustrates that completeness is equivalent to uniqueness in the appropriate dual.

An outline of the paper is as follows. Section 1 introduces the economy we consider. In Section 2 we establish the general form of the second fundamental theorem of asset pricing, i.e., the equivalence of market completeness with the uniqueness of a signed state price density. This result permits arbitrage opportunities. Section 3 considers the case of the weak topologies and establishes the equivalence of the uniqueness of a positive state price density in the appropriate dual to the completeness of markets. This result excludes arbitrage opportunities. Section 4 considers the norm topologies for \( q \) below \( \infty \). In Section 5
we provide counterexamples showing that when call options do not trade, one may have uniqueness of a positive state price density without completeness. We also show, by way of a counterexample, that completeness in a particular topology, though it implies uniqueness in the dual space, does not imply uniqueness in the space of all state price densities. Section 6 comments on the Artzner and Heath (1995) counterexample. Section 7 concludes the paper.

1. THE GENERAL ECONOMY

The economy has two dates \([0, 1]\). Let \((\Omega, \mathcal{F}, P)\) be a probability space at time 1. A cash flow is defined to be a real-valued \(\mathcal{F}\) measurable function \(c: \Omega \to \mathbb{R}\), such that \(P(\{\omega: |c(\omega)| \leq \theta\}) = 1\) for some real number \(\theta\). We denote by \(\mathbb{C}\), the vector space of all such bounded cash flows.

Traded in this economy are a possibly infinite collection of assets indexed by the elements of \(a\) of an abstract set \(A\). Let \(\mathcal{A}\) be a \(\sigma\)-field of measurable subsets of \(A\). For each \(a \in A\), ownership of the asset gives the holder the claim to a cash flow in \(\mathbb{C}\). We denote this cash flow by the equivalence class of functions \(c[a]\), such that elements of \(c[a]\) are almost surely equal to some function \(c(\omega, a)\), where \(c(\omega, a)\) is \(\mathcal{A}\mathcal{F}\) measurable.

Let \(\nu\) be a positive measure defined on \(\mathcal{A}\), with \(\nu(A) < \infty\). The measure \(\nu\) represents a reference portfolio. We define a trading strategy to be real-valued \(\mathcal{A}\) measurable function, \(y: \mathcal{A} \to \mathbb{R}\) defining the position the strategy takes in the asset \(a\), relative to the reference portfolio \(\nu\). Positive values represent long positions and negative values represent short positions. The position obtained by the reference portfolio can be obtained by the trading strategy \(y = 1_{\mathcal{A}}\), which is the constant function that takes the value 1 for all \(a \in A\). The vector space of admissible trading strategies consists of all \(\mathcal{A}\) measurable functions that are bounded in the essential supremum norm of the measure \(\nu\). We further suppose that the asset cash flows satisfy an integrability condition across \(a \in A\), with respect to the reference measure \(\nu\), specifically requiring that

\[
\text{ess sup}_{\omega} \int_{A} |c(\omega, a)| \nu(da) < \infty. \tag{1.1}
\]

For each \(y \in \mathcal{Y}\), we define the cash flow obtained by the trading strategy as \(\Phi(y)\), where

\[
\Phi(y)(\omega) = \int_{A} y(a)c(\omega, a)\nu(da). \tag{1.2}
\]

Under the integrability condition, equation (1.2) defines a linear operator from the vector space \(\mathcal{Y}\) to the vector space \(\mathbb{C}\).

For each asset \(a \in A\) there is a market price denoted by \(\pi(a)\). We require that market price functionals, \(\pi: \mathcal{A} \to \mathbb{R}\), are \(\mathcal{A}\) measurable with \(\int_{A} |\pi(a)|\nu(da) < \infty\). The integrability condition implies that the reference portfolio has a finite value. Our candidate price functionals are in \(L^{1}(A, \mathcal{A}, \nu)\).

For each \(a \in A\) there is a unique \(c(\omega, a) \in \mathbb{C}\). Thus, we can think of \(\pi\) as giving the price of the cash flow \(c[a]\). This identification motivates us to consider the space of possible price functionals as the space \(\mathcal{Q}\) of all linear functionals on \(\mathbb{C}\) that are continuous with respect to some topology weaker than the \(L^{\infty}\) norm topology. This topology is specified below.
Set theoretically we make the following identifications, $\mathbb{C} = L^\infty(\Omega, \mathcal{F}, P)$ and $\mathbb{Y} = L^\infty(A, \mathcal{A}, \nu)$. We endow $\mathbb{C}$ with some topology $\tau$ such that $\mathbb{C}$ is a locally convex topological vector space. The topology we place on $\mathbb{Y}$ is the one induced by the operator $\Phi$. That is, a subset $V$ of $\mathbb{Y}$ is a neighbourhood of 0 if and only if there exists a neighbourhood $U$ of zero in $\mathbb{C}$ such that $V = \{y \in \mathbb{Y} \mid \Phi(y) \in U\}$. We denote this topology on $\mathbb{Y}$ by $\tau_\Phi$, and note that by construction $\mathbb{Y}$ is a locally convex topological vector space under $\tau_\Phi$ and $\Phi$ is a continuous linear operator.

Once $\mathbb{C}$ and $\mathbb{Y}$ are defined as locally convex topological vector spaces, the topological dual spaces of all continuous linear functionals on $\mathbb{C}$ and $\mathbb{Y}$ are identified. We shall restrict attention to topologies on $\mathbb{C}$ for which the dual of $\mathbb{Y}$ with its induced topology, is contained in $L^1(A, \mathcal{A}, \nu)$. We denote these dual spaces $\mathbb{X}$ and $\mathbb{Q}$, where by construction we shall have $\mathbb{X} \subseteq L^1(A, \mathcal{A}, \nu)$ and $\mathbb{Q} \subseteq ba(\Omega, \mathcal{F}, P)$. Our candidate price functionals are potentially elements of $\mathbb{X}$. We place on $\mathbb{Q}$ and $\mathbb{X}$ the weakest topologies consistent with $\mathbb{C}$ and $\mathbb{Y}$ being the biduals. Hence, the topology on $\mathbb{Q}$ is the weak topology $\sigma(\mathbb{Q}, \mathbb{C})$ and that on $\mathbb{X}$ is the topology $\sigma(\mathbb{X}, \mathbb{Y})$ (for further details on the definitions of these topologies the reader is referred to Horvath 1966, p. 185, or Schaefer 1971, p. 52). With these topologies, $\mathbb{Q}$ and $\mathbb{X}$ are locally convex topological vector spaces. Furthermore, $(\mathbb{Q}, \mathbb{C})$ and $(\mathbb{X}, \mathbb{Y})$ form Hausdorff dual pairings provided the initial topology on $\mathbb{C}$ is Hausdorff. We shall let $\langle \cdot, \cdot \rangle$ denote the bilinear form for both pairings.

**Definition 1.1.** Algebraic completeness and market completeness with respect to a topology. The market is said to be algebraically complete if for all $c \in \mathbb{C}$ there exists $y \in \mathbb{Y}$ such that $\Phi(y) = c$. The market is said to be complete with respect to some topology $\tau$, if for all $c \in \mathbb{C}$ there exists a net $y_\alpha$ for $\alpha \in D$, for $D$ a directed set, such that $\lim_\alpha \Phi(y_\alpha) = c$ with respect to the topology $\tau$.

We note that the concept of algebraic completeness is algebraic in the mathematical sense and does not use any topological structure. This definition is too strong and would hardly ever be satisfied in practice. A weaker definition of completeness depends on a topology for the space of cash flows, $\mathbb{C}$. From an economic perspective, market participants would certainly view any sequence of trading strategies converging to a particular cash flow in the essential sup norm on $\mathbb{C}$, as approximately equal to the limit cash flow. What is at issue here is whether any weaker topologies would suffice for the purpose of constructing approximations. Definition 1.1 allows for this flexibility.

**Definition 1.2.** Signed state price density and positive state price density. A signed state price density is an element $q \in \mathbb{Q} \subseteq L^1(\Omega, \mathcal{F}, P)$, the topological dual of $\mathbb{C}$, such that for all $a \in A$,

$$\pi(a) = \langle q, c[a] \rangle,$$

where $\pi$ is the market price function. A state price density is said to be positive if $P\{\omega \mid q(\omega) > 0\} = 1$.

The second fundamental theorem relates completeness with respect to a topology $\tau$ to uniqueness of signed state price densities. Price functionals induced by state price densities are linear functions on $\mathbb{C}$ and, as such, they belong to the algebraic dual of $\mathbb{C}$. Topological considerations arise when we consider the continuity of the state price densities. If two cash flows are deemed similar in some sense, then their prices should also be close in some
sense, that is, in some topology. We will be concerned with topologies on \( \mathbb{C} \) for which the dual \( \mathcal{Q} \) is contained in \( L^1(\Omega, \mathcal{F}, \mathbb{P}) \). For example, for any subspace of \( L^1(\Omega, \mathcal{F}, \mathbb{P}) \), say \( L^q(\Omega, \mathcal{F}, \mathbb{P}) \) for \( q > 1 \), one may define on \( \mathbb{C} \) the topology \( \sigma(\mathbb{C}, L^q(\Omega, \mathcal{F}, \mathbb{P})) \). In this topology on \( \mathbb{C} \), the dual \( \mathcal{Q} \) of \( \mathbb{C} \) is \( L^q(\Omega, \mathcal{F}, \mathbb{P}) \).

We consider weak topologies on \( \mathbb{C} \) consistent with dual \( L^q(\Omega, \mathcal{F}, \mathbb{P}) \), \( q \geq 1 \), and the norm topologies generated by the inclusion of \( \mathbb{C} \) in \( L^q(\Omega, \mathcal{F}, \mathbb{P}) \). It is important to note that these topologies are not necessarily comparable. The norm topology of \( L^q \) is stronger than the weak topology when \( q \) exceeds 2, but for \( q < 2 \), the topologies are not comparable (the appendix provides examples for the case \( q = 1 \)). We also note that for the \( \sigma(\mathbb{C}, L^q(\Omega, \mathcal{F}, \mathbb{P})) \) topology the dual \( \mathcal{Q} \) is \( L^q(\Omega, \mathcal{F}, \mathbb{P}) \), while for the \( L^q(\Omega, \mathcal{F}, \mathbb{P}) \) norm topology, by the Hahn–Banach extension theorem, the dual \( \mathcal{Q} \) is \( L^p(\Omega, \mathcal{F}, \mathbb{P}) \) where \( p \) is the conjugate exponent \((\frac{1}{q} + \frac{1}{p} = 1)\).

The signed state price density is not required to be positive and may take negative or zero values on sets of positive probability. This implies that arbitrage opportunities may exist. Although our concern here is with the second fundamental theorem, the question of the existence of signed state price densities and positive state price densities requires us to review the various concepts of no arbitrage.

In this regard, let \( V = \{v(a) = (\sigma(a), c(a)) \mid a \in A\} \) be the infinite collection of asset prices and cash flows, at times 0 and 1 respectively. Suppose that \((1, 1_\Omega) \in V \) where \( 1_\Omega \) is the constant function 1 on the set \( \Omega \). Denote by \( Lin(V) \) the linear span generated by \( V \) and set

\[
\mathbb{C}_0 = \{v_1 \mid (0, v_1) \in Lin(V)\} \subseteq \mathbb{C}.
\]

The set \( \mathbb{C}_0 \) represents those cash flows that can be generated at zero cost. We denote by \( \overline{\mathbb{C}}_0 \) the closure of \( \mathbb{C}_0 \) in the topology \( \sigma(\mathbb{C}, \mathcal{Q}) \).

**Definition 1.3.** No weak arbitrage and no weak arbitrage with free disposal. There are no weak arbitrage opportunities if and only if

\[
\overline{\mathbb{C}}_0 \cap L^\infty_+ = \{0\}.
\]

Furthermore, there are no weak arbitrages with free disposal if and only if

\[
\overline{\mathbb{C}}_0 - L^\infty_+ \cap L^\infty_+ = \{0\}.
\]

These are the standard notions of no weak arbitrage and no weak arbitrage with free disposal used in the literature; see Delbaen (1992), Lakner (1993), and Delbaen and Schachermayer (1994) for further explanation. From this literature the following lemma holds.

**Lemma 1.1.** Existence of signed state price densities and positive state price densities. There exists a state price density if and only if there are no weak arbitrage opportunities. Moreover, there exists a positive state price density if and only if there are no weak arbitrages with free disposal.

**Proof.** Since this lemma can be proved in exactly the same method as that of Theorem 2.1 in Lakner (1993), we omit the proof. \( \square \)
2. MARKET COMPLETENESS WITH RESPECT TO A TOPOLOGY AND THE UNIQUENESS OF (SIGNED) STATE PRICE DENSITIES

In this section we establish the general form of the second fundamental theorem of asset pricing. To state this theorem, we first need to define what we mean by uniqueness of the state price density. Uniqueness of the state price density means that, given an initial price function \( p(a) \), there exists only one state price density \( q \), up to equivalence with respect to differences on \( P \) null sets, such that \( p(a) = \int \Omega q(\omega)c(\omega, a)P(d\omega) \), for all \( a \in A \).

The next theorem relates to market completeness with respect to a topology \( \tau \).

**Theorem 2.1.** Suppose that there exists a state price density. Then the market is complete with respect to \( \sigma(\mathbb{C}, \mathbb{Q}) \) if and only if the state price density is unique in \( \mathbb{Q} \).

**Proof.** Consider the operator \( \Psi \) defined from \( \mathbb{Q} \) to \( \mathbb{X} \) by

\[
\Psi[q](a) = \int \Omega q(\omega)c(\omega, a)P(d\omega).
\]

The operator \( \Psi \) is the pricing operator and employs state price densities to price assets. The topology on \( \mathbb{X} \) is the weak topology induced by elements of \( \mathbb{Y} \). For any \( y \in \mathbb{Y} \), define \( \tilde{c} = \Phi[y] \) and observe that \( \langle \Psi[q], y \rangle = \langle q, \Phi(y) \rangle \) and hence, if a net \( q_n \) tends to zero in the weak topology on \( \mathbb{Q} \), then \( \Psi[q_n] \) tends to zero in \( \mathbb{X} \), in the weak topology. The operator \( \Psi \) is a continuous linear operator.

Consider now the adjoint of \( \Phi \), that is an operator \( \Phi' \) from \( \mathbb{Q} \) to \( \mathbb{X} \) defined by the condition that for \( y \in \mathbb{Y} \)

\[
\langle y, \Phi'[q] \rangle = \langle \Phi[y], q \rangle.
\]

It follows from the observation \( \langle \Psi[q], y \rangle = \langle q, \Phi[y] \rangle \), that \( \Psi \) is the adjoint of \( \Phi \). From the properties of continuous linear maps on locally convex topological vector spaces (Grothendieck (1973, p. 82, Prop. 27) that \( \Psi \) is \( 1 \rightarrow 1 \) if and only if the range of \( \Phi \) is dense in \( \mathbb{C} \). The theorem follows on noting that \( \Psi \) is \( 1 \rightarrow 1 \) only if the state price density is unique, and the market is complete only if \( \Phi \) has dense range. \( \square \)

Crucial in the proof of Theorem 2.1 is the identification of the adjoint operator \( \Psi \) to \( \Phi \), mapping \( \mathbb{Q} \) to \( \mathbb{X} \). This adjoint operator is \( 1 \rightarrow 1 \) if and only if the state price density is unique. This insight gives us the following corollary.

**Corollary 2.1.** The adjoint operator \( \Psi: \mathbb{Q} \rightarrow \mathbb{X} \) is \( 1 \rightarrow 1 \) if and only if the market is complete with respect to \( \sigma(\mathbb{C}, \mathbb{Q}) \).

This corollary is useful in applications because it is often easier to verify that the adjoint operator is \( 1 \rightarrow 1 \) in a particular topological dual than it is to show that the range of \( \Phi \) is dense. An application of this corollary gives market completeness for these market structures.

The fact that the state price density may be negative on sets of positive probability \( P \) is crucial. When this occurs, the market has arbitrage opportunities. As pointed out in the introduction, our concept of market completeness is independent of the notion of no arbitrage and depends only on the statistical or true probability measure, as opposed to a presumed-to-exist martingale measure. Markets do not have to be arbitrage-free just because they happen to be complete. However, given additional restrictions on the economy, stronger
results can be obtained—yielding markets that are both complete and arbitrage free. This analysis is conducted in the next section.

3. ARBITRAGE FREE COMPLETE MARKETS AND UNIQUENESS OF POSITIVE STATE PRICE DENSITIES: THE CASE OF THE WEAK TOPOLOGIES

This section studies the second fundamental theorem for the weak topologies on $C$. For this case, let $M$ be some subspace of $L^1(\Omega, \mathcal{F}, P)$ and consider the pairing $(C, M)$ obtained by placing on $C$ the $\sigma(C, M)$ topology. Let $\mathcal{M}$ be such that both topologies are Hausdorff.

**Theorem 3.1.** If the market is complete with respect to $\sigma(C, M)$ then there exists at most one positive state price density in $M$.

**Proof.** In Theorem 2.1, by taking $Q = M \in L^1(\Omega, \mathcal{F}, P)$, we get the conclusion of this theorem. $\square$

This theorem yields a simple corollary.

**Corollary 3.1.** If the market is complete for the topology $\sigma(C, L^q(\Omega, \mathcal{F}, P))$ for $1 \leq q \leq \infty$, then there exists at most one state price density in $L^q(\Omega, \mathcal{F}, P)$.

We see from Theorem 3.1 that completeness in a particular topology does not guarantee the existence of a positive state price density, but if it exists, then it is unique. For results in the reverse direction of Theorem 3.1, we need additional assumptions. First we need to know that a positive state price density exists (see Lemma 1.1 of the previous section). Second, we need to guarantee that additional (signed) nonpositive state prices densities do not exist. Assumptions A and B are necessary in this regard.

**Assumption A.** If, for $m \in \mathcal{M}^+$, there exists $\lambda \in (0, 1)$ and $m', m'' \in [L^1(\Omega, \mathcal{F}, P)]^+$ such that $m = \lambda m' + (1 - \lambda)m''$, then $m' \in \mathcal{M}^+$ and $m'' \in \mathcal{M}^+$.

Assumption A guarantees that the set of positive price functionals is closed under convex combinations from the larger space $L^1(\Omega, \mathcal{F}, P)$. For example, this assumption is satisfied for $\mathcal{M} = L^q(\Omega, \mathcal{F}, P)$ for all $q$, $1 \leq q \leq \infty$.

Our second assumption is a structural assumption on the space of cash flows. We basically require that if a cash flow trades, then call options on this cash flow also trade.

**Assumption B.** Let $1_{[\Omega]} \in \Phi(\mathbb{Y})$, and suppose that if a cash flow $u \in \Phi(\mathbb{Y})$ then

$$\text{Max} [u - K, 0] \in \Phi(\mathbb{Y})$$

for all positive or negative integers $K$.

Assumption B states that the market must include trading in a riskless asset and all call options on the traded cash flows, $\Phi(\mathbb{Y})$. Essentially, $C$ is a lattice, (see Green and Jarrow 1987 and Nachman 1988).

The role of Assumption B in the following lemma is analogous to closure under stopping times in the continuous time analysis of Delbaen (1992).
Lemma 3.1. If a probability measure $\mathcal{Q}$ equivalent to $\mathcal{P}$ induces a complete market in the topology of the $L^1(\Omega, \mathcal{F}, \mathcal{Q})$ norm, then the space of traded assets is dense in $L^\infty(\Omega, \mathcal{F}, \mathcal{P})$ for the weak topology $\sigma(L^\infty(\Omega, \mathcal{F}, \mathcal{Q}), L^1(\mathcal{Q}))$.

Proof. Suppose $c \in \mathbb{C}$ and the market is complete in the $L^1(\Omega, \mathcal{F}, \mathcal{Q})$ norm. Let $\{y_n\}$ be such that $0 < C - \Phi(y_n) < 0$. Let $\|c\|_\infty = M$, and let $c_n = \Phi(y_n)$ and define $c'_n = (\Phi(y_n) \wedge M) \vee (-M)$. Note that as $(\Phi(y_n) \wedge M) \vee (-M) = -M + \max\{\Phi(y_n) + M, 0\} - \max\{\Phi(y_n) - M, 0\}$, by Assumption B, there exists $y'_n$ such that $c'_n = \Phi(y'_n)$. Further note that $|c - c'_n| \leq |c - c_n|$ and so $\|c - \Phi(y'_n)\|_1 \to 0$. By construction, $\Phi(y'_n)$ is a bounded sequence and so every $m \in L^1(\Omega, \mathcal{F}, \mathcal{Q})$, $E^2[m(c - \Phi(y'_n))] \to 0$ as $n \to \infty$. The space of traded assets is therefore dense in the $\sigma(L^\infty(\Omega, \mathcal{F}, \mathcal{Q}), L^1(\Omega, \mathcal{F}, \mathcal{Q}))$ topology. \qed

Given the above lemma, the following theorem holds.

Theorem 3.2. Suppose that there exists a unique positive state price density $m_0 \in \mathbb{M}^+$, and $\mathbb{M}^+$ satisfies Assumption A. Then, under Assumption B, the market is complete with respect to the $\sigma(\mathcal{C}, \mathcal{M})$ topology on $\mathcal{C}$.

Proof. From the uniqueness of the state price density $m_0 \in \mathbb{M}^+$ and Assumption A, one deduces that $m_0$ is an extreme point of the set of all martingales measures $L^1(\Omega, \mathcal{F}, \mathcal{P})$. The absence of completeness in the norm topology $L^1(\Omega, \mathcal{F}, \mathcal{P})$ implies the existence of an element $f$ of $L^\infty(\Omega, \mathcal{F}, \mathcal{P})$ that is orthogonal to the asset cash flows. Such an element $f$ can be used to violate the extremality of $m_0$ as described in Delbaen (1992). By Assumption B and Lemma 3.1 we extract a bounded and convergent subsequence from an $L^1$ convergent subsequence. We therefore deduce that the range of $\Phi$ is dense in $\mathcal{C}$ under the topology $\sigma(\mathcal{C}, L^1(\Omega, \mathcal{F}, \mathcal{P}))$. Hence $\Phi(\mathcal{Y})$ is dense in $\mathcal{C}$ under the $\sigma(\mathcal{C}, L^1(\Omega, \mathcal{F}, \mathcal{P}))$. Therefore $\Phi(\mathcal{Y})$ is also dense in $\sigma(\mathcal{C}, \mathcal{M})$ as $\mathbb{M} \subseteq L^1(\Omega, \mathcal{F}, \mathcal{P})$. \qed

By Theorem 3.2 we see that if we consider arbitrage free economies in the weak topology induced by $L^q(\Omega, \mathcal{F}, \mathcal{P})$ on $\mathcal{C}$, for any $q$, $1 \leq q \leq \infty$, then provided the space of traded assets contains a riskless asset and is closed under call options, the state price density is unique in $L^q(\Omega, \mathcal{F}, \mathcal{P})$ if and only if markets are complete in the $\sigma(\mathcal{C}, L^q(\Omega, \mathcal{F}, \mathcal{P}))$ topology. This corresponds to the traditional theorems found in the literature.

4. ARBITRAGE FREE COMPLETE MARKETS AND UNIQUENESS OF POSITIVE STATE PRICE DENSITIES: THE CASE OF THE NORM TOPOLOGIES

This section studies arbitrage free complete markets using norm topologies. For the case of the various norm topologies $L^q(\Omega, \mathcal{F}, \mathcal{P})$ for $1 \leq q \leq \infty$ on $\mathcal{C}$ we have the following results.

Theorem 4.1. Completeness of markets in the norm topology for $1 \leq q \leq \infty$ implies that there exists at most one positive state price density in the dual space $L^p(\Omega, \mathcal{F}, \mathcal{P})$ for $p = q/(q - 1)$.

Proof. Completeness in the norm topology of $L^q(\Omega, \mathcal{F}, \mathcal{P})$ implies completeness in the weak topology of $\sigma(\mathcal{C}, L^p(\Omega, \mathcal{F}, \mathcal{P}))$ and the result follows by Theorem 3.1. \qed

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For a result in the reverse direction, we have the following theorem.

**Theorem 4.2.** If $m_0$ is a positive state price density that is unique in $L^p(\Omega, \mathcal{F}, P)$ for $1 < p \leq \infty$, and Assumption B holds, then the market is complete in the norm topology $L^q(\Omega, \mathcal{F}, P)$, where $q = p/(p - 1)$.

**Proof.** Uniqueness of $m_0$ in $L^p(\Omega, \mathcal{F}, P)$, along with Assumption B, yields by the same arguments as in the proof of Theorem 3.2 that the traded asset cash flows are dense in $\mathcal{C}$ under $\sigma(\mathcal{C}, L^1(\Omega, \mathcal{F}, P))$. Since the set of traded assets is a convex set, the closure under the $\sigma(\mathcal{C}, L^1(\Omega, \mathcal{F}, P))$ topology coincides with the closure under the Mackey topology $\tau(\mathcal{C}, L^1(\Omega, \mathcal{F}, P))$, that is, the traded assets are dense in this Mackey topology (or the topology on $\mathcal{C}$ of uniform convergence on $\sigma(L^1(\Omega, \mathcal{F}, P), \mathcal{C}$) compact convex circled subsets of $L^1(\Omega, \mathcal{F}, P)$; see Horvath 1966, p. 206). As on $\mathcal{C}$, the $L^q(\Omega, \mathcal{F}, P)$ norm topology is coarser than the Mackey topology for $q < \infty$, and the traded assets are dense in the norm topology $L^q(\Omega, \mathcal{F}, P)$.

Thus, combining Theorems 3.2 and 4.2 we see that uniqueness of a positive state price density in $L^q(\Omega, \mathcal{F}, P)$ delivers two types of completeness. The first is in the topology $\sigma(\mathcal{C}, L^q(\Omega, \mathcal{F}, P))$ (by Theorem 3.2), but also in the norm topology of $L^p(\Omega, \mathcal{F}, P)$ provided $q > 1$ and $p < \infty$ (by Theorem 4.2).

5. COUNTEREXAMPLES WHEN CALL OPTIONS ARE NOT AVAILABLE

In this section we provide counterexamples showing that positive state price densities can be unique and markets need not be complete in the appropriate topology when markets do not include call options—that is, when Assumption B does not hold.

5.1. Counterexamples for Uniqueness Implies Completeness in $\sigma(\mathcal{C}, L^1(\Omega, \mathcal{F}, P))$

The event set is $\Omega = \{n \mid n$ is a positive integer}, the sigma-field is all subsets of $\Omega$, and the probability measure is $P(\{n\}) = k_p p^n$ for $n = 1, 2, \ldots$, for the constant $k_p = (1 - p)/p$ where $p$ is a positive real below unity. Let $q < \sqrt{p}$ and let $Q(\{n\}) = k_q q^n$ with $k_q = (1 - q)/q$ be another equivalent probability measure.

Traded in the economy are a countable infinity of securities indexed by $i$ that have time 1 cash flows given by $v_i(n)$. All these securities have a market price of unity at time 0 and we shall ensure that the $Q$ measure has a unique positive state price density. The measures associated with positive state price densities are termed martingale measures.\(^2\) The cash flows of security $i$ are zero except in states $i$ and $i + 1$.

For $Q$ to be a positive state price density we must have that

\begin{equation}
\tag{5.1}
k_q q^i v_i(i) + k_q q^{i+1} v_i(i + 1) = 1.
\end{equation}

\(^2\)For a positive state price density $q'$, the associated equivalent martingale measure is $Q$, where $q_i = q_i' p_i / (\sum q_i' p_i)$. On the other hand, given an equivalent martingale measure $Q$ the associated positive state price density $q'$ is $q_i' = \pi(1_Q q_i / p_i)$. 

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We also require that

\[(5.2) \quad k_p p^i v_i(i) + k_p p^{i+1} v_i(i + 1) = a_i,\]

where the sequence \(a_i\) will be fixed later. The solution of equations (5.1) and (5.2) yields

\[v_i(i) = \frac{k_p p^{i+1} - a_i k_q q^{i+1}}{k_p k_q p^{i} q^{i}(p - q)}\]

and

\[v_i(i + 1) = \frac{k_p p^{i} - a_i k_q q^{i}}{k_p k_q p^{i} q^{i}(p - q)}.\]

The sequence \(a_i\) is chosen so as to ensure that for the odd-numbered securities the ratio of the two payouts is \(\sqrt{p}\), and for the even-numbered securities the ratio is \(-\sqrt{p}\). This gives the equations for \(i \geq 1\) that

\[(5.3) \quad \frac{v_{2i-1}(2i - 1)}{v_{2i-1}(2i)} = \sqrt{p},\]

and for \(i > 1\)

\[(5.4) \quad \frac{v_{2i}(2i)}{v_{2i}(2i + 1)} = -\sqrt{p}.\]

It follows that for all \(i \geq 1\)

\[a_{2i-1} = \frac{k_p p^{2i-1}(p + \sqrt{p})}{k_q q^{2i-1}(q + \sqrt{p})},\]

and that for all \(i > 1\)

\[a_{2i} = \frac{k_p p^{2i}(p - \sqrt{p})}{k_q q^{2i}(q - \sqrt{p})}.\]

We now show that the martingale measure defined by \(Q\) is unique. Suppose that \(Q'\) is a martingale measure with probabilities \(Q'(|n|) = q'_n\). Then we must have that \(q'_n \geq 0\) for all \(n\) and that for all \(i\)

\[q'_{i+1} v_i(i + 1) + q'_i v_i(i) = 1.\]

This implies that

\[(5.5) \quad q'_{i+1} = \frac{1}{v_i(i + 1)} - \frac{v_i(i)}{v_i(i + 1)} q'_i \quad \text{for all } i.\]
Equation (5.5) is a first-order difference equation for which $k_q q^i$ is a particular solution by construction. The homogeneous equation is given by

$$q_{i+1}' = -\frac{v_i(i)}{v_i(i+1)} q_i'$$

and has from equations (5.3) and (5.4) the solution

$$q_i' = (-1)^{i+1} (\sqrt{p})^{i-1} q_1.$$

Hence the general solution is

$$q_i' = k_q q^i + \lambda (-1)^{i+1} (\sqrt{p})^i q_1.$$

Since $q < \sqrt{p}$, if $\lambda$ is non-zero $q_i'$ must be negative for sufficiently large $i$ and $Q'$ is not a probability measure. Hence the martingale measure $Q$ is unique.

To demonstrate the lack of completeness in $\sigma(C, L^1(\Omega, \mathcal{F}, P))$ it suffices to show that there exists $\xi \in L^1(\Omega, \mathcal{F}, P)$, $\xi \neq 0$, such that

(5.6) \hspace{1cm} E^P[\xi v_i] = 0 \quad \text{for all} \ i.

When such a $\xi$ exists, the closure of the span is contained in the orthogonal complement of $\xi$ and we do not have completeness.

Equation (5.6) implies that we must have

$$\xi_i k_p p^i v_i(i) + \xi_{i+1} k_p p^{i+1} v_i(i+1) = 0.$$

This yields the difference equation

$$\xi_{i+1} = -\frac{v_i(i)}{v_i(i+1)} \xi_i$$

which has the solution

$$\xi_i = (-1)^{i+1} (\sqrt{p})^{-i+1} \xi_1, \quad \text{for all} \ i \geq 1.$$

Clearly $E^P[|\xi|] < \infty$, and therefore $\xi \in L^1(\Omega, \mathcal{F}, P)$.

We note that for our counterexample, since $\xi \in L^\alpha(\Omega, \mathcal{F}, P)$ for $\alpha < 2$, it follows that the market is not complete under the $\sigma(C, L^\alpha(\Omega, \mathcal{F}, P)$ topology for all $\alpha < 2$. For $\alpha \geq 2$ we have completeness in the $\sigma(C, L^\alpha(\Omega, \mathcal{F}, P)$ topology. Furthermore, we have completeness in the norm topology of $L^\alpha(\Omega, \mathcal{F}, P)$ for $\alpha \leq 2$.

This counterexample for completeness has a martingale measure $Q$ that is in fact unique in $L^\infty(\Omega, \mathcal{F}, P)$. This is because the change of measure density is $(q/p)^i$ and this is
bounded for $q < p$ and belongs to $L^\infty(\Omega, F, P)$. We should therefore be able to construct an example in which the martingale measure is unique in $L^\infty(\Omega, F, P)$ and markets are complete in $\sigma(C, L^\infty(\Omega, F, P))$ but the martingale measure is not unique in $L^1(\Omega, F, P)$ and the market is not complete in $\sigma(C, L^1(\Omega, F, P))$.

5.2. Counterexample for Completeness and Uniqueness in a Weaker Topology that also Illustrates Non-uniqueness and Non-completeness in a Stronger Topology

The structure is the same as in the previous example except that we take $q < p$ as noted above and this time we require that for all $i$

$$\frac{v_i(i)}{v_i(i+1)} = -\sqrt{p}.$$  

The solution for the sequence $a_i$ is now given by

$$a_i = \frac{k_p p^i (p - \sqrt{p})}{k_q q^i (q - \sqrt{p})}.$$  

As in the last example the general solution for all martingale measures is now

$$q'_i = k_q q^i + \lambda(\sqrt{p})^i q_i.$$  

Hence, if $\lambda > 0$ and $q'_i > 0$ we have another martingale measure. The corresponding densities with respect to $P$ are, for $Q$ and $Q'$ respectively,

$$\xi_i = \frac{k_q}{k_p} \left( \frac{q}{p} \right)^i$$  

and

$$\xi'_i = \frac{k_q}{k_p} \left( \frac{q}{p} \right)^i + \frac{\lambda q'_i}{k_p (\sqrt{p})^{-i}}.$$  

Clearly $\xi \in L^\infty(\Omega, F, P)$, but $\xi'$ fails to belong to $L^\infty(\Omega, F, P)$ although it belongs to $L^1(\Omega, F, P)$. It follows from the separation theorem that the market is $\sigma(C, L^\infty(\Omega, F, P))$ complete, but not $\sigma(C, L^1(\Omega, F, P))$ complete. As in the last counterexample, we have both completeness and uniqueness for the weaker topology, and both properties fail for the stronger topology.

6. THE ARTZNER AND HEATH EXAMPLE

The intuition behind the construction of the Artzner and Heath example is the classic result, used in the earlier sections, that one has completeness in the norm topology of $L^1(\Omega, F, Q)$ only if $Q$ is an extreme point of the set of martingale measures. The key insight of the
Artzner and Heath (1995) construction is that it is possible in infinite asset, infinite state models for extremal martingale measures to be equivalent. This allows the construction of two equivalent extremal probability measures that are both norm $L^1(\Omega, \mathcal{F}, Q)$ complete under their own respective $Q$ measures. Hence, completeness and non-uniqueness.

From the perspective of this paper, we consider the non-uniqueness to be established in $L^1(\Omega, \mathcal{F}, P)$. By Theorem 4.2, we should have the absence of completeness in the dual norm topology of $L^\infty(\Omega, \mathcal{F}, P)$ and by Theorem 3.2, a lack of completeness in the topology $\sigma(C, L^1(\Omega, \mathcal{F}, P))$. We prove this next.

The event space is the set of integers, excluding 0. Two probability measures are constructed on this event space termed $P_0$ and $P_1$ and the asset space is so formulated that all martingale measures are convex linear combinations of these two.

For $p < q$, the unnormalized measure $P_0$ is defined by $p^i$ for $i > 0$ and $q^{-i}$ for $i < 0$. The normalization factor is therefore $p/(1 - p) + q/(1 - q)$ and $\kappa = (1 - p - q - pq)/(p + q - 2pq)$. The measure $P_1$ reverses the roles of $p$ and $q$ with the unnormalized probability element being $q^i$ for $i > 0$ and $p^{-i}$ for $i < 0$. All convex linear combinations are also probability measures and $P_0, P_1$ are extremal in this class of measures $\mathcal{M} = \{P_\alpha | P_\alpha = \alpha P_0 + (1 - \alpha) P_1\}$.

Let us choose the statistical or true probability measure to be $P_0$. If $H$ is some other equivalent probability measure with probabilities $h_i$ for $i \in \mathbb{Z} - \{0\}$, then the change of measure density process is $m_i = h_i/P_0(i)$. This belongs to the space $L^1(\Omega, \mathcal{F}, P_0)$, where $\mathcal{F}$ is the collection of all subsets of $\Omega = \mathbb{Z} - \{0\}$. From the perspective of this paper, the cash flow space is the set $C = L^\infty(\Omega, \mathcal{F}, P_0)$.

The traded assets in the Artzner and Heath (1995) economy are a bond paying $1_{\Omega}$ and a countable collection of assets indexed by $\mathbb{Z}$. The interest rate is zero and all assets have a time 0 price of unity under all measures $P_\alpha \in \mathcal{M}$. Asset $v_0$ pays only in states $e_1$ and $e_{-1}$, and the payoff in these states must be $\kappa(p + q)^{-1}$, to ensure the unit time 0 price. For $i > 0$, asset $i$ pays only in states $e_i$ and $e_{i+1}$ the amounts $v_i(i)$ and $v_i(i+1)$. For the time zero price to be unity we must have that

$$\kappa p^i v_i(i) + \kappa p^{i+1} v_i(i + 1) = 1$$

and that

$$\kappa q^i v_i(i) + \kappa q^{i+1} v_i(i + 1) = 1.$$ 

Solving these equations for $v_i(i)$ and $v_i(i + 1)$ we obtain that

$$v_i(i) = \frac{q^{i+1} - p^{i+1}}{\kappa p^i q^i(q - p)}$$

and

$$v_i(i + 1) = \frac{-(q^i - p^i)}{\kappa p^i q^i(q - p)}.$$

For negative $i$, the securities are defined by reflection about zero so that $v_i(j) = v_{-i}(-j)$. Note by construction that all cash flows are in $C$. 

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By construction all the $P_i \in \mathcal{M}$ are martingale measures with densities $m_i = m_n(i) = P_n(i)/P_0(i)$, $i \in \Omega$ that are elements of $L^1(\Omega, \mathcal{F}, P_0)$. Artzner and Heath show that these are all the martingale measures with $P_0$ and $P_1$ being extremal.

Let us consider now the operator $\Psi$ that maps potential martingale measures to asset prices. For $m \in [L^1(\Omega, \mathcal{F}, P_0)]^+$ we know that $h$ defined by $h_i = m_i P_0(i)$ is an element of $L^1(\Omega)$ in that $\sum_i h_i < \infty$ and $m$ is a martingale measure only if

\begin{align}
\sum_{i \in \Omega} h_i &= 1 \quad \tag{6.1} \\
\Psi[m](0) &= \frac{1}{\kappa(p+q)}[h(-1) + h(1)] = 1 \quad \tag{6.2} \\
\Psi[m](i) &= \frac{q^{i+1} - p^{i+1}}{\kappa p^i q^i(q-p)} h_i - \frac{q^i - p^i}{\kappa p^i q^i(q-p)} h_{i+1} = 1 \quad \text{for } i > 0 \quad \tag{6.3} \\
\Psi[m](i) &= \frac{q^{-i-1} - p^{-i-1}}{\kappa p^{-i} q^{-i}(q-p)} h_i - \frac{q^{-i} - p^{-i}}{\kappa p^{-i} q^{-i}(q-p)} h_{i-1} = 1 \quad \text{for } i < 0. \quad \tag{6.4}
\end{align}

Equation (6.1) restricts attention to the unit simplex of $L^1(\Omega)$. The operator $\Psi$ may equivalently be viewed as acting on

$$S = \left\{ h \mid h \in [L^1(\Omega)]^+ \text{ with } \sum_{i \in \Omega} h_i = 1 \right\}$$

with range given by a sequence, indexed by $i \in \mathbb{Z}$, that gives the prices of assets $i \in \mathbb{Z}$.

Let $\nu$ be a reference portfolio measure defined on the set of assets indexed by $\mathbb{Z}$ with the number of units of asset $i$ held in the reference portfolio being $\phi^{(i)}$ for all $i$, where $\phi$ is less than $p$. The operator $\Psi$ then maps into $L^1(\mathbb{Z}, \nu)$. $\Psi$ is in fact a continuous linear operator from $L^1(\Omega)$ into $L^1(\mathbb{Z}, \nu)$ and has an adjoint operator $\Psi'$ that maps $L^\infty(\mathbb{Z}, \nu)$ into $\mathbb{C}$. Since $\Psi$ is not $1 - 1$ (which will be proved later), the range of the adjoint is not dense in the cash flow space with the weak topology and we do not have completeness in the $\sigma(\mathbb{C}, L^1(\Omega, \mathcal{F}, P_0))$ topology.

To investigate further this lack of completeness, we first identify the null space of $\Psi$. Let $n \in L^1(\Omega)$ be a candidate for the null space of $\Psi$. The equation for asset 0 is

$$\Psi[n](0) = \frac{1}{\kappa(p+q)}(n(-1) + n(1)) = 0,$$

and this implies that $n(-1) = n(1)$.

The equation for asset $i$, for $i > 0$, yields

$$\Psi[n](i) = \frac{q^{i+1} - p^{i+1}}{\kappa p^i q^i(q-p)} n(i) - \frac{q^i - p^i}{\kappa p^i q^i(q-p)} n(i + 1) = 0,$$
and this implies that
\[ n(i + 1) = \frac{q^{i+1} - p^{i+1}}{q^i - p^i} n(i) = \frac{q^{i+1} - p^{i+1}}{q - p} n(1). \]

The equations for \( i < 0 \) yield
\[ \Psi[n](i) = \frac{q^{-i-1} - p^{-i-1}}{\kappa p^{-i} q^{-i} (q - p)} n(i) - \frac{q^{-i} - p^{-i}}{\kappa p^{-i} q^{-i} (q - p)} n(i - 1) = 0 \]
and imply that
\[ n(i - 1) = \frac{q^{-i-1} - p^{-i-1}}{q^{-i} - p^{-i}} n(i) = \frac{q^{-i} - p^{-i}}{q - p} n(-1) = -n(-i + 1). \]

Setting \( n(1) = q - p \), we deduce that \( n(i) = q^i - p^i \) for \( i > 0 \) and \( n(i) = -q^{-i} + p^{-i} \) for \( i < 0 \) defines a one-dimensional subspace of \( L^1(\Omega) \) that constitutes the null space of \( \Psi \). For every cash flow in the range of \( \Psi' \), say \( c = \Psi'[\alpha] \), we must have that
\[ \langle c, n \rangle = \langle \Psi'[\alpha], n \rangle = \langle \alpha, \Psi[n] \rangle = 0, \]
and so every cash flow satisfying \( \langle c, n \rangle \neq 0 \) cannot be in the closure of the range of the adjoint. Consider the cash flow \( e_1 \). By construction, \( \langle e_1, n \rangle = n(1) = q - p \neq 0 \). So \( e_1 \) is not in the closure of the range of \( \Psi' \).

However, Artzner and Heath (1995) show that \( e_1 \) is in the range of the space of traded assets in the norm topology of \( L^1(\Omega, \mathcal{F}, P_0) \). In fact, they show that
\[ e_1 = \sum_{i=1}^{k} \mu_i v_i + \frac{q - p}{q^{k+1} - p^{k+1}} e_{k+1}. \]

On evaluating the \( L^1(\Omega, \mathcal{F}, P_0) \) norm of \( e_1 - \sum_{i=1}^{k} \mu_i v_i \), we obtain that
\[ \left\| e_1 - \sum_{i=1}^{k} \mu_i v_i \right\|_{L^1(P_0)} = \frac{q - p}{q^{k+1} - p^{k+1}} p^{k+1} = \frac{q - p}{(q/p)^{k+1} - 1} \to 0 \]
as \( k \to \infty \), and in this topology we have completeness.

If we use the topology of the sup norm then we observe that
\[ \left\| e_1 - \sum_{i=1}^{k} \mu_i v_i \right\|_{L^\infty(P_0)} = \frac{q - p}{q^{k+1} - p^{k+1}} \to \infty. \]
as \( k \to \infty \) and we do not have completeness in this topology. If we use the \( \sigma(\mathbb{C}, L^1(\Omega, \mathcal{F}, P_0)) \) topology then consider \( \tilde{n} \in L^1(\Omega, \mathcal{F}, P_0) \) defined by

\[
\tilde{n}(i) = \frac{n(i)}{p^i} = \frac{q^i - p^i}{p^i} \quad \text{for} \ i > 0
\]

and

\[
\tilde{n}(i) = \frac{n(i)}{q^{-i}} = \frac{q^{-i} - p^{-i}}{q^{-i}} \quad \text{for} \ i < 0.
\]

Now let us evaluate

\[
\left\langle e_1 - \sum_{i=1}^{k} \mu_i v_i, \tilde{n} \right\rangle_{P_0} = \left\langle \frac{q - p}{q^{k+1} - p^{k+1}} e_{k+1}, \tilde{n} \right\rangle_{P_0}
\]

\[
= \frac{q - p}{q^{k+1} - p^{k+1}} \frac{q^{k+1} - p^{k+1}}{p^{k+1}} p^{k+1}
\]

\[
= q - p,
\]

which is constant \( k \to \infty \). So \( e_1 \) is not in the weak limit of the trading strategies either. From the perspective of this paper, the Artzner and Heath (1995) economy has both non-uniqueness and lack of completeness using the dual topologies.

For a fuller understanding of the types of completeness offered in this economy we observe that as all the martingale measures are linear combinations of \( P_0 \) and \( P_1 \), for \( P_0 \) taken as the statistical measure, for all \( \alpha, 1 \leq \alpha < \infty \) we have the absence of uniqueness of the martingale measures in \( L^\alpha \). Thus, there is no completeness in \( \sigma(\mathbb{C}, L^\alpha(\Omega, \mathcal{F}, P_0)) \) and no completeness in the norm topology for \( L^p(\Omega, \mathcal{F}, P_0) \) for \( p = \alpha/(\alpha - 1) \) and \( 1 \leq \alpha < \infty \). However, since the martingale measure is unique in \( L^\infty(\Omega, \mathcal{F}, P_0) \) as all the changes from \( P_0 \) are unbounded and this is the dual of \( L^1(\Omega, \mathcal{F}, P_0) \), by Theorem 4.2 one must have completeness in the norm topology of \( L^1(\Omega, \mathcal{F}, P_0) \). This fact is demonstrated by Artzner and Heath (1995).

7. CONCLUSION

This paper studies the second fundamental theorem of asset pricing. The key economic insight of this paper is that the notion for "closeness" of two cash flows (in the definition of market completeness) is intimately related to the notion of "closeness" of prices—that is, to the continuity of the price functional. If two cash flows are deemed similar, then their prices should also be close. To break this dual relationship doesn't make economic sense. Thus, when searching for uniqueness of linear price functionals, as in the second fundamental theorem of asset pricing, one should restrict attention to the continuous ones, in the dual topology.

This economic insight explains why the counterexample proposed by Artzner and Heath (1995) is uninteresting. Their example, as shown before, breaks this duality between the notion of "closeness" for cash flows and the continuity of the linear price functional, and, when broken, the second fundamental theorem of asset pricing need not hold. But, even
for the Artzner and Heath (1995) economy, if this duality is imposed, then, as shown here, the second fundamental theorem of asset pricing is satisfied.

APPENDIX

Example Showing that $\sigma(\mathbb{C}, L^1(\Omega, \mathcal{F}, P))$ and the $L^1(\Omega, \mathcal{F}, P)$ Norm Topology on $\mathbb{C}$ Are Not Comparable

We first show that one may have convergence in the norm topology without convergence in the weak topology. Consider the set $\Omega$ to be the nonnegative integers and define the probability measure $P$ by $(1-p)p^k$. Let $x_k = (1/q^k)e_k$ for $q > p$ and observe that the $\|x_k\|_{L^1(P)} = (p/q)^k$ tends to 0 and hence $x_k \to 0$ in $L^1(P)$. However, the $L^\infty$ norm of $x_k$ is $(1/q^k)$ and this tends to $\infty$ and we have divergence. For the weak topology consider $n = (n_k, k \geq 0) \in L^1(\Omega)$ defined by $n_k = (q/p)^k$ with $|n|_{L^1(P)} = 1 - q$. We have that $\langle n, x \rangle$ is given by

$$\langle n, x \rangle = \frac{q^k}{p^k} \frac{1}{q^k} \frac{p^k}{q^k} = 1$$

and so $x_k$ does not converge to zero in the weak topology either.

For an example in the other direction consider the linear space $\mathbb{C}$ over some probability space. Consider on $\mathbb{C}$ four topologies: $\tau_1$ is the weak topology $\sigma(\mathbb{C}, L^\infty(P))$; $\tau_2$ is the $L^1$ norm topology; $\tau_3$ is the weak topology $\sigma(\mathbb{C}, L^1)$; and $\tau_4$ is the $L^\infty$ norm topology.

Obviously, by the dominated convergence theorem, $\tau_2$ and $\tau_4$ are stronger than $\tau_1$ and $\tau_3$ respectively. It is also clear that $\tau_3$ is stronger than $\tau_1$ as the latter requires convergence only on bounded functions whereas the former requires convergence on a larger set of functions.

What remains is the comparison of $\tau_2$ and $\tau_2$. If we have convergence on all $L^1$ functions of a net of bounded functions, do we have $L^1$ convergence of the net? We know from the above example that we may have $L^1$ convergence without $\sigma(L^\infty, L^1)$ convergence.

Let us first consider the $L^1 \cap L^\infty$ functions with respect to the Lebesgue measure on the positive half line. Let $f_n(x)$ be $1/n$ from $n$ to $2n$, then the integral of $f_n$ is unity and $f_n$ does not converge to zero in the $L^1$ norm. For any $L^1$ function $y$, however,

$$\left| \int_0^\infty f_n y \, dx \right| \leq \frac{1}{n} \int_n^{2n} |y| \, dx \to 0$$

and we have convergence in the $\sigma(L^\infty, L^1)$ topology. Hence these two topologies are not comparable.

For a probability measure, consider the negative exponential density on the positive half line as the probability $P$. Convergence in $\sigma(L^\infty, L^1)$ topology requires that for all $y$ in $L^1(P)$

$$\left| \int f_n ye^{-x} \, dx \right| \to 0,$$

(A.1)
and $L^1$ convergence requires that

(A.2) \[ \int |f_n| e^{-x} \, dx \to 0. \]

With respect to Lebesgue, $1_{R^+}$ is not in $L^1$ but here $y = \text{sign}(f_n)$ is an $L^1$ function and if this is independent of $n$ then applying equation (A.1) to $y = \text{sign}(f_n)$ yields equation (A.2), and so for such functions $\sigma(\mathbb{C}, L^1)$ convergence implies $L^1$ norm convergence. Counterexamples with functions $f_n(x)$ must have $\text{sign}(f_n(x))$ dependent on $n$.

The argument is more involved and goes back to the Riemann–Lebesgue theorem. Let $P$ be the uniform distribution on $[0, 1]$ and define $f_n(x) = \sin(2\pi nx)$. For every $L^1$ function $g(x)$ we have that $\int g f_n(x) \, dx \to 0$ but $\int |f_n(x)| \, dx = 1$. Hence we have weak convergence or convergence in the $\tau_2$ topology but not in the $\tau_1$ topology. The weak and norm topologies on $\mathbb{C}$ for the case $q = 1$ are therefore not comparable.

To show that $\int g f_n \, dx \to 0$, we observe that $g$ may be approximated by step functions for any $L^1$ function $g$. Consider any interval $(a, b)$ of the unit interval and observe that

\[
\int_a^b \sin(2\pi nx) \, dx = \frac{1}{2\pi n} \int_{2\pi na}^{2\pi nb} \sin(y) \, dy = \frac{1}{2\pi n} [\cos(2\pi na) - \cos(2\pi nb)],
\]

and so

\[
\left| \int_a^b \sin(2\pi nx) \, dx \right| \leq \frac{1}{n\pi} \to 0.
\]

On the other hand,

\[
\int_a^b |\sin(2\pi nx)| \, dx = \frac{1}{2\pi} \int_{2\pi a}^{2\pi b} |\sin(y)| \, dy \to (b - a).
\]

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4, 223–245.

