An Empirical Analysis of the Jarrow-van Deventer Model for Valuing Non-Maturity Demand Deposits

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Imperfect competition in the banking markets can cause non-maturity demand deposits to have positive net present values. The Jarrow-van Deventer model based on an arbitrage-free term structure methodology can be used to determine this value.

This research uses both a commercial bank's demand deposit data and Federal Reserve aggregate NOW account data to fit the model over an eight-and-a-half-year period. Robust time series models for the demand deposit balances and interest rates paid are formulated and fit to the data. The extended Vasicek model is used to model evolution of the term structure of interest rates.

The premium on the bank's NOW account deposits ranges between -16% to +12% over the sample period, with an average life (of the deposit's liability) of between 7.8 and 9.8 years. For the Federal Reserve aggregate NOW account data, the premium ranges between -1.4% and +13% over the sample period, with an average life of between 8.4 and 10 years. These results are consistent with estimates obtained in previous studies of demand deposit premiums and with the premiums paid on Resolution Trust Corporation auctions of demand deposits.

Asset/liability management in commercial banks is in a state of evolution, as is risk management in general. Given the recent advances in the pricing and hedging of interest rate-sensitive instruments, it is now possible for a commercial bank, in theory, to manage the interest rate risk of its entire balance sheet (see Jarrow and Turnbull [1996]). Interest rate risk in fixed-income securities — OTC derivatives (caps, floors, swaps), mortgage-backed securities, corporate debt (loans) — has been thoroughly investigated. The remaining balance sheet item, non-maturity demand deposits, is the topic of our analysis.

Recent theoretical advances now enable one to estimate and hedge the interest rate risk in demand deposits (see Jarrow and van Deventer [1998] and Hutchison and Pennacchi [1996]). Hutchison and Pennacchi estimate an equilibrium-based model for two types of accounts. Their equilibrium-based model is distinct from the arbitrage-based approach of Jarrow and van Deventer. Jarrow and van Deventer use the Heath, Jarrow, and Morton [1992] interest rate methodology for valuation and hedging.

The two approaches differ in specification of the evolution of the term structure of interest rates and of demand deposit balances, and in specification of the demand deposit rate paid process. These differences lead to distinct valuation formulas.

We provide an empirical investigation of the Jarrow-van Deventer model [1998] for valuing non-maturity demand deposits. Four different types of consumer accounts are studied for a commercial bank: NOW (negotiable orders of withdrawal) accounts,
passbook accounts, statement accounts, and DDA (demand deposit accounts). These accounts differ with respect to interest paid, fees, checking services, and minimum deposit required.\(^1\)

For comparison purposes, aggregate data on consumer NOW accounts for all commercial banks are obtained from the Federal Reserve Bulletin. The Federal Reserve Bulletin data can be interpreted as describing an “average” commercial bank.

The observation period is January 1988 through June 1996. Using monthly observations of the various accounts’ demand deposit balances and rates paid, we obtain estimates of the accounts’ deposit premiums, durations, and average lives. The results provide evidence consistent with the validity of the model. The bank’s average NOW account deposit premiums are consistent with estimates obtained in previous studies and in observed auctions of demand deposits.

Our analysis indicates that it is time for regulatory agencies to reconsider banking capital requirements formulated when hedging the risk of demand deposits was not possible. Imposing hedging restrictions on demand deposits rather than providing capital to cover losses in default would be more efficient, and help to stabilize the banking system. This increased stabilization would enable the Federal Reserve to better manage the monetary sector, and hence the entire economy.

I. THE MODEL

We present the theoretical model first, followed by the statistical model.

Theory

First, we briefly review the Jarrow-van Deventer [1998] model. Let \([0, t]\) be the trading horizon.\(^2\) Traded are default-free zero-coupon bonds and a money market account. Let \(P(t, T)\) be the time \(t\) price of a default-free zero paying a sure dollar at time \(T\). Continuously compounded forward rates \(f(t, T)\) are implicitly defined by

\[
P(t, T) = \exp \left( - \int_t^T f(t, s) ds \right)
\]

(1)

The spot rate of interest is defined by \(r(t) = f(t, t)\). The money market account earns interest at the spot rate:

\[
B(t) = \exp \left( \int_0^t r(s) ds \right)
\]

(2)

Let the bank’s demand deposit balances at time \(t\) be denoted by \(D(t)\), and the demand deposit rate be denoted \(i(t)\). The rate \(i(t)\) includes the non-interest costs of servicing the accounts, except for any reserve requirements. The percentage reserve requirements are denoted by \(m\).

Assuming no arbitrage and complete markets, Jarrow and van Deventer [1998] value the demand deposit as an interest rate derivative. They show that the net present value of the demand deposit, \(V(0)\), can be written as:

\[
V(0) = \hat{E}_0 \left( \int_0^t \frac{(1-m)D(t)r(t)}{B(t)} dt \right) - \hat{E}_0 \left( \int_0^t \frac{D(t)i(t)}{B(t)} dt \right)
\]

(3)

where \(\hat{E}_0(\cdot)\) is the time 0 expectation using an equivalent martingale measure \(\hat{Q}\).

Equation (3) has a useful economic interpretation. Benefits are obtained from the interest earned on the deposits after reserve requirements, while costs are due to the interest paid. The first integral represents the present value of the benefits, and the second integral represents the present value of the costs.

The term inside the first integral, \((1-m)D(t)r(t)/B(t)\), represents the discounted cash inflows from investing \((1-m)D(t)\) dollars at \(r(t)\), as \(mD(t)\) dollars are held in reserve and can earn no interest. \(B(t)\) is the discount factor. These cash inflows are summed across time and an expectation taken. The term inside the second integral, \(D(t)i(t)/B(t)\), represents the discounted costs from the deposits. \(B(t)\) is the discount factor. These cash outflows are summed across time and an expectation taken.

Equation (3) holds under very general stochastic processes for the term structure of interest rates [represented by \(r(t)\)], demand deposit balances \(D(t)\), and demand deposit rates \(i(t)\). The only restriction imposed on the evolution of demand deposit balances and rates paid is that these evolutions depend on (at most) the information incorporated in the evolution of the spot rate \(r(t)\); see Jarrow and van Deventer [1998] for further
clarification. For empirical implementation, additional structure is needed.

**Term Structure Evolution.** As a first pass for the analysis, we use a one-factor term structure model with deterministic volatilities (see Heath, Jarrow, and Morton [1992]). This model has proven useful in many applications, and is sometimes known as the extended Vasicek model. (The analysis is easily generalized to a multiple-factor model, but such an extension awaits research.)

The term structure evolution is described by the evolution of the spot rate of interest under the martingale measure $\tilde{Q}$:

$$dr(t) = a[\tilde{r}(t) - r(t)]dt + \sigma d\tilde{W}(t) \tag{4}$$

where $a$ and $\sigma$ are constants; $\tilde{r}(t)$ is a deterministic function of $t$, and $\tilde{W}(t)$ is a standard Brownian motion under $\tilde{Q}$ initialized at $\tilde{W}(0) = 0$.

The spot rate of interest follows a mean-reverting process under the martingale measure. It has a mean-reverting parameter $a$, a long-run spot rate of $\tilde{r}(t)$, and a volatility of $\sigma$. It is important to emphasize that although the spot rate is mean-reverting under the martingale probabilities, it need not be mean-reverting under the empirical probabilities. This difference is important in the statistical estimation.

As shown in Heath, Jarrow, and Morton [1992], to match an arbitrary initial forward rate curve $\{f(0, t) \text{ for } t \in [0, T]\}$, one must set the long-run spot rate equal to

$$\tilde{r}(t) = f(0, t) + \frac{\partial f(0, t)}{\partial t} + \frac{\sigma^2}{2a^2}(1 - e^{-2at}) / 2a \tag{5}$$

Equation (5) states that to match the initial forward rate curve the long-run spot rate must be equated to the forward rate plus an adjustment term involving the slope of the initial forward rate curve and the forward rate’s volatility parameters.

We can rewrite the evolution for the spot rate of interest as depending on only the initial forward rate curve and the forward rate’s volatility parameters:

$$r(t) = f(0, t) + \sigma^2(e^{-2aT} - 1)^2 / 2a^2 + \int_0^t \sigma e^{-2a(s-t)} d\tilde{W}(s) \tag{6}$$

Equation (6) is a standard result that is useful in the valuation equations.

**Demand Deposit Balance Evolution.** For empirical implementation, we assume that the demand deposit balances follow a stochastic process given by:

$$D(t) = C\left(D_0 / C\right)^{1/e} e^{\mu t + \frac{1}{2} \alpha_{r-s}^2 t^2} \tag{7-A}$$

where $C$, $D_0$, $\epsilon$, and $\mu$ are constants, and $\alpha_{r-s}$ is a deterministic function of $t - s \geq 0$.

$C$ represents the long-run or perpetual deposits. Demand deposit balances are assumed to grow in an exponential manner, from initial balances of $D_0$. The parameter $\epsilon$ measures the retention rate of existing balances. The growth rate of new balances, and $\alpha_{r-s}$ measures the interest rate sensitivity of balance growth to an “average” of the past spot interest rates. The average is taken over the entire time period $[0, t]$.

There are two “growth” rate terms in Equation (7-A). The first involves the retention rate of existing deposits $\epsilon$ raised to the $t$-th power. The second involves the growth rate of new deposits $\mu$ multiplied by time $t$. The first growth rate evolves exponentially through time, while the second evolves linearly. These differing time dynamics (linear versus exponential) enable the two different growth rates to be statistically distinguished. Inclusion of both types of growth rates in Equation (7-A) allows a rich evolution for the demand deposits $D(t)$.

Define $c = C / D_0$ as the percentage of the initial demand deposit balances that the perpetual deposits represent. This does not represent the percentage of core deposits, but the long-run/steady-state deposits as a percentage of current balances. If the bank’s deposits are expected to increase in the long run, then $c \geq 1$ is possible.

Using the percentage of perpetual deposits $c$, Equation (7-A) can be rewritten as:

$$D(t) = cD_0(1 / c) e^{\mu t + \frac{1}{2} \alpha_{r-s}^2 t^2} \tag{7-B}$$

The advantage of this form of the demand deposit evolution equation is that it demonstrates a scale invariance; i.e., the same parameters apply whether demand deposits $D(t)$ are measured in dollars or thousands of dollars.
For empirical estimation, we need to discretize Equations (7). Let $\Delta$ be a fixed time interval, the time interval between data observations measured in years (say, $\Delta = 1/12$, or one month). We discretize the interest rate sensitivity function $\alpha_{t-s}$ by making it constant over successive intervals; that is, we set:

$$
\alpha_{t-s} =
\begin{cases}
\alpha_0 & \text{for } t-s \in [0, \Delta] \\
\alpha_{-1} & \text{for } t-s \in [\Delta, 2\Delta] \\
\alpha_{-2} & \text{for } t-s \in [2\Delta, 3\Delta] \\
\vdots
\end{cases}
$$

(8)

Since it is impossible to observe (or measure) the spot rate continuously through time, we also need to discretize the term involving the integral of the spot rates. This is done through the use of averages. Define the average spot rate $R(t-j\Delta)$ over the time interval $[t-(j+1)\Delta, t-j\Delta]$ as:

$$
R(t-j\Delta) = \frac{1}{t-j\Delta} \int_{t-(j+1)\Delta}^{t-j\Delta} r(s) ds \quad \text{for } t \geq \Delta
$$

(9)

and $j = 0, 1, \ldots, N-1$ where $N\Delta = t$.

Combining Equations (8) and (9), the demand deposit evolution simplifies to:

$$
D(t) = cD_0 (1/c)^{t} e^{{\mu + \sum_{i=N+1}^{N+1} \alpha_i R((i-1)\Delta)}} \quad \text{for } t \geq \Delta
$$

(10)

Equation (10) forms the basis for the statistical model.

**Demand Deposit Rate Evolution.** The demand deposit rate $i(t)$ is assumed to satisfy a stochastic process similar to that satisfied by the logarithm of the demand deposit balances $\log[D(t)]$. Hence, we let

$$
i(t) = k + \pi'(i_0 - k) + \int_{0}^{t} \beta_{t-s} r(s) ds
$$

(11)

where $k$, $\pi$, and $i_0$ are constants, and $\beta_{t-s}$ is a deterministic function of $t-s \geq 0$. $k$ represents the steady-state floor on the interest rate paid.\(^5\)

A negative value for $k$ is possible. To see this, suppose that in steady-state there is a competitive banking environment for demand deposits where banks earn zero
profits. In steady-state, if \( \pi < 0 \), the second term in Equation (11) disappears. Then, if spot interest rates \( r(t) \) are zero, the third term disappears, and \( k \) equals the interest rate paid on the account. Since banks earn no interest on the account deposits \( r(t) \) is zero), and profits are zero, customers would need to pay back to the bank the service costs from holding their deposits: an interest rate paid of \( k < 0 \% \).

\( \pi \) measures the rate of decay of the current deposit rate \( i_0 \) to the "long-run" floor rate \( k \). The function \( \beta_{t-s} \) is the sensitivity of deposit rates to an average level of past spot interest rates \( r(t) \). The average is taken over the time interval \([0, t] \).

To implement Equation (11), we need to discretize the integral. This is done by assuming that the sensitivity of deposit rates to the average level of past spot interest rates \( \beta_{t-s} \) is piecewise constant, and by using the average spot rate \( R(t) \). In this regard, define

\[
\beta_{t-s} = \begin{cases} 
\beta_0 & \text{for } t-s \in [0, \Delta] \\
\beta_{-1} & \text{for } t-s \in [\Delta, 2\Delta] \\
\beta_{-2} & \text{for } t-s \in [2\Delta, 3\Delta] \\
\vdots
\end{cases}
\]  

Substitution of this expression and the definition of an average spot rate into Equation (11) gives the final result:

\[
i(t) = k + \pi \left( i_0 - k \right) + \sum_{j=-N+1}^{0} \beta_{j} R(t+j\Delta) \Delta
\text{ for } t \geq \Delta
\]  

Equation (13) proves useful in the empirical estimation.

**Demand Deposit Valuation.** Given the stochastic evolutions for the term structure of interest rates in Equation (6), demand deposit balances in Equation (10), and demand deposit rates in Equation (13), the demand deposit valuation Equation (3) can be evaluated in closed form. The expression for the net present value of the demand deposits is:

\[
V(0) = cD_0 \int_0^\tau \left( 1/c \right)^{\tau-t} e^{\mu t+\mu_1(t)+\sigma_1^2(t)/2} \times
\{(1-m)[\mu_3(t)+\sigma_3^2(t)] - \\
[k+\pi(i_0-k)+\mu_2(t)+\sigma_2^2(t)] \} dt
\]

where \( \mu_1(t), \mu_2(t), \mu_3(t), \sigma_1^2(t), \sigma_2^2(t), \sigma_3^2(t) \) are deterministic functions of time and the spot rate evolution's parameters, for which explicit formulas are in the appendix.

The derivation of Equation (14) is also in the appendix. This simple closed-form solution facilitates the computation of values and risk management statistics.

For comparison across deposit types and time, we express the net present value of demand deposits \( V(0) \) in percentage terms; that is, \( V(0)/D_0 \). This is called the \textit{percentage deposit premium}.

Another quantity of interest is the demand deposit's cost, defined as the entire demand deposit liability \( D(0) \) less its net present value \( V(0) \):

\[
L(0) = D(0) - V(0)
\]  

This is the quantity most often used in traditional bank management discussions of demand deposits.

**Demand Deposit Deltas, Durations, and Average Lives.** For risk management purposes, it is useful to analyze the risk characteristics of both the net present value of the demand deposits and their costs.

**Definition of delta:**

\[
\text{NPV Delta} = \frac{\partial V(0)}{\partial r(0)}
\]

\[
\text{Cost Delta} = \frac{\partial L(0)}{\partial r(0)}
\]

This is the standard definition of the delta used in interest rate option modeling (see Jarrow and Turnbull [1996]).

To define the demand deposit's duration, we need to modify the notation. Let \( V[0; f(0, \bullet)] \) indicate the demand deposit's net present value at time 0, given the initial forward rate curve of \( f(0, \bullet) \). Similarly, define \( L[0; f(0, \bullet)] \) as the demand deposit's cost at time 0, given the initial forward rate curve of \( f(0, \bullet) \).

**Definition of duration:**

\[
\text{NPV Duration} = \lim_{\delta \to 0} \left[ \frac{V[0; f(0, \bullet) + \delta] - V[0; f(0, \bullet)]}{\delta |V[0; f(0, \bullet)]|} \right]
\]

\[
\text{Cost Duration} = \lim_{\delta \to 0} \left[ \frac{L[0; f(0, \bullet) + \delta] - L[0; f(0, \bullet)]}{\delta |L[0; f(0, \bullet)]|} \right]
\]

This is the standard duration calculation. It corresponds to the percentage change in the net present...
value of the demand deposits when the initial forward rate curve is shifted in parallel by a small ($\delta$). The absolute value of the initial NPV or cost is included in the denominator to accommodate negative values for these quantities.

For simple models of the evolution of the term structure of interest rates (those with parallel shifts of the forward rate curve), the duration corresponds to the average life of the financial security. This interpretation, however, does not hold for more complex evolutions [including that given by Equation (6)]; see Jarrow and Turnbull [1996]. For this reason, we also compute the demand deposit's average life.

Definition of average life:

\[ \text{NPV Average Life} = \] 
\[ \hat{E}_0 \left[ \int_0^\infty \frac{1}{B(t)} t \hat{D}(t) [(1 - m) \hat{r}(t) - i(t)] dt \right] / V(0) \]  \hspace{1cm} (18-A)

\[ \text{Cost Average Life} = \hat{E}_0 \left[ \int_0^\infty \frac{1}{B(t)} t \hat{D}(t) [i(t) + m \hat{r}(t)] dt \right] / V(0) \] \hspace{1cm} (18-B)

The NPV average life is seen to be a weighted average of the time periods $t$ in which the cash flows $D(t)/[(1 - m)\hat{r}(t) - i(t)]$ are received. The cash flows equal the total deposits $D(t)$ times the interest earned after reserve requirements $(1 - m)\hat{r}(t)$ less the rate paid $i(t)$. The weights correspond to the percentage of the present value that each time period's cash flow represents $D(t)/[(1 - m)\hat{r}(t) - i(t)]/B(t)V(0)$. The weights integrate to one over the time period $[0, \tau]$. Recall that $B(t)$ is the time $t$ discount factor to time $0$, and $V(0)$ is the time $0$ net present value of the demand deposits.

A similar interpretation follows for the cost average life. Closed-form solutions for the average lives in Equations (18-A) and (18-B) are easily obtained using the same mathematics underlying the valuation formula in Equation (14).

From an economic point of view, the net present value of the demand deposit is the most relevant quantity to analyze and risk-manage, although traditional bank management focuses on the total cost measures. We concentrate on the NPV in our discussion, although both measures are reported.

Statistical Model

The statistical model uses the term structure evolution in conjunction with the demand deposit balance and rate paid evolutions to simplify the regression equations. In their current form, the discretized evolutions for the demand deposits [Equation (10)] and the rate paid [Equation (13)] include too many lagged variables. The lagged variables are the past values for the average spot rates $R(t)$. Our knowledge of the evolution for the average spot rate can be used to significantly simplify these expressions.

Average Spot Rate Equation. The simplest evolution for the average spot rate $R(t)$ is that:

\[ R(t) = R(t - \Delta) + u_t \quad \text{for } t \geq \Delta \] \hspace{1cm} (19)

where $u_t$ is a normal random variable with zero mean.

This expression states that the average spot rate follows a random walk. It is consistent with the term structure model in Equation (4), and it enables us to simplify the evolutions for the demand deposit balances and the interest rates paid.\(^{7}\)

Demand Deposit Balance Evolution. To specify the statistical model, we introduce some additional notation. As before, time is indexed by $t$, and time progresses through intervals of size $\Delta$. $\Delta$ is measured in years and it will equal 1/12 (one month). The regression index is denoted by $j = 0, 1, 2, \ldots$. The correspondence between time and the regression index is that $t = j\Delta$.

The statistical model for the demand deposit balance evolution starts with Equation (10), using Equation (19). Direct substitution yields:

\[ D(t) = cD_0((1/c)^\tau') e^{\mu t + \alpha R(t) \Delta + v_t} \quad \text{for } t \geq \Delta \] \hspace{1cm} (20)

where

\[ \alpha = \sum_{j=-N+1}^0 \alpha_j \]

and

\[ v_t = -\sum_{j=-N+1}^0 \alpha_j u_{t+j\Delta} \Delta \]
The demand deposit evolution thus takes a simpler form that is more conducive to estimation. The essential difference between Equation (20) and Equation (10) is that the summation of past average spot rates is replaced by only the current rate. The coefficient is appropriately modified. Rather than having to estimate \((N - 1)\) interest rate sensitivity coefficients \((\alpha_{-N+1}, \alpha_{-N+2}, \ldots, \alpha_0)\), we need to measure only one:

\[
\alpha = \sum_{j=-N+1}^0 \alpha_j
\]

the sum of the coefficients.

The difference form of Equation (20) provides the basis for a linear regression model. To get this expression, take logarithms of (20), and subtract \(\varepsilon^\lambda \log D(t - \Delta)\) from both sides of the resulting equation. After simplification, one obtains:

\[
\log D(j\Delta) = \varepsilon^\lambda \log D(0) + (1 - \varepsilon^\lambda) \log c D_0 + \\
\varepsilon^\lambda \mu \Delta + \mu(1 - \varepsilon^\lambda) j \Delta + \alpha R(j\Delta) \Delta - \\
\varepsilon^\lambda \alpha R((j - 1)\Delta) \Delta + x_j \quad \text{for} \ j \geq 1
\]

where \(x_j = v_j - \varepsilon^\lambda v_{(0-1)\Delta}\).

Equation (21) describes the evolution of the logarithm of the demand deposits \(D(j\Delta)\). Its value at time \(j\Delta\) equals its value at time \((j - 1)\Delta\) after an adjustment for the fraction of deposits retained over the period \(\varepsilon^\lambda\) plus terms involving the growth rate in new deposits \(\mu\), the interest rate sensitivity parameter \(\alpha\), and the average spot rates \(R(j\Delta)\) and \(R((j - 1)\Delta)\).

After the inclusion of measurement error in both demand deposits \(D(t)\) and the average spot rates \(R(t)\), it is assumed that the error terms in Equation (21) satisfy \(x_0 \equiv 0\) and \(x_j = \rho x_{j-1} + x_j^*\) where \(x_j^*\) are iid normal \((0, \sigma^2_{x^*})\) for \(j \geq 1\). In this specification, the error terms can be autocorrelated. This situation occurs frequently in time series regressions with lagged dependent variables on the right-hand side of the equation. Equation (21) gives the initial linear regression equation.

**Demand Deposit Rate Evolution.** A similar simplification can be obtained for the rate paid evolution. Substituting Equation (19) into Equation (13) generates the statistical model for the demand deposit rate evolution:

\[
i(t) = k + \pi^\lambda(i_0 - k) + \beta R(t\Delta) + w_t \quad \text{for} \ t \geq \Delta
\]

where

\[
\beta = \sum_{j=-N+1}^0 \beta_j
\]

and

\[
w_t = -\sum_{j=-N+1}^0 \beta_j u_{t+j\Delta} \Delta
\]

As with the demand deposit balances, the use of the term structure equation simplifies the evolution considerably. Rather than having to estimate \((N - 1)\) interest rate coefficients \((\beta_{-N+1}, \beta_{-N+2}, \ldots, \beta_0)\), we need only to estimate one:

\[
\beta = \sum_{j=-N+1}^0 \beta_j
\]

the sum of the coefficients.

To get the linear regression equation, subtract \(\pi^\lambda i(t - \Delta)\) from both sides of Equation (22). Simplification yields:

\[
i(j\Delta) = \pi^\lambda i[(j - 1)\Delta] + (1 - \pi^\lambda) k + \\
\beta R(j\Delta) - \pi^\lambda \beta R[(j - 1)\Delta] + z_j \quad \text{for} \ j \geq 1
\]

where \(z_j = w_{j\Delta} - \pi^\lambda w_{(0-1)\Delta}\).

Equation (23) is the basis for the interest rate paid regression. It states that the interest rate paid \(i(j\Delta)\) equals its value last period \(i[(j - 1)\Delta]\) after a fractional adjustment \(\pi^\lambda\) plus terms involving the steady-state rate paid \(k\), the interest rate sensitivity coefficient \(\beta\), and the average spot rates \(R(j\Delta)\) and \(R((j - 1)\Delta)\).

After the inclusion of measurement error in both the interest rate paid \(i(t)\) and the average spot rate \(R(t)\), it is assumed that the error terms in Equation (23) satisfy \(z_0 \equiv 0\) and \(z_j = \rho z_{j-1} + z_j^*\) where \(z_j^*\) are iid normal \((0, \sigma^2_{z^*})\) for \(j \geq 1\). In Equation (23), the residuals can be autocorrelated. This gives the initial regression equation.

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**Demand Deposit Valuation.** Having simplified the demand deposit and rate paid evolutions, we now need to revisit the net present valuation Equation (14). Two changes need to be made. First, Equation (14) formally applies when all the demand deposit and rate paid evolution parameters are known. But, we can obtain only statistical estimates of these parameters. Substitution of these unbiased parameter estimates into the valuation equation can produce biased estimates for this value.8

Second, Equation (14) is derived using all the interest rate sensitivity parameters (α_\text{N+1}, α_\text{N+2}, ..., α_0) and (β_\text{N+1}, β_\text{N+2}, ..., β_0) rather than their sums (α, β). This modification also needs to be included.

Fortunately, both these changes can be incorporated by explicitly including the additional error structure from the demand deposit evolution [Equation (20)] and the rate paid evolution [Equation (22)] in the valuation formula.

Our estimate for the net present value is the expectation of the formula given by Equation (14) including the additional error structure in the parameter estimates:

\[
\hat{V}(0) = \hat{E}_0 \left[ \frac{e^{\int_0^T CD_0 (1/c) t^* e^{\mu t + \alpha (t/\Delta + v_t)}}}{B(t)} \times \frac{B(t)}{(1 - m)\pi(t) - k - \pi^*(i_0 - k) - \beta R(t)\Delta - w_t} \right] \ 
\]

\[
\text{(24-A)}
\]

where \( \hat{E}_0 \) denotes expectation over the parameter estimate errors (\( v_t, w_t \)).

In the appendix, it is shown that \( \hat{V}(0) \) equals

\[
\hat{V}(0) = e^{\int_0^T \frac{se}{(1 - e^{2s})} \cdot CD_0 (1/c) e^{\mu t + \alpha (t/\Delta + v_t)}} \times \frac{(1 - m)[\mu_3(t) + \sigma_{13}(t)] - [k + \pi^*(i_0 - k) + \mu_2(t) + \sigma_{12}(t)] dt}{(1 - m)[\mu_3(t) + \sigma_{13}(t)] - [k + \pi^*(i_0 - k) + \mu_2(t) + \sigma_{12}(t)] dt} \ 
\]

\[
\text{(24-B)}
\]

where \( \sigma_{13}^2 \) is the variance of the error term in the demand deposit evolution regression (21); and the expressions for \( \mu_1(t), \mu_2(t), \mu_3(t), \sigma_{12}^2(t), \sigma_{13}^2(t) \) are simplifications of those given in Equation (14) involving only the sums of the interest rate sensitivity parameters (α, β).

Equation (24-B) is similar to the net present value formula in Equation (14). The difference is that it explicitly incorporates the error due to uncertainty of the parameter estimates. It differs from Equation (14) by inclusion of a term (the first) involving the exponential of the variance of the demand deposit regression \( \sigma_{13}^2 \), divided by one minus the demand deposit retention rate \( e^{2\alpha} \). This term adjusts for the bias in the net present value introduced by using unbiased estimates of the parameters, rather than the known values. For reasonable values of these parameters, this adjustment is nearly one. The remaining term in Equation (24-B) is identical (in appearance) to that in Equation (14). Equation (24-B) is the equation used in the empirical estimation of the demand deposit’s net present value.

**II. DESCRIPTION OF THE DATA**

There are three types of data: the bank’s deposits and rates paid; the Federal Reserve aggregate NOW account deposits and rates paid; and the Treasury bond data.

**Bank Data**

We obtain from the commercial bank monthly averages of non-maturity account balances and interest rates paid for January 1988 through June 1996. The consumer accounts include: 1) NOW accounts, 2) statement accounts, 3) passbook accounts, and 4) demand deposit accounts (DDA). These accounts differ with respect to the interest paid, fees, checking services, and minimum deposits required. The most important differences are: 1) all accounts pay interest except for the DDA, and 2) the NOW accounts and DDA have a reserve requirement of 10%, while the statement and passbook accounts have no reserve requirement. No non-interest costs are available for the commercial bank’s accounts.

Exhibit 1 is a time series graph of the bank’s NOW accounts, statement accounts, passbook accounts, and DDA. During the observation period, the bank experienced acquisitions of other commercial banks. The non-maturity demand deposits of the acquired
banks, after acquisition, are included in the account balance data. The acquisitions cause the account data to experience occasional discrete jumps.

The NOW and statement account balances appear to exhibit significant growth only during the middle of the observation period. The passbook accounts experience an unusually large increase in size due to merger activity around months 37 through 41. This significantly influences the standard errors of the parameter estimates. Otherwise, passbook accounts appear to be declining over time. Finally, the DDA appear to be the fastest-growing accounts, especially over the second half of the observation period.

Estimates based on this demand deposit data are performed in two ways. The first estimates are based on the combined data, including the mergers. The impact is thus that the future account balances are assumed to evolve in a manner similar to that observed over the sample period. Hence, future acquisitions are implied.

The second set of estimates is based on adjusted demand deposit balance data, smoothed to eliminate the acquisitions. The remaining deposits evolve over time according to a growth rate inherited from the non-smoothed data.

We estimate the net present values for both demand deposit series to provide a type of sensitivity analysis or “sanity check.” We know that the combined data provide an upper bound on the demand deposit growth. Because the merger removal procedure is subject to error, we err on the conservative side in order to obtain a lower bound for deposit growth. The true estimate of the bank’s demand deposit growth will lie somewhere between the two estimates produced.

Removing the merged deposit balances is a matter of some work. After the acquisitions, throughout the following fiscal year, the deposit balances are merged into the acquiring bank’s balance sheet in non-equal installments. It is not trivial to separate the merger installments from the ongoing growth in both the acquired and the acquiring banks’ deposits. Unfortunately, it is almost impossible to isolate this growth, especially for the acquired bank’s deposits.

To obtain a conservative estimate, we remove all increases due to the acquired bank deposits, thereby underestimating the actual growth in the demerged/smoothed balance series and providing a lower bound. All account types could be smoothed, except for the statement accounts (because of ambiguities in the recognition dates of the acquired statement account deposits).

Exhibit 2 is a time series graph of the bank’s interest rates paid, excluding costs, on the various accounts. The rates paid on the various accounts follow similar patterns over this observation period. Early in the
**Exhibit 2**
Bank NOW, DDA, Passbook, and Statement Account Rates (percent, costs included)

**Exhibit 3**
Federal Reserve Aggregate NOW Accounts
Balances (millions of dollars), Rates Paid (percent), and Costs (percent)
observation period, the rates paid were fairly constant. As market rates declined, however, so did the rates paid. At the end of the period, the rates paid appear to stabilize again, but at a lower level.

**Federal Reserve Bulletin Data**

From various issues of the *Federal Reserve Bulletin* and a yearly publication of the Federal Reserve Board called the “Functional Cost Analysis,” we obtain for all commercial banks monthly aggregates of consumer NOW accounts: 1) balances; 2) average rates paid; and 3) average non-interest costs. The Federal Reserve data divide commercial banks into three groups according to total deposits: up to $50 million, $50-$200 million, and more than $200 million. Our non-interest cost estimate is the average costs (net of fees earned) across all three groups. These non-interest costs are obtained for all four types of accounts. This is identical to the procedure used by Hutchison and Pennacchi [1996].

The sample period is January 1988 through December 1995 (shorter than for the individual bank data. Exhibit 3 shows the Federal Reserve aggregate NOW account balances, aggregate NOW account rates paid (costs not included), and aggregate NOW account costs. The balances exhibit a steady growth over the sample period. The rates paid are stable, until they decline and then stabilize again at a lower level. Costs, however, are almost constant across the sample period.

**Term Structure Data**

The commercial bank supplied end-of-month Treasury yields for maturities as follows: three months, six months, one year, three years, five years, ten years, and thirty years. Exhibit 4 shows a time series plot of the spot rate (three-month yield) and the ten-year and thirty-year yields for the sample period. Over the observation period, rates first declined, and then increased, but to a level lower than at the start of the period.

From these term structures, smoothed forward rate curves and extended Vasicek parameters are estimated.

**III. STATISTICAL ANALYSIS**

The statistical results obtained include parameter estimates, net present values, and risk management statistics: deltas, durations, and average lives.
Term Structure Estimation

The first step in valuing demand deposits is to generate smoothed forward rate curves and to estimate the term structure evolution parameters given in Equation (6) and to fit the spot rate Equation (19).

Smoothed Forward Rate Curves. Since the purpose of this analysis is to value demand deposits, the simplest procedure for generating forward rate curves is employed. Although more sophisticated procedures are available (see, for example, Adams and van Deventer [1994]), their application to this exercise awaits subsequent research.

Treasury yields are implicitly defined by:

\[ P(t, T) = e^{-\gamma(t, T)(T-t)} \]  

(25)

where \( P(t, T) \) is the time \( t \) price of a zero-coupon bond that matures at time \( T \), and \( \gamma(t, T) \) is the yield at time \( t \) on a zero-coupon bond that matures at time \( T \).

The first task is to transform the yields to zero-coupon bond prices. Given these zero-coupon bond prices, we next compute the forward rate curve. For simplicity, we assume that the forward rate curve is piecewise-constant between the observed maturities:

\[
\begin{align*}
  f(t, t) & \quad \text{for} \quad s \in [t, t+1/4) \\
  f(t, t+1/4) & \quad \text{for} \quad s \in [t+1/4, t+1/2) \\
  f(t, t+1/2) & \quad \text{for} \quad s \in [t+1/2, t+1) \\
  f(t, s) = & \begin{cases} 
  f(t, t+1) & \text{for} \quad s \in [t+1, t+3) \\
  f(t, t+3) & \text{for} \quad s \in [t+3, t+5) \\
  f(t, t+5) & \text{for} \quad s \in [t+5, t+10) \\
  f(t, t+10) & \text{for} \quad s \in [t+10, t+30) 
  \end{cases}
\end{align*}
\]  

(26)

Then, using the definition of forward rates in Equation (1), we obtain the estimate of the forward rate by solving:

\[ \frac{P(t, T)}{P(t, T+\delta)} = e^{-\int_{t}^{T+\delta} f(s)ds} = e^{\delta f(t, T)} \]  

(27)

for \( T < T + \delta \)

and \( T, T + \delta \in \{t, t+1/4, t+1/2, t+1, t+3, t+5, t+10, t+30\} \)

\[ \begin{array}{c|c|c|c}
\text{Exhibit 5} & \text{Extended Vasicek Parameters — January 1988-June 1996} \\
\hline
\text{Estimate} & \text{Standard Error} & R^2 & N \\
\hline
a & 0.007200 & 0.000946 & 0.99 & 102 \\
\sigma & 0.010059 & 0.000137 & & \\
\hline
\end{array} \]

\[ \begin{array}{c|c|c|c}
\text{Exhibit 6} & \text{Estimates of One-Month Treasury Rate Monthly Data — January 1988-June 1996} \\
\hline
a(0) & a(1) & R^2 \\
\hline
(standard error) & (standard error) & \\
\hline
r(t) & 0.000396 & 0.991808 & 0.98 \\
(0.000760) & (0.012979) & \\
\hline
\end{array} \]

These smoothed forward rate curves are computed for each month in the observation period.

Extended Vasicek Parameters. To determine the extended Vasicek parameters \((a, \sigma)\) from Equation (6), we follow Heitmann and Trautmann [1995], who estimate these parameters for German bond data. The procedure was later used by Henn [1997] for U.S. Treasury data.

It is well known that under the extended Vasicek model:

\[
\var[\log(P(t+\Delta, T)/P(t, T))] = r(t)\Delta = \sigma^2(e^{-a(T-t)} - 1)^2\Delta/a^2
\]  

(28)

Equation (28) represents the variance of the continuously compounded return on the \( T \)-th maturity zero-coupon bond less the spot rate over the time period \( [t, t + \Delta] \).

Using the time series of zero-coupon bond price observations from Equation (25), for each maturity \( i = \) three months, six months, one year, three years, five years, ten years, and thirty years, we compute the sample variance of the left-hand side of Equation (28) over the entire observation period (January 1988-June 1996). We then run a cross-sectional non-linear regression across the different maturities to estimate the parameters \((a, \sigma)\):
\[ v_i = \sigma^2 (e^{-ai} - 1)^2 \Delta / a^2 + e_i \]
for all \( i \) \hspace{1cm} (29)

where \( v_i \) are the sample variances on the left-hand side of Equation (28) for the \( i \)-th maturity bond, and \( e_i \) are independent and identically distributed errors with zero means and constant variances.

The estimates for these parameters are given in Exhibit 5. The mean reversion coefficient is 0.007200, and the volatility is 0.010059. These estimates are similar to those obtained by Henn [1997].

**Average Spot Rate Equation Estimation.** To investigate the satisfaction of the average spot rate Equation (19), a regression is estimated as follows:9

\[ r(t) = a(0) + a(1)r(t - \Delta) \]
for \( t \geq \Delta \) \hspace{1cm} (30)

Exhibit 6 gives the regression estimates of Equation (30). The constant \( a(0) \) is insignificantly different from zero, and \( a(1) \) is insignificantly different from one, confirming Equation (19). Hence, under the empirical probabilities, the spot rate process appears to follow a random walk.10

The correlation between \( r(t) \) and \( r(t - \Delta) \) is estimated to be 0.95. The high correlation between \( r(t) \) and \( r(t - \Delta) \) will cause multicollinearity problems in the estimation of the demand deposit balance regression (21) and the interest rate paid regression (23) (addressed further below).

**Demand Deposit Account's Evolution Estimation**

Given the multicollinearity problem between the average spot rate variables \( R(t) \) and \( R(t - \Delta) \) on the right-hand side of this regression equation, we need to modify Equation (21) one more time. The goal is to eliminate the lagged spot rate variable from the right-hand side of the regression.

This can be accomplished by using the average spot rate evolution (19). Substituting Equation (19) into Equation (21) for the lagged average spot rate variable yields the reduced-form expression:11

\[ \log D(j\Delta) = \varepsilon^\Delta \log D((j - 1)\Delta) + (1 - \varepsilon^\Delta) \log cD_0 + \varepsilon^\Delta \mu \Delta + \mu (1 - \varepsilon^\Delta) j \Delta + \alpha \Delta (1 - \varepsilon^\Delta) R(j\Delta) + \bar{x}_j \]
for \( j \geq 1 \) \hspace{1cm} (31)

where

\[ \bar{x}_j = x_j - \alpha \Delta \Delta u_{j\Delta}, \text{ and } \bar{x}_0 = 0 \]

with

\[ \bar{x}_j = \rho \bar{x}_{j-1} + \bar{x}_j^* \]

and \( \bar{x}_j^* \) iid normal \((0, \sigma_\Delta^2)\) for \( j \geq 1 \).

Equation (31) is identical to Equation (21) except that the lagged average spot rate variable is eliminated, and the error structure is modified. In the new error structure, if \( \rho \neq 0 \), a standard autocorrelation adjustment is necessary for the estimation.12

The regression equation without the autocorrelation adjustment is:

\[ \log D(j\Delta) = a(0) \log D((j - 1)\Delta) + a(1) + a(2)j + a(3)r(j\Delta) \]
for \( j \geq 1 \) \hspace{1cm} (32)

Equation (32) is quite simple. It states that the logarithm of the demand deposits at time \( j\Delta \) equals a constant \( a(0) \) times its value one period earlier, plus a constant \( a(1) \), plus a time trend term \( a(2)j \), plus a constant \( a(3) \) times the spot rate at time \( j\Delta \).

The identifications below determine the parameters from the regression coefficients:

\[
\begin{align*}
\varepsilon^\Delta &= a(0) \\
\Delta &= (1 - \varepsilon^\Delta) \log cD_0 + \varepsilon^\Delta \mu \Delta \\
\mu &= \mu \Delta (1 - \varepsilon^\Delta) \\
\alpha &= \alpha \Delta (1 - \varepsilon^\Delta)
\end{align*}
\]

\[ c = D_0^{-1} \exp \left\{ \frac{a(1)}{1 - a(0)} - \frac{a(0)a(2)}{1 - a(0)^2} \right\} \hspace{1cm} (33) \]

\[ \mu = \frac{a(2)}{\Delta[1 - a(0)]} \]

\[ \alpha = \frac{a(3)}{\Delta[1 - a(0)]} \]

The results of fitting Equation (32) to the bank's various demand deposit balances over the observation period are reported in Exhibit 7. Results are reported
### Exhibit 7
Estimates of Deposit Balance Evolutions

<table>
<thead>
<tr>
<th>Account</th>
<th>a(0)</th>
<th>a(1)</th>
<th>a(2)</th>
<th>a(3)</th>
<th>R²</th>
<th>h¹</th>
<th>ρ</th>
<th>N</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(std. error)</td>
<td>(std. error)</td>
<td>(std. error)</td>
<td>(std. error)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Federal Reserve-</td>
<td>0.93196*</td>
<td>1.069072*</td>
<td>0.000005</td>
<td>-0.62622*</td>
<td>0.99</td>
<td>-0.32</td>
<td>-0.21</td>
<td>95</td>
<td>0.000357</td>
</tr>
<tr>
<td>(1/88-12/95)</td>
<td>(0.03516)</td>
<td>(0.526058)</td>
<td>(0.000191)</td>
<td>(0.164963)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bank Combined</td>
<td>0.825412*</td>
<td>2.31621*</td>
<td>0.001422</td>
<td>-2.09675*</td>
<td>0.99</td>
<td>-1.11</td>
<td>NA</td>
<td>101</td>
<td>0.001745</td>
</tr>
<tr>
<td>(1/88-6/96)</td>
<td>(0.051542)</td>
<td>(0.67299)</td>
<td>(0.000524)</td>
<td>(0.60982)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passbook</td>
<td>0.990881*</td>
<td>0.141224</td>
<td>-0.00025</td>
<td>-0.00813</td>
<td>0.99</td>
<td>0.69</td>
<td>NA</td>
<td>101</td>
<td>0.010935</td>
</tr>
<tr>
<td>Statement</td>
<td>0.824656*</td>
<td>1.72078*</td>
<td>0.002242</td>
<td>-2.49204*</td>
<td>0.98</td>
<td>-0.16</td>
<td>0.24</td>
<td>101</td>
<td>0.004892</td>
</tr>
<tr>
<td></td>
<td>(0.05196)</td>
<td>(0.496576)</td>
<td>(0.00066)</td>
<td>(1.273342)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>DDA</td>
<td>0.90889*</td>
<td>1.20069*</td>
<td>0.001131</td>
<td>-0.7907*</td>
<td>0.99</td>
<td>-0.55</td>
<td>NA</td>
<td>101</td>
<td>0.001898</td>
</tr>
<tr>
<td></td>
<td>(0.033607)</td>
<td>(0.426028)</td>
<td>(0.000503)</td>
<td>(0.318689)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bank Demerged</td>
<td>0.831819*</td>
<td>2.803825*</td>
<td>0.000661</td>
<td>-0.81844*</td>
<td>0.94</td>
<td>-0.01</td>
<td>-0.30</td>
<td>101</td>
<td>0.001367</td>
</tr>
<tr>
<td>NOW</td>
<td>(0.061321)</td>
<td>(1.10848)</td>
<td>(0.00143)</td>
<td>(0.352995)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passbook</td>
<td>0.862701*</td>
<td>1.65372*</td>
<td>-0.00041</td>
<td>-1.45013*</td>
<td>0.92</td>
<td>0.40</td>
<td>NA</td>
<td>101</td>
<td>0.002835</td>
</tr>
<tr>
<td></td>
<td>(0.048206)</td>
<td>(0.576306)</td>
<td>(0.000251)</td>
<td>(0.596409)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DDA</td>
<td>0.819885*</td>
<td>2.280135*</td>
<td>0.001317</td>
<td>-0.17448</td>
<td>0.99</td>
<td>1.57</td>
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<td>101</td>
<td>0.000641</td>
</tr>
<tr>
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<td>(0.05354)</td>
<td>(0.673671)</td>
<td>(0.000405)</td>
<td>(0.176314)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*The small H-statistic tests for autocorrelation (after correcting for non-zero autocorrelation). If an adjustment is necessary, the estimated correlation (ρ) is given in the next column.

*Significantly different from zero at 95% confidence level.

NA — not adjusted.

both for the combined data and the “demerged” data.

We see that Equation (32) provides an excellent fit to the data, with an R² of between 0.92 and 0.99 for all the accounts. An h-statistic is computed to test for autocorrelation in the residuals (see Maddala [1992, p. 249]). If zero autocorrelation is rejected, an estimate of ρ is obtained, and the equation is reestimated to remove the autocorrelation. The h-statistics for the final regressions and correlation coefficients (if an adjustment is necessary) are reported. Only three of the deposit balances exhibit correlation in the residuals [Federal Reserve NOW, Statement (combined), and NOW (combined)].

The coefficients in the regressions have the appropriate signs. First, consider the bank’s NOW accounts using the combined data. All coefficients are significantly different from zero. The first coefficient (0.825412) indicates that the retention rate of NOW account deposits per month is about 82.5%. That is, about 17.5% of existing deposits decay after each month. Offsetting this outflow, however, new deposits enter with a growth rate determined by a(2), 0.001422.

The negative coefficient a(3), -2.09675, indicates that disintermediation occurs, because as interest rates rise, deposits decrease.

The demerged NOW accounts data are similar, except that the retention rate (0.831819) is higher while the growth rate of deposits (0.000061) and the interest rate sensitivity (-0.81844) are lower. The lower growth rate is to be expected after the removal of the acquired bank’s deposits.

A comparison of the individual bank coefficients with those of the “average” Federal Reserve Bank indicates that the average bank has a higher retention rate of deposits (0.93196), a slower growth rate of deposit inflows (0.000005), and less sensitive disintermediation (-0.62622). This suggests that the subject bank may be in a more competitive banking environment than the average bank.13

A comparison of the bank’s NOW accounts to the passbook and statement accounts and DDA reveals some interesting patterns. The remaining three accounts have similar retention rates of deposits as indicated by...
**EXHIBIT 8**
Estimates of Parameters for Accounts

<table>
<thead>
<tr>
<th>Account</th>
<th>$\epsilon^A$ (std. error)</th>
<th>$\mu$ (std. error)</th>
<th>$\alpha$ (std. error)</th>
<th>$c$ (std. error)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Federal Reserve</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1/88-12/95)</td>
<td>0.93196*</td>
<td>0.000803</td>
<td>-110.4443*</td>
<td>2.3989*</td>
</tr>
<tr>
<td>(1/88-6/96)</td>
<td>(0.03516)</td>
<td>(0.033371)</td>
<td>(41.1333)</td>
<td>(0.92614)</td>
</tr>
<tr>
<td>Bank Combined</td>
<td>0.825412*</td>
<td>0.097743*</td>
<td>-144.116*</td>
<td>2.4103*</td>
</tr>
<tr>
<td>(1/88-6/96)</td>
<td>(0.051542)</td>
<td>(0.0134177)</td>
<td>(20.7637)</td>
<td>(0.356177)</td>
</tr>
<tr>
<td>Passbook</td>
<td>0.990881*</td>
<td>-0.32776</td>
<td>-10.6911</td>
<td>1,454.3</td>
</tr>
<tr>
<td></td>
<td>(0.037226)</td>
<td>(2.05577)</td>
<td>(1,639.93)</td>
<td>(59,308)</td>
</tr>
<tr>
<td>Statement</td>
<td>0.824656*</td>
<td>0.15341*</td>
<td>-170.547*</td>
<td>2.8578*</td>
</tr>
<tr>
<td></td>
<td>(0.05196)</td>
<td>(0.029851)</td>
<td>(46.8739)</td>
<td>(0.9537)</td>
</tr>
<tr>
<td>DDA</td>
<td>0.90889*</td>
<td>0.148981*</td>
<td>-104.142*</td>
<td>1.470284*</td>
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<tr>
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<td>(0.033607)</td>
<td>(0.025762)</td>
<td>(47.3120)</td>
<td>(0.472657)</td>
</tr>
<tr>
<td>Bank Demerged</td>
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<td>0.004388</td>
<td>-58.3973*</td>
<td>1.607607*</td>
</tr>
<tr>
<td>(1/88-12/95)</td>
<td>(0.061321)</td>
<td>(0.009568)</td>
<td>(14.2649)</td>
<td></td>
</tr>
<tr>
<td>(1/88-6/96)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Passbook</td>
<td>0.862701*</td>
<td>-0.03572</td>
<td>-126.742*</td>
<td>2.439358*</td>
</tr>
<tr>
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<td>(0.048206)</td>
<td>(0.02065)</td>
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</tr>
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<td>DDA</td>
<td>0.819858*</td>
<td>0.087701*</td>
<td>-11.6226</td>
<td>0.95591*</td>
</tr>
<tr>
<td></td>
<td>(0.05354)</td>
<td>(0.007331)</td>
<td>(12.0525)</td>
<td>(0.078224)</td>
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</tbody>
</table>

*Significantly different from zero at the 95% level.

the $a(0)$ coefficients. For the demerged data, the passbook account has a higher retention rate ($0.862701$) and the DDA has a lower one ($0.819858$). Since the DDAs pay no interest, this is to be expected.

The combined statement accounts have the highest growth rate of balances as measured by $a(2)$, while the passbook accounts' growth rate is negative. This negative growth rate can be verified by looking at Exhibit 1. All the various accounts reflect disintermediation ($a(3) < 0$), but of varying sensitivities.

The regression coefficients provide the desired estimates of the parameters ($\epsilon^A$, $\mu$, $\alpha$, $c$) as given in Equation (33). These estimates and their standard errors are recorded in Exhibit 8. Most of these parameters are significantly different from zero, with the exception of the passbook account's parameters when mergers are included. As mentioned earlier, this is to be expected because of the unusually large spikes in the passbook deposit balances due to the mergers (see Exhibit 1).

As spot rates also changed during the spikes in deposits, the regression equation is not able to accurately distinguish growth in the deposits ($\mu$) from either disintermediation (interest rate sensitivity $\alpha$) or the perpetual deposits ($c$). Hence, all $\mu$, $\alpha$, and $c$ are insignificantly different from zero.

Encouraging, however, are the estimates for the passbook accounts after the mergers are removed. Here, the parameters are statistically distinguishable, and similar in level to the other demand deposit accounts' parameter values.

It is interesting to note that the growth rates in all the individual bank deposits are drastically reduced by the removal of the acquisitions. For example, for the NOW accounts, the growth rate per year with mergers is 0.097743; without mergers, it is only 0.004388. Without mergers, the bank's deposits are still growing faster than those of the "average bank," as measured by the Federal Reserve NOW account data (a growth rate per year of only 0.000803).

The estimate of the steady-state percentage deposits ($c$) for the Federal Reserve NOW accounts is 2.3989. For the bank's NOW accounts, with mergers it is 2.4103; without mergers it is 1.607607. These values of the steady-state percentage deposits greater than one can be verified by noting that the NOW account balances exhibit an upward growth trend during the observation period. The other accounts' steady-state percent-
age deposit point estimates all exceed one, except for the DDAs after removal of the mergers.

**Rate Paid Evolution Estimation**

To estimate the demand deposit interest rate paid regression (23), we again need to address the multico-linearity between the average spot rate independent variables \( R(t) \) and \( R(t - \Delta) \). The average spot rate evolution Equation (19) is again substituted into Equation (23) for the lagged average spot rate variable to obtain the reduced-form expression:

\[
i(j\Delta) = \pi^\Delta i[(j - 1)\Delta] + (1 - \pi^\Delta)k + \beta\Delta(1 - \pi^\Delta)\overline{R}(j\Delta) + \tilde{z}_j \quad \text{for } j \geq 1 \tag{34}
\]

where \( \tilde{z}_j = z_j - \beta\pi^\Delta u_{j\Delta} \), and \( \tilde{z}_0 = 0; \tilde{z}_j \) satisfy \( \tilde{z}_j = \rho \tilde{z}_{j-1} + \tilde{z}_j^* \); and \( \tilde{z}_j^* \) are iid normal \( (0, \sigma_{\tilde{z}}^2) \) for \( j \geq 1 \).

Equation (34) is a simplification of the interest rate paid regression (23). The only difference is that Equation (34) does not have the lagged average spot rate variable, and it has a modified error term. The modified error structure can have autocorrelation. A standard autocorrelation adjustment to Equation (34) yields: \(^{14}\)

\[
i(j\Delta) - \rho i[(j - 1)\Delta] = \pi^\Delta [i[(j - 1)\Delta] - \rho i[(j - 2)\Delta]] + \beta\Delta(1 - \pi^\Delta)[R(j\Delta) - \rho R((j - 1)\Delta)] + (1 - \pi^\Delta)k(1 - \rho) + \tilde{z}_j^* \quad \text{for } j \geq 2 \tag{35}
\]

The regression equation for (35) is:

\[
i(j\Delta) - \rho i[(j - 1)\Delta] = a(0)[i[(j - 1)\Delta] - \rho i[(j - 1)\Delta]] + a(1) + a(2)[r(j\Delta) - \rho r((j - 1)\Delta)]
\]

for \( j \geq 2 \tag{36}\)

Equation (35) is quite straightforward. It states that the interest rate paid at time \( j\Delta \), after adjusting for autocorrelation, is equal to a constant \( a(0) \) times its value last period, plus a constant \( a(1) \), plus a constant \( a(2) \) times the spot rate variable.

The coefficients from the regression can be used to estimate the parameters as follows:

\[
a(0) = \pi^\Delta \quad \left\{ \begin{array}{l} a(1) = (1 - \pi^\Delta)k(1 - \rho) \\ a(2) = \beta\Delta(1 - \pi^\Delta) \end{array} \right. \quad \left\{ \begin{array}{l} \pi^\Delta = a(0) \\ k = a(1) / (1 - \pi^\Delta)(1 - \rho) \\ \beta = a(2) / [1 - a(0)\Delta] \end{array} \right. \tag{37}
\]

**E X H I B I T 9**

**Estimates of Deposit Rate Paid Evolutions**

<table>
<thead>
<tr>
<th>Account</th>
<th>( a(0) ) (std. error)</th>
<th>( a(1) ) (std. error)</th>
<th>( a(2) ) (std. error)</th>
<th>( R^2 )</th>
<th>( h^* )</th>
<th>( \rho )</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Federal Reserve (1/88-12/95) NOW</td>
<td>0.953043** (0.011416)</td>
<td>-0.00033* (0.000193)</td>
<td>0.045988** (0.008719)</td>
<td>0.99</td>
<td>-0.22</td>
<td>0.50</td>
<td>94</td>
</tr>
<tr>
<td>Bank (1/88-6/96) NOW</td>
<td>0.962144** (0.011825)</td>
<td>-0.00097** (0.000347)</td>
<td>0.049232** (0.011252)</td>
<td>0.99</td>
<td>-0.13</td>
<td>0.28</td>
<td>100</td>
</tr>
<tr>
<td>Passbook Statement</td>
<td>0.943639** (0.020452)</td>
<td>0.001035 (0.000885)</td>
<td>0.03744** (0.011816)</td>
<td>0.98</td>
<td>-0.94</td>
<td>NA</td>
<td>100</td>
</tr>
<tr>
<td>DDA</td>
<td>0.93333** (0.030482)</td>
<td>0.00062 (0.000722)</td>
<td>0.021024** (0.009587)</td>
<td>0.94</td>
<td>0.05</td>
<td>NA</td>
<td>100</td>
</tr>
</tbody>
</table>

\(^{14}\)The small H-statistic tests for autocorrelation (after correcting for non-zero autocorrelation). If an adjustment is necessary, the estimated correlation (\( \rho \)) is given in the next column.

\(^a\)Significantly different from zero at 90% confidence level.

\(^{**}\)Significantly different from zero at 95% confidence level.

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**Exhibit 10**

Estimates of Demand Deposit Rate Paid Parameters

<table>
<thead>
<tr>
<th>Account</th>
<th>$\pi^d$ (std. error)</th>
<th>$\beta$ (std. error)</th>
<th>$k$ (std. error)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Federal Reserve (1/88-12/95) NOW</td>
<td>0.953043** (0.011416)</td>
<td>11.75223** (1.9095)</td>
<td>-0.014073 (0.01033)</td>
</tr>
<tr>
<td>Bank (1/88-6/96) NOW</td>
<td>0.962144** (0.011825)</td>
<td>15.60623** (3.4742)</td>
<td>-0.035523* (0.018725)</td>
</tr>
<tr>
<td>Passbook</td>
<td>0.943639** (0.020452)</td>
<td>7.971476** (2.060489)</td>
<td>0.018372* (0.01073)</td>
</tr>
<tr>
<td>Statement</td>
<td>0.945707** (0.013912)</td>
<td>8.459275** (1.49824)</td>
<td>0.015721** (0.007904)</td>
</tr>
<tr>
<td>DDA</td>
<td>0.93333** (0.030482)</td>
<td>3.784177** (1.598968)</td>
<td>0.009301 (0.008017)</td>
</tr>
</tbody>
</table>

*Significantly different from zero at 90% level.
**Significantly different from zero at 95% level.

The theoretical model for the demand deposit valuation requires the interest rate paid $i(t)$ to include all non-interest servicing costs (per dollar deposit). While for the Federal Reserve aggregate data, these non-interest costs are available, they are not for the individual bank. Consequently, as a first estimate, we use the average non-interest costs for each account type for all commercial banks as reported in the Federal Reserve data as the costs for each of the individual bank's accounts.

The results of fitting the interest rate paid regression (36) are reported in Exhibit 9. The regression provides an excellent fit to the data, with an $R^2$ of between 0.94 and 0.99 for all the accounts. After adjusting for autocorrelation (if necessary), the $h$-statistic indicates no remaining correlation in the residuals. Autocorrelation adjustments are necessary for all the NOW and statement accounts.

First, consider the bank’s NOW accounts. The coefficients $a(0)$ and $a(1)$ are significantly different from zero. The coefficient $a(2) > 0$ shows the sensitivity of NOW account rates paid to market spot interest rates. All these coefficients are significantly different from zero.

The magnitudes of $a(2)$ coefficients appear low (for example, $a(2) = 0.045988$ for the Federal Reserve NOW accounts), but in fact this is not the case. This coefficient does not directly measure the interest rate sensitivity of the deposit rate paid $\beta$. Instead it is the interest rate sensitivity parameter $\beta$ multiplied by $\Delta(1 - \pi^d)$, which is a small number lower than one. A comparison of the bank’s NOW account regressions with those of the Federal Reserve aggregate NOW accounts, passbook accounts, and statement accounts shows similar results. Although the DDAs pay no interest, the results are reported for the DDAs based on only the non-interest servicing costs.

The regression coefficients in Exhibit 9 generate the desired estimates for the interest rate paid parameters ($\pi^d$, $\beta$, $k$) as given in expression (37) and recorded in Exhibit 10. The steady-state interest rate paid ($k$) is negative for both the bank and the Federal Reserve NOW accounts. The bank’s NOW account steady-state interest rate paid is seen to be significantly higher than the average bank’s, consistent with the bank’s presumably more competitive banking environment, where significantly higher expenditures are required to retain deposits (e.g., advertising expenses). In contrast, both the bank’s passbook and statement account floors are positive.

All the parameters are significantly different from zero, except for the Federal Reserve NOW account’s long-run interest rate paid. Notice that the interest rate sensitivity parameter $\beta$ is much higher than the $a(2)$ coefficients in Exhibit 9. Furthermore, the convergence parameter $\pi^d$ for the interest rate paid to the long-run interest rate paid is quite high, over 0.93 for all the accounts.
Demand Deposit Valuation Estimates

With the demand deposit balance and rates paid evolution parameters, we can determine the net present value of the various demand deposits. The parameter values given in Exhibit 5 for the term structure of interest rates and in Exhibits 8 and 10 for the various accounts are fixed for the entire time period. The forward rate curve, however, varies across time as estimated by Equation (27). The reserve requirement is set at \( m = 0.1 \) for both the NOW and DDA, while \( m = 0 \) for both the passbook and statement accounts.

The percentage demand deposit premiums are plotted in Exhibit 11 for the parameter estimates after the removal of the mergers. They exhibit significant fluctuation through time. Summary statistics of the

Exhibit 12
Summary Statistics for Monthly Observations of Percentage Demand Deposit Premiums

<table>
<thead>
<tr>
<th>Account</th>
<th>Deposit Premiums in Percentages</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Median</td>
<td>Std. Dev.</td>
<td>Max</td>
<td>Min</td>
</tr>
<tr>
<td>Federal Reserve (1/88-12/95)</td>
<td>NOW</td>
<td>3.81</td>
<td>2.72</td>
<td>4.27</td>
<td>12.88</td>
</tr>
<tr>
<td>Bank Combined (1/88-6/96)</td>
<td>NOW</td>
<td>38.30</td>
<td>26.19</td>
<td>33.68</td>
<td>121.10</td>
</tr>
<tr>
<td></td>
<td>Passbook</td>
<td>14.30</td>
<td>15.75</td>
<td>7.45</td>
<td>26.00</td>
</tr>
<tr>
<td></td>
<td>Statement</td>
<td>-15.24</td>
<td>0.26</td>
<td>41.95</td>
<td>30.62</td>
</tr>
<tr>
<td></td>
<td>DDA</td>
<td>232.00</td>
<td>234.24</td>
<td>18.22</td>
<td>262.42</td>
</tr>
<tr>
<td>Bank Demerged</td>
<td>NOW</td>
<td>-5.73</td>
<td>-6.94</td>
<td>9.02</td>
<td>12.38</td>
</tr>
<tr>
<td></td>
<td>Passbook</td>
<td>-1.20</td>
<td>-0.65</td>
<td>2.71</td>
<td>2.97</td>
</tr>
<tr>
<td></td>
<td>DDA</td>
<td>109.50</td>
<td>109.38</td>
<td>2.63</td>
<td>115.28</td>
</tr>
</tbody>
</table>
percentage demand deposit premiums are provided in Exhibit 12.

First, consider the bank’s NOW accounts. The median percentage NOW account premium for the combined data with mergers is 26.19%. This implies that for each dollar NOW account deposit the bank obtains, 26.19 cents is pure “profit,” flowing directly into shareholders’ equity. The maximum net present value over this period is 121.10%, and the minimum is –2.69%.

For the demerged data, the median NOW account premium is –6.94%. The maximum is 12.38%, and the minimum is –16.78%.

The combined NOW account deposit balance series, due to the mergers, overestimates the actual premium. The smoothed deposit balance series, due to the conservative method of removing the acquired firms’ deposits, underestimates the actual premium. Consequently, the true premium lies between these two.

Nonetheless, both the NOW account premiums fall within the range of premiums reported in Jarrow and van Deventer [1998] from auctions of failed banks

### EXHIBIT 13
Summary Statistics for Monthly Observations of Demand Deposit Durations and Average Lives

<table>
<thead>
<tr>
<th></th>
<th>Federal Reserve (1/88-12/95)</th>
<th>Bank — Combined (1/88-6/96)</th>
<th>Bank — Demerged</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>NOW</td>
<td>NOW</td>
<td>DDA</td>
</tr>
<tr>
<td>Number of Observations</td>
<td>96</td>
<td>102</td>
<td>102</td>
</tr>
<tr>
<td>NPV Delta</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Max</td>
<td>16,567</td>
<td>28,416</td>
<td>–129,735</td>
</tr>
<tr>
<td>Min</td>
<td>1,312</td>
<td>–211,603</td>
<td>–547,780</td>
</tr>
<tr>
<td>Median</td>
<td>12,460</td>
<td>–14,035</td>
<td>–226,654</td>
</tr>
<tr>
<td>Average</td>
<td>10,093</td>
<td>–40,829</td>
<td>–298,797</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>4,785</td>
<td>65,755</td>
<td>147,786</td>
</tr>
<tr>
<td>NPV Duration</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Max</td>
<td>–38.03</td>
<td>–56.91</td>
<td>3.60</td>
</tr>
<tr>
<td>Median</td>
<td>–85.21</td>
<td>–102.01</td>
<td>–7.83</td>
</tr>
<tr>
<td>Average</td>
<td>–631.16</td>
<td>–2.289</td>
<td>–6.59</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>3,858</td>
<td>20,823</td>
<td>3.92</td>
</tr>
<tr>
<td>Cost Duration</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Max</td>
<td>5.65</td>
<td>5,088</td>
<td>23.21</td>
</tr>
<tr>
<td>Min</td>
<td>1.18</td>
<td>12.51</td>
<td>–6.05</td>
</tr>
<tr>
<td>Median</td>
<td>2.37</td>
<td>37.42</td>
<td>13.94</td>
</tr>
<tr>
<td>Average</td>
<td>2.68</td>
<td>233.2</td>
<td>12.03</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>1.21</td>
<td>699.9</td>
<td>7.46</td>
</tr>
<tr>
<td>NPV Average Life</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Max</td>
<td>18,054</td>
<td>44,399</td>
<td>20.35</td>
</tr>
<tr>
<td>Median</td>
<td>39.47</td>
<td>29.38</td>
<td>19.50</td>
</tr>
<tr>
<td>Average</td>
<td>300.05</td>
<td>491.52</td>
<td>19.52</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>1,839</td>
<td>4,391</td>
<td>0.44</td>
</tr>
<tr>
<td>Cost Average Life</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Max</td>
<td>10.04</td>
<td>14.81</td>
<td>20.89</td>
</tr>
<tr>
<td>Min</td>
<td>8.36</td>
<td>13.56</td>
<td>19.00</td>
</tr>
<tr>
<td>Median</td>
<td>9.09</td>
<td>14.01</td>
<td>19.83</td>
</tr>
<tr>
<td>Average</td>
<td>9.10</td>
<td>14.14</td>
<td>19.91</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>0.51</td>
<td>0.31</td>
<td>0.51</td>
</tr>
</tbody>
</table>
as reported by the Resolution Trust Corporation (an average of 2.32%, a minimum of −16.87%, and a maximum of 25.33%).

A comparison of the bank’s NOW account percentage premium with the average bank from the Federal Reserve NOW account data is informative. The average bank has a median premium of 2.72%, not too different from those reported by Hutchison and Pennacchi [1996] (3.49% for all NOW accounts). Their sample is comparable to our Federal Reserve data estimate.

A comparison of the bank’s NOW account premiums with the other account premiums shows that DDAs are the most profitable, with mean premiums of 234.24% (with mergers) and 109.38% (without mergers). As the DDAs pay no interest and the NOW accounts do, one would expect the DDA premiums to be higher. Our DDA premium estimates are consistent with this intuition. Passbook account premiums are less than for the NOW accounts. Statement account premiums are close to zero (0.26%).

These premiums for the bank are subject to three potential biases. First, the average costs for all commercial banks for each account type may not represent the true costs of the subject bank. The subject bank may be more efficient than the average bank (because of the competitive environment in which it operates), and its costs may be lower than the average bank.

Second, the opportunity cost of funds for this bank may be the Eurodollar rate and not the Treasury rate. Eurodollar rates are higher. Third, properly accounting for the acquired firms’ deposits (as discussed previously) would raise the estimated growth rate and increase long-run deposits.

We conjecture that the combined effect of these three biases would generate a positive NOW account premium for the bank. To validate this conjecture would require new and more refined data and further research.

Demand Deposit, Deltas, Durations, and Average Life Estimates

For risk management considerations, the demand deposit deltas, durations, and average lives for each type of account are given in Exhibit 13.

We first consider the demand deposit deltas. When spot rates rise, there are two offsetting effects. The first is that increased disintermediation occurs, decreasing demand deposit balances and therefore decreasing premiums. The second effect is that existing deposits become more profitable (as market rates rise faster than rates paid), thereby increasing premiums. Either effect could dominate. In a competitive banking environment, one would expect the first effect to dominate, because the rates paid respond more rapidly to market rate movements.

The data support this intuition. The median NPV deltas are of different signs for the Federal Reserve and the bank’s NOW accounts. For the Federal Reserve data, the delta is positive, while for the bank’s NOW accounts, the delta is negative. This is consistent with a more competitive environment for the subject bank than for the average bank. Of the remaining accounts, the passbook and statement accounts have positive deltas, while the DDA delta is negative.

The median NPV duration for the Federal Reserve NOW account is −85.21 years. This measure demonstrates the significant interest rate sensitivity of the deposit NPVs. The median NPV duration for the bank’s NOW account is −102.01 years (with mergers) and −80.27 years (without mergers). These values span the Federal Reserve’s.

The durations for the bank’s DDA are −7.83 years (with mergers) and −0.60 years (without mergers). The passbook account’s durations are 56.02 years (with mergers) and 150.23 years (without mergers). The statement account’s duration is 159.38 years.

These durations are not the quantities that bank management is typically concerned with. Bank management more traditionally concentrates on the cost durations. The cost durations are all much smaller than the NPV durations. This is because the deposit balances are much less sensitive to rate changes than are the deposit profits.

The median cost durations for the bank’s NOW accounts are 37.42 years (with mergers) and 5.21 years (without mergers). The cost duration without the mergers is the more appropriate statistic. The median cost duration for the Federal Reserve average bank is 2.37 years.

It is often believed that duration is a measure of a financial instrument’s average life. This is false, in general, and true only if interest rates follow a model in which parallel shifts in the yield curve are the only evolutions possible (see Jarrow and Turnbull [1996]). For this analysis, the term structure estimates of the interest rate evolution in Exhibit 5 are inconsistent with a parallel shift model for the spot rate evolution. Consequently, duration does not provide a valid measure of the deposit’s...
average life. We therefore compute average lives of the various deposits directly.

The average lives of the various demand deposits in Exhibit 13 indicate that the median value for the average life of the NPV for the bank's NOW accounts is 29.38 years with mergers and 7.32 years without mergers. The median NPV average life is 39.47 years for the Federal Reserve NOW account. Again, these are not the quantities normally studied by traditional bank management.

Traditional bank management is concerned with the average life of the demand deposit's cost, which is a measure of how long the demand deposit liability remains within the bank. The median cost (or liability) average lives for the bank's NOW accounts are 14.01 years (with mergers) and 8.80 years (without mergers). The median Federal Reserve NOW account's cost average life is 9.09 years. These estimates indicate that NOW accounts are not as short-lived as is commonly believed.

The DDA cost average life is 19.83 years (with mergers) and 15.58 years (without mergers). The DDAs are significantly longer-lived than the NOW accounts. The statement account has the longest average life of 20.03 years (with mergers). The passbook accounts are the shortest-lived, with an average life of 7.81 years (with mergers) and 7.38 years (without mergers).

IV. SUMMARY

This is the first empirical application of the Jarrow and van Deventer model for valuing various demand deposits of a commercial bank over the time period January 1988 through June 1996. Estimates of deposit premiums, durations, and average lives are consistent with previous studies, and confirm the validity of the model.

APPENDIX

Derivations

Derivation of Equation (14)

The spot rate evolution is:

\[ r(t) = f(0, t) + b(0, t)^2 / 2 + \int_0^t \rho(s, t) d\bar{W}(s) \]

where \( \rho(s, t) = \sigma e^{-\alpha(t-s)} \).

and \( b(s, t) = \sigma(1 - e^{-\alpha(t-s)}) / a = \int_t^s \rho(s, v) dv \)

The following can be proven:

\[ \int_0^t b(0, s)^2 ds = \int_0^t b(s, t)^2 ds / 2 \]

\[ \int_0^t \rho(v, s)d\bar{W}(v)ds = \int_0^t b(v, s)d\bar{W}(v) \]

\[ \int_0^t r(s)ds = \int_0^t f(0, s)ds + \int_0^t \rho(s) d\bar{W}(s) \]

\[ \int_0^t \alpha_{t-r}(s)ds = \int_0^t \rho(0, s)ds + \int_0^t \alpha_{t-r}(0, s)^2 ds / 2 \]

\[ \int_0^t \alpha_{t-r}(v, s)d\bar{W}(v)ds \]

Similarly, one can determine \( \int_0^t \beta_{t-r}(s)ds \).

Define

\[ x_1 = \int_0^t (\alpha_{t-r} - 1)r(s)ds \]

\[ x_2 = \int_0^t \beta_{t-r}(s)ds \]

\[ x_3 = \rho(t) \]

Then, \( (x_1, x_2, x_3) \) are joint-normal with parameters:

\[ \mu_1(t) = \bar{E}(x_1), \quad \sigma_1^2(t) = \text{var}(x_1), \quad \sigma_{12}(t) = \text{cov}(x_1, x_2) \]

\[ \mu_2(t) = \bar{E}(x_2), \quad \sigma_2^2(t) = \text{var}(x_2) \]

\[ \mu_3(t) = \bar{E}(x_3), \quad \sigma_3^2(t) = \text{var}(x_3), \quad \sigma_{13}(t) = \text{cov}(x_1, x_3) \]

These functions are explicitly evaluated below. Using the
moment-generating functions gives (see Jarrow and van Deventer [1998]):

\[
\hat{E}_0\left(e^{r(s)}\right) = e^{\mu_s(t)+\sigma_s^2(t)/2}
\]

\[
\hat{E}_0\left(e^{r(s)}\right) = \int_0^t \beta_t r(s) ds
\]

\[
\hat{E}_0\left(e^{r(s)}\right) = e^{\mu_s(t)+\sigma_s^2(t)/2} [\mu_s(t)+\sigma_s^2(t)]
\]

Use of these expressions in the valuation of Equation (3) gives the result.

**Derivation of Valuation Formula**

**Based on Equation (24)**

From Equation (24), interchanging the order of expectations, and using the statistical independence of \(v_t, w_t\), we obtain:

\[
\hat{V}(0) = \hat{E}_0 \times \left( \int_0^\tau \int_0^\tau e^{\gamma_t(s)} \hat{E}_0(e^{v_t}) \frac{f(0,s)ds + \alpha \int_0^{\max[0,t-\Delta]} b(0,s)^2 ds}{\beta(t)} \right)
\]

But, since \(v_t\) is normal:

\[
\hat{E}_0(e^{v_t}) = e^{\hat{E}_0(v_t)+\sigma^2(v_t)/2}
\]

where \(\hat{E}_0(v_t) = 0\) and \(\hat{E}_0(w_t) = 0\).

But, from Equation (21), assuming \(\rho = 0\), one can see that:

\[
v_t = \sum_{i=0}^{t-\Delta} e^{\Delta t} X_{t-i\Delta}
\]

Since \(\text{var}(x_t) = \sigma_x^2\), for all \(t\), we get that:

\[
\text{var}(v_t) = \sum_{i=0}^{t-\Delta} e^{2\Delta t} \sigma_x^2
\]

\[
= \sigma_x^2 (1-e^{2\Delta t}) / (1-e^{2\Delta})
\]

\[
\leq (\sigma_x^2) / (1-e^{2\Delta})
\]

Substituting this into the first equation above, we obtain:

\[
\hat{E}_0\left( \int_0^\tau \int_0^\tau e^{\mu_t(s)+\sigma_t^2(t)/2} [\mu_t(t)+\sigma_t^2(t)] \right) \times
\]

The expectation on the right-hand side of this expression has the form given in Equation (14), with

\[
\alpha_{t-s} = \begin{cases} \alpha & \text{for } t-s \in [0, \Delta] \\ 0 & \text{otherwise} \end{cases}
\]

and

\[
\beta_{t-s} = \begin{cases} \beta & \text{for } t-s \in [0, \Delta] \\ 0 & \text{otherwise} \end{cases}
\]

Using these definitions one obtains:

\[
\mu_t(t) = \alpha \int_{\max[0,t-\Delta]} f(0,s)ds + \alpha \int_{\max[0,t-\Delta]} b(0,s)^2 ds
\]

\[
\int_0^\tau f(0,s)ds - \frac{1}{2} \int_0^\tau b(s, t)^2 ds
\]

\[
\sigma_t^2(t) = \alpha \int_{\max[0,t-\Delta]} \left( \frac{e^{\alpha t} - 1}{\alpha} \rho(s, t) - b(s, t) \right)^2 ds + \frac{1}{\alpha} \int_{\max[0,t-\Delta]} [\alpha - 1]^2 b(s, t)^2 ds
\]

\[
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\]

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\[
\mu_2(t) = \beta \int_{\text{max}(0, t - \Delta)} f(0, s)ds + \beta \int_{\text{max}(0, t - \Delta)} \frac{b(0, s)^2}{2} ds.
\]

\[
\sigma^2_2(t) = \frac{\beta^2}{a^2} \int_{\text{max}(0, t - \Delta)} \frac{(e^{t\Delta} - 1)^2}{2} \rho(s, t)^2 ds + \beta^2 \int_{\text{max}(0, t - \Delta)} b(s, t)^2 ds.
\]

\[
\sigma_{12}(t) = \frac{\alpha \beta}{a^2} \int_{0}^{t} \rho(s, t)b(s, t)ds - \beta \int_{0}^{t} \frac{(e^{s\Delta} - 1)}{a} \rho(s, t)b(s, t)ds.
\]

\[
\mu_3(t) = f(0, t) + \frac{b(0, t)^2}{2}, \quad \sigma^2_3(t) = \int_{0}^{t} \rho(s, t)^2 ds.
\]

\[
\sigma_{13}(t) = \frac{\alpha \beta}{a} \int_{0}^{t} \rho(s, t)^2 ds - \int_{\text{max}(0, t - \Delta)} \rho(s, t)b(s, t)ds + \max(0, t - \Delta) \int_{0}^{t} \rho(s, t)b(s, t)ds.
\]

\[
(\alpha - 1) \int_{\text{max}(0, t - \Delta)} b(s, t)\rho(s, t)ds.
\]

ENDNOTES

Helpful comments from Donald van Deventer and the Finance Workshop at the University of Virginia are gratefully acknowledged.

1 For confidentiality reasons, we do not reveal the bank's identity or sensitive dates regarding merger activity.

2 A finite (but long) trading horizon is a standard hypothesis necessary to accommodate a martingale measure that is used for valuation (see Jarrow and Madan [1999]).

3 In arbitrage pricing theory, the drift of an asset's price process is different under the empirical and martingale probabilities. The difference in drifts is due to a risk premium. Since the spot rate's mean reversion is characterized through the drift term, it is possible for the spot rate to be mean-reverting under the martingale probabilities and not mean-reverting under the empirical probabilities.

4 This can be seen by taking the limit as \( t \to \infty \) in Equation (7-A), noting that \( \epsilon < 1 \), setting \( \mu = 0 \) and \( \rho_t = \rho \equiv 0 \). Then, \( \lim_{t\to\infty} D(t) = C \).

5 This is seen by noting that \( \lim_{t\to\infty} \delta(t) = \kappa \) when \( \kappa = 0 \) and \( \pi < 1 \).

6 Since \( D(0) \) is a constant with respect to \( r(0) \), NPV Delta = Cost Delta.

7 It is easy to show that the average spot rate can be written as \( R(t) = R(t - \Delta) + \lambda(t) + u_t \), where \( u_t \) is normally distributed with zero mean, and \( \lambda(t) \) is a stochastic process dependent on a risk premium and the spot rate's drift and volatility parameters. Depending upon the risk premium selected, the \( \lambda(t) \) term could be identically zero.

8 An analogous problem arises when using the Black-Scholes formula with an unknown volatility. The usual procedure is to substitute an unbiased estimate of the volatility in the Black-Scholes equation. But, because the Black-Scholes equation is convex with volatility, Jensen's inequality reveals that this will give a biased estimate for the option's value.

9 The equation has additional measurement error because \( r(t) \) is used instead of \( R(t) \) in Equation (19). Estimates of \( R(t) \) are not available.

10 Under the martingale probabilities, it follows Equations (6) with \((\alpha, \sigma)\) as estimated in Exhibit 5.

11 Equation (21) also has an overidentification problem in that the parameter \( \alpha \) is independently determined by the coefficients for both the \( R(t) \) and \( R(t - \Delta) \) terms. The substitution of (19) into (21) removes this difficulty as well.

12 If \( \rho \neq 0 \), the equation estimated is:

\[
\log D(j\Delta) - \rho \log D((j-1)\Delta) = \epsilon^\Delta [\log D((j-1)\Delta) - \rho \log D((j-2)\Delta)] + \epsilon^\Delta \rho \log D(0) + \epsilon^\Delta [\log D((j-1)\Delta) - \rho \log D((j-2)\Delta)] + \epsilon^\Delta [\log D((j-2)\Delta) - \rho \log D((j-3)\Delta)] + \epsilon^\Delta [\log D((j-3)\Delta) - \rho \log D((j-4)\Delta)]
\]

\[
\times [(1 - \epsilon^\Delta) \log D(0) + \epsilon^\Delta [\log D((j-1)\Delta) - \rho \log D((j-2)\Delta)] + \epsilon^\Delta [\log D((j-2)\Delta) - \rho \log D((j-3)\Delta)] + \epsilon^\Delta [\log D((j-3)\Delta) - \rho \log D((j-4)\Delta)]] + \epsilon^\Delta [\log D((j-4)\Delta) - \rho \log D((j-5)\Delta)] + \epsilon^\Delta [\log D((j-5)\Delta) - \rho \log D((j-6)\Delta)] + \epsilon^\Delta [\log D((j-6)\Delta) - \rho \log D((j-7)\Delta)]
\]

The parameter estimates are appropriately modified.

13 The reduced interest rate sensitivity of the Federal Reserve NOW account data is partially expected. Although a single bank may lose deposits to other banks, deposits stay in the banking system; these deposit transfers would not register in the Federal Reserve NOW account data. Hence, we would expect the interest rate sensitivity of the Federal Reserve data to be lower.

14 This adjustment requires an estimate for \( \rho \), which is obtained by first running a regression of (34) without the lagged adjustment; see Johnston [1972].

15 For the DDA, premiums include the costs of servicing these accounts (from the Federal Reserve data).
REFERENCES


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