Arbitrage, martingales, and private monetary value

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This paper reevaluates the mathematical and economic meaning of no arbitrage in frictionless markets. Contrary to the traditional view, no arbitrage is not generally equivalent to the existence of an equivalent martingale measure. Departures from this equivalence allow asset prices to contain a monetary component. The refined view is that no arbitrage and no private monetary value components are equivalent to the existence of an equivalent martingale measure. The implications of prices having a monetary value component for option pricing are discussed.

1. INTRODUCTION

The conventional wisdom in finance is that a stock’s value equals the present value of its future cashflows (dividends or earnings), called the intrinsic or fundamental value. Recent market experience, with respect to many internet stocks, seems inconsistent with the view. Prices for selected internet stocks, such as Yahoo, Amazon, and Netscape, have been extraordinarily high in the face of zero dividend payments and continued reports of zero or negative earnings. A related market-wide phenomenon has been the steady climb in price-to-earnings ratios from around 11 in the 1976–82 period to a value of around 27 in July 1998 (see Lee, Myers, and Swaminathan 2000). This difference between the stock price and the intrinsic value has been called a bubble. The terminology itself reflects the belief that any difference between the price and intrinsic value must be short lived and will surely burst, like a soap bubble. Embedded in this wisdom is the feeling that stock price bubbles reflect a market imperfection or a market inefficiency. Sometimes it is even argued that bubbles reflect implicit arbitrage opportunities and, therefore, they cannot be long lasting.

These conventional views are strongly embedded in the profession. However, there exists economic theory, albeit quite abstract in its formulation (see Bewley 1972; Gilles 1989; Back and Pliska 1991; Gilles and LeRoy 1992; Becker and Boyd 1993), that can be used to explain this phenomenon. The purpose of this

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1 For an alternative reconciliation based on growth rate projections, see Schwartz and Moon (2000).
paper is to use this abstract theory to argue that the conventional wisdom is too narrow and incorrect. In contrast, we believe that the difference between a stock's price and intrinsic value can be significant and long lasting, never bursting. We believe that this difference reflects the use of the stock as a type of private money.

Money has long been regarded in economic theory as an anomalous asset that has positive value, even though it never pays any cashflow. It has positive value because of the public's confidence that it can be used for the future purchase of goods and services. These considerations apply to the liquidity benefits of cash, given its status as legal tender for the satisfaction of debts and other obligations. However, such benefits also extend to the promissory notes of individuals that have served as private monies in times past, e.g., the notes of Rothschild discussed by Galbraith (1975). The source of liquidity in an asset may well reside in the confidence placed by market participants in the general ability of the issuer to meet their obligations. One may well imagine Bill Gates issuing a paper entitled the holder to a share in his estate upon eventual liquidation. Such shares may then circulate and trade at a market-determined price. These shares may then be used to meet their holders' debts and obligations, much like money itself. The market price of these shares may well be based on the market's assessment of the eventual or limiting size of Gates's estate. This view can also be applied to stock certificates.

Stock certificates are a type of privately issued money that can also be used to purchase goods and services, in addition to the cashflows that they provide. For example, the use of stock options as employee compensation is well established. Furthermore, the use of stock certificates as payment in mergers is also quite common; the recent purchases of MCI by WorldCom or of Geocities by Yahoo are cases in point.

Money, as issued by the state, gets its value from its general acceptability as legal tender for the payments of debt and obligations. Stock certificates, seen as private money, can get its monetary value from a general acceptability based on the view that the certificate can be resold at a later date owing to a long-term earning potential of the issuer. This monetary value part of the stock price can be long lasting, never bursting, and it need not represent an arbitrage opportunity. It can represent a permanent difference between the stock price and its intrinsic value that arbitrageurs cannot bid away.

Yet another aspect of near money substitutes, especially in its role as a store of value, are commodities like gold that are often held as a possible hedge against the effects of monetary crisis and inflation on the value of paper monies. Stock certificates of corporations with an accepted limiting earning potential can inherit a comparable component to their valuations.

Generally, we define a stock's monetary value as the benefit derived from its acceptability as a means for meeting one's financial obligations and as a source of liquidity. This benefit is not directly related to a stream of promised regular cashflows, but instead is connected with the stock's cashflows in either the distant future or in times of crisis. The implications of such benefits for asset pricing have also been studied by Bansal and Coleman (1996).
To understand why this monetary value does not represent an arbitrage opportunity, one must critically reexamine the first fundamental theorem of asset pricing (see Dybvig and Ross 1987). This theorem states that the absence of arbitrage is equivalent to stock prices being equal to their expected discounted cashflows (using risk-adjusted probabilities and discounting at the default-free spot rate of interest). The expected discounted cashflows represent the intrinsic value. So, at first blush, no arbitrage appears to imply that stock prices must equal their fundamental value, and there can be no bubbles or monetary value.

A reconciliation of the first fundamental theorem of asset pricing with a stock’s monetary value requires a fine understanding of the phrase “an arbitrage opportunity”. Implicit in the statement of the first fundamental theorem of asset pricing is a very weak notion of arbitrage, so that excluding these weak arbitrage opportunities imposes a strong restriction on the economy (see Harrison and Kreps 1979; Kreps 1981; Harrison and Pliska 1981; Dalang, Morton, and Willinger 1989; Lakner 1993; Delbaen and Schachermeyer 1994; Delbaen 1992, 1997; Huang and Pages 1992). These strong restrictions exclude bubbles and monetary values. Stronger notions of an arbitrage opportunity lead to weaker restrictions on the economy. As shown below, weaker restrictions allow bubbles and monetary values.

It is possible that some market participants view weak arbitrage as real and act on them, e.g., by shorting or writing call options on stocks with large deviations in price relative to fundamental value. Yet, the market may accept only strong arbitrages and, as a consequence, these deviations may not only persist but can widen as a consequence of increased confidence in the assessment of limiting cashflows. Such strategies were recently reported as being responsible for the failure of Tiger Management and one of the Soros funds. Similar problems arise in traditional arbitrages that exclude the impact of market valuations for limiting eventualities or extreme events. A case in point is the failure of Long Term Capital Management (see Liu and Longstaff 2000).

The distinction between these different notions of an arbitrage opportunity only becomes relevant when there are an infinite number of assets trading across time and/or the state space. It could be due to an infinite horizon with the ability to trade a finite number of assets an infinite number of times; or it could be due to an infinite number of assets trading at a single point in time, an example being the trading in options with a continuum of different strikes and maturities. The strongest notion of an arbitrage opportunity involves trading in portfolios consisting of only a finite number of assets, called finite-asset arbitrage opportunities. Weaker notions of arbitrage opportunities involve limiting sequences of finite-asset portfolios that converge to portfolios with an infinite number of assets. An assumed equivalence between the portfolio in the limit and the limiting portfolio distinguishes the various notions of infinite-asset arbitrage opportunities.

Intuitively, infinite-asset arbitrages are executed by a position in finitely many assets that approximates the cashflow of an infinite-asset portfolio. Weak arbitrages occur when the approximation holds only “on average”. For
TABLE 1

<table>
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<tr>
<th>Arbitrage excluded</th>
<th>Strength of economic restriction</th>
<th>Implication for values</th>
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<tbody>
<tr>
<td>Weak infinite-asset arbitrages</td>
<td>Strong condition on economy</td>
<td>No bubbles, no monetary values</td>
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<tr>
<td>Finite-asset arbitrages and some strong infinite-asset arbitrages</td>
<td>Weak condition on economy</td>
<td>Allows bubbles and monetary values</td>
</tr>
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example, for weak arbitrage approximations, it is possible to have a small expected difference between the cashflows but a large discrepancy in some low-probability states. In contrast, strong arbitrages restrict the cashflow differences uniformly across all states. The situation is summarized in Table 1.

For example, a portfolio that pays a dollar except for a catastrophic loss on an event whose probability is approaching zero as the portfolios grows, may be viewed as equivalent to a sure dollar in the limit. If so, then the price of the limiting portfolio must approach the price of a sure dollar. This is what we mean by a weak finite-asset approximation to the limiting portfolio that pays a sure dollar. If economies exclude such weak infinite-asset arbitrages, as in the work of Kreps (1981), Schachermayer (1994), and Delbaen and Schachermayer 1994, then bubbles and monetary values are excluded. Instead, if only finite-asset arbitrages (and a slight relaxation called no Jarrow–Madan arbitrage) are excluded, then bubbles and monetary values can exist.

As option-pricing theory is based on the first fundamental theorem of asset pricing, a relaxation of this theorem has important implications for option pricing and risk management in general. Existing option-pricing theory implicitly excludes the existence of stock's having monetary value. Consequently, an option's value will not reflect this component of the stock's price. If stocks have a monetary value component, then standard option-pricing formulas will be misspecified. This misspecification could, for example, explain the well-known biases in the Black–Scholes option-pricing formula. This misspecification also has implications for implicit estimation procedures that attempt to estimate the stock's risk-neutral density (see Rubinstein 1994; Derman and Kani 1994; Dupire 1994; Ait-Sahalia and Lo 1995). Ignoring a stock's monetary value may generate fat tails in these implicit procedures. These implications are detailed below.

An outline of this paper is as follows. In Section 2, we first provide a simple example of an economy where assets have a long-lasting monetary value. This example can be used to generate the intuition needed to understand the more general theory. Section 3 presents the reexamination of the first fundamental theorem of asset pricing. For pedagogical reasons, this theory is cast in a single-period setting. We illustrate the general theory with an example. Section 4 discusses the implications for option-pricing theory, while Section 5 concludes the paper. All proofs are contained in the Appendix.
2. A SIMPLE EXAMPLE OF MONETARY VALUE

This section provides a simple example of an economy where financial assets can have long-lasting monetary value. The economy is deterministic, with a single representative consumer, but of infinite horizon.

Consider a discrete-time economy with infinitely many time points denoted \( t = 0, 1, \ldots \). Let the economy consist of a single individual endowed with a unit of the consumption good (which we identify with cashflows) in perpetuity. The endowment is \( \omega = (1, 1, \ldots) \). The individual will choose from a collection of cashflows denoted by \( C = \{ c = (c_0, c_1, \ldots) \} \); each \( c_t \) is uniformly bounded. The agent's utility function over consumption is given by

\[
U(c) = \sum_{t=0}^{\infty} \beta^t \ln c_t + \ln T[c],
\]

(1)

where \( T[c] = \lim_{t \to \infty} c_t \) if the limit exists (otherwise \( T[c] \) is the limit of an appropriately chosen subsequence of \( c \) that defines \( T \) on all of \( C \), called a Banach limit).\(^2\)

The investor is seen to have the standard logarithmic utility function with time-preference rate \( \beta \) (\( 0 < \beta < 1 \)), except that he also places a value on the limiting cashflow represented by the last term in expression (1). The time-preference rate reflects the individual's personalized discount rate for future cashflows. This last term represents a personalized value for the permanence of the cashflows.

The individual's preferences can be understood by comparing two distinct consumption bundles. The first is his endowment \( \omega \). The second is the consumption bundle that has cashflows of one unit up to time \( n \) and zero units thereafter, i.e., \( c^n = (1, 1, \ldots, 1, 0, 0, \ldots) \). The second consumption bundle converges to the endowment as \( n \) gets large, i.e.,

\[
\lim_{n \to \infty} c^n = \omega.
\]

But, the representative investor does not see them as equivalent, even in the limit \( n \to \infty \), since

\[
U(c^n) = \sum_{t=0}^{n} \beta^t \ln(1) + \sum_{t=n+1}^{\infty} \beta^t \ln(0) + \ln(0) = -\infty,
\]

and

\[
U(\omega) = \sum_{t=0}^{\infty} \beta^t \ln(1) + \ln(1) = 0
\]

because\(^3\)

\[ T[c^n] = \lim_{t \to \infty} c^n = 0 \quad \text{and} \quad T[\omega] = \lim_{t \to \infty} \omega = 1. \]

\(^2\) For a definition of a Banach limit, see Bhaskara Rao and Bhaskara Rao (1983, pp. 39-43).

\(^3\) Note that the \( T[c] \) operator involves the limit as \( t \to \infty \) for a fixed \( n \) in \( c^n \) and for \( \omega \).

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The investor sees the approximating consumption bundle as having infinite disutility, even in the limit as \( n \to \infty \), whereas the initial endowment has zero utility.

Given a representative individual, equilibrium for this economy is determined by the condition that the individual's utility-maximizing consumption bundle equals his endowment. It is easy to show (see the Appendix) that the equilibrium price \( p[c] \) for the consumption bundle \( c \in C \) is

\[
p[c] = \sum_{t=0}^{\infty} \beta^t c_t + T[c].
\]  

(2)

This equilibrium price has two components. The first is represented by the sum of the discounted cashflows. This is the intrinsic value. The second is represented by the limit of the cashflow. This is a permanent and nonbursting monetary value. It exists because the investor's preferences assign a value to cashflows in perpetuity.

At the equilibrium prices, there are no arbitrage opportunities in the economy. This is because the existence of an arbitrage opportunity would imply that the investor was not at his optimum consumption position.

Let us draw a distinction that will be useful later, between finite-element consumption bundles (such as \( c^n \)) and infinite-element consumption bundles (such as \( \omega \)). We saw earlier that, although \( \lim_{n \to \infty} c^n = \omega \), the individual does not view these bundles as equivalent, even in the limit. This is also reflected in the equilibrium prices, since

\[
p[c^n] = \frac{1 - \beta^n}{1 - \beta}, \quad \text{so} \quad \lim_{n \to \infty} p[c^n] = \frac{1}{1 - \beta}, \quad \text{but} \quad p[\omega] = \frac{1}{1 - \beta} + 1.
\]

If the individual had viewed both consumption bundles as equal in the limit, then the fact that the limiting prices are different would imply the existence of an infinite-asset arbitrage opportunity. But, the individual does not view both consumption bundles as equal in the limit, so there are none.

Although simple, this example illustrates all of the issues raised in the introduction: that stock certificates can have a permanent monetary value; that stock prices can deviate from intrinsic value; and that there can be no arbitrage opportunities in such equilibrium. The key insight is that the valuation of the limit is divorced from the timing of the receipt of the cash values. As long as investors foresee positive cashflows in the distant future from the activities of corporate entities, they may value the equity substantially, even though current earnings are zero or negative for possibly extended periods of time (Frankel and Lee 1996). It may be argued that the current stock market is in such a phase with high price-to-earnings and price-to-dividend ratios. Preference orderings of the type described by expression (1) cannot be ruled out by assumption.
3. THE FIRST FUNDAMENTAL THEOREM OF ASSET PRICING REVISITED

Having illustrated an equilibrium economy where stock prices can have monetary value, this section now moves on to a detailed investigation of the relationship between no arbitrage opportunities and intrinsic values. The purpose of this section is to clarify the first fundamental theorem of asset pricing and to show that no arbitrage is, in fact, consistent with the existence of monetary values. The reconciliation of the conventional wisdom regarding arbitrage opportunities and monetary values is through understanding the subtle meaning of the phrase “no infinite-asset arbitrage opportunities”. Imposing a weak definition of no infinite-asset arbitrage opportunities implies strong restrictions on the economy and excludes monetary values. In contrast, imposing a strong definition of no infinite-asset arbitrage opportunities implies weak restrictions on the economy and allows monetary values. The imposition of these definitions is not a choice of the model builder, but an intrinsic characteristic of investor preferences in an economy, i.e., whether or not investors see two portfolios as equivalent in some limiting sense. These are the topics we now explore.

For stock certificates to have monetary values, it is essential to have an infinite number of assets trading either (i) by trading in a finite number of assets over an infinite number of trading dates (as in the previous section), or (ii) by trading in an infinite number of assets at a single point in time due to an infinite state space (e.g., stock options with a continuum of strikes). The necessity of an infinite number of assets trading is seen by applying the finite-state, finite-time version of the fundamental theorem of asset pricing, as in the work of Harrison and Kreps (1979). The strongest notion of no arbitrage possible is that there are no finite-asset trading strategies that generate arbitrage opportunities (without the consideration of limits of portfolios). In the finite-state, finite-time models, there are no other types of trading strategies. And, for finite-state, finite-time models, no finite-asset arbitrage opportunities is equivalent to the existence of an equivalent martingale measure that computes values as an expectation over discounted cashflows. This is an intrinsic-value condition. So, for the finite-state, finite-time models, no arbitrage excludes bubbles and monetary value.4

To revisit this theorem, we choose the simpler setting—a single-period model with trading in an infinite number of assets. The insights we obtain generalize to either continuous-trading or discrete-time infinite-horizon models with only a finite number of assets trading at any date.

We consider an economy with trading at date 0 and with all uncertainty resolved at time 1. The uncertainty in the economy is characterized by the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the elements $\omega \in \Omega$ referred to as states. Traded in this economy are assets from the infinite set $I$ indexed by $i \in I$.

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4 The case of finite time and infinitely many states but only a finite number of assets trading is studied by Dalang, Morton, and Willener (1989), where it is shown that no arbitrage is equivalent to the existence of an equivalent martingale measure, and hence to no bubbles.
An asset is represented by a random cashflow (an \( F \)-measurable real-valued function) generated at time 1 and denoted by \( D_i(\omega) \). Each asset has a market price \( \pi_i \) determined by trading at time 0.

For the aggregate economy, each asset is held in either positive or zero net supply, with \( n_i \geq 0 \) representing the aggregate supply of shares for asset \( i \). For derivative assets in zero net supply, \( n_i = 0 \). For other assets in positive net supply, cashflows must be nonnegative, i.e., \( D_i(\omega) \geq 0 \) for all \( \omega \in \Omega \). The aggregate wealth in the economy at time 1 is

\[
V(\omega) = \sum_{i \in I} n_i D_i(\omega). 
\]

The sum in expression (3) is well defined only if a finite number of the traded assets are in positive supply, a condition we now assume.

Finally, we let the riskless asset trade with a one-period risk-free rate denoted by \( r \). This is the standard set-up for discussing the first fundamental theorem of asset pricing.

The first step in defining an arbitrage opportunity is to define the set of random variables accessible via finite-asset portfolios. Let\(^5\)

\[
C = \left\{ c(\omega) = \sum_{i \in I} a_i D_i(\omega) : \text{all but finitely many } a_i \text{'s are } 0 \text{ and } \frac{c(\omega)}{V(\omega)} \text{ is bounded with probability } 1 \right\}.
\]

The set \( C \) represents those time-1 payoffs that can be generated by a finite portfolio in the traded assets. These represent the finite-asset trading strategies. The payoffs of these portfolios are bounded relative to the aggregate value of all assets traded in the economy.

Two subsets of the set \( C \) are useful. The first set is those finite-asset portfolios that can be constructed with zero initial investment,\(^6\) i.e.,

\[
C_0 = \left\{ c \in C : \sum_{i \in I} a_i \pi_i = 0 \right\}.
\]

The second set is those finite-asset portfolios that have nonnegative payoffs for sure and strictly positive payoffs with positive probability, i.e.,

\[
C^+ = \left\{ c \in C : \mathbb{P}(\omega \mid c(\omega) \geq 0) = 1 \text{ and } \mathbb{P}(\omega \mid c(\omega) > 0) > 0 \right\}.
\]

A finite-asset arbitrage opportunity is a zero-investment trading strategy that has no risk of a loss and some chance of earning positive profits, i.e., it is an element \( c \in C_0 \cap C^+ \). This is the strongest notion of an arbitrage opportunity that can be constructed because it does not involve approximating one random variable.

\(^5\) In symbols, \( u(\omega) = c(\omega)/V(\omega) \in L_{\infty}(\Omega, \mathcal{F}, \mathbb{P}) \), where \( L_{\infty} \) represents the essentially bounded functions on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with the essential sup norm defined by \( \| u \|_{\infty} = \operatorname{esssup}_{\omega \in \Omega} |u(\omega)| > 0 \).

\(^6\) Clearly \( C_0 \) is a linear subspace of \( C \).
with a sequence (or net) of alternative random variables. A finite-asset arbitrage opportunity represents a trading strategy that can be constructed exactly for all possible states. We say that the economy has no finite-asset arbitrage opportunities (NA) if \( C_0 \cap C^+ = \emptyset \).

To clarify the traditional first fundamental theorem of asset pricing, we also need to introduce some weaker notions of arbitrage opportunities involving sequences of finite-asset trading strategies. We introduce these alternative definitions from strongest to weakest.

Two random variables \( c_1(\omega) \) and \( c_2(\omega) \) can be viewed as approximately equal if, for a small \( \varepsilon > 0 \), \( |c_1(\omega) - c_2(\omega)| < \varepsilon V(\omega) \) for all \( \omega \in \Omega \) except perhaps on a set of probability 0. This corresponds to a “uniform closeness” of the payoffs across (almost) all possible states.\(^7\) We utilize this notion of approximately equal in the subsequent notions of infinite-asset arbitrage opportunities.

Intuitively, a Jarrow–Madan infinite-asset arbitrage opportunity is a sequence of zero-investment trading strategies that, in the limit, includes within its payoffs a finite-asset arbitrage opportunity with substantial payoffs across all states. Formally, a Jarrow–Madan infinite-asset arbitrage opportunity is a sequence of zero-investment trading strategies from which one can subtract a nonnegative payoff across all states and a positive payoff in some states (of positive probability), averaging a unit, and which converges (in the sense of uniform closeness) to a portfolio with zero values in all states, i.e., \( c_n \in C_0 - G \to 0 \) as \( n \to \infty \) where \( G = \{ c \in C^+ : \int_\Omega c(\omega)/V(\omega) d\mathbb{P}(d\omega) = 1 \} \). We say that the economy has no Jarrow–Madan arbitrage opportunities (NJMA) if \( 0 \notin \text{cl}(C_0 - G) \), where \( \text{cl} \) represents the closure in the sense of uniform closeness. The requirement that the imbedded finite-asset arbitrage opportunity has an average unit payoff makes this a strong notion of an infinite-asset arbitrage opportunity. A weaker notion removes this unit payoff requirement.

A no free lunch with vanishing risk infinite-asset arbitrage opportunity is a sequence of zero-investment trading strategies from which one can subtract a nonnegative payoff across all states and a positive payoff in some states (of positive probability) and which converges (in the sense of uniform closeness) to an arbitrage opportunity in the limit, i.e., \( c_n \in C_0 - C^+ \to c_\text{\infty} \in C_0 \cap C^+ \) as \( n \to \infty \). We say that the economy has no free lunch with vanishing risk arbitrage opportunities (NFLVR) if \( \text{cl}(C_0 - C^+) \cap C^+ = \emptyset \), where the closure is in the sense of uniform closeness.

Finally, an equivalent martingale measure is a probability measure \( \mathbb{Q} \) defined on \( (\Omega, \mathcal{F}) \) such that \( \mathbb{Q}(A) = 0 \) if and only if \( \mathbb{P}(A) = 0 \) for all \( A \in \mathcal{F} \) and

\[
\pi[c] = \frac{1}{1+r} \int_\Omega c(\omega) \mathbb{Q}(d\omega) \quad \text{for all } c \in C. \tag{4}
\]

The “equivalent” condition is that the probability measure \( \mathbb{Q} \) agrees with \( \mathbb{P} \) on zero probability events. The martingale condition is expression (4). Expression (4)\(^7\) This involves convergence in the sup norm or \( L_\infty \) norm.

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is also an intrinsic-value formula where expected future cashflows using risk-adjusted probabilities are discounted to the present.

The traditional fundamental theorem of asset pricing is given by Delbaen and Schachermayer (1994). It uses a weak notion of an infinite-asset arbitrage opportunity.

**Traditional Fundamental Theorem of Asset Pricing** Assume that there are no finite-asset (NA) arbitrage opportunities nor any free lunch with vanishing risk (NFLVR) arbitrage opportunities. Then there exists an equivalent martingale probability measure.

This theorem states that NFLVR implies the existence of an equivalent martingale probability measure, and hence that prices must equal intrinsic values. That is, monetary values are inconsistent with this weak notion of no infinite-asset arbitrage opportunities.\(^8\)

A generalization of the fundamental theorem of asset pricing uses the stronger notions of arbitrage opportunities.

**Generalized Fundamental Theorem of Asset Pricing** No finite-asset (NA) arbitrage opportunities holds if and only if the price operator \( \pi \) can be decomposed as

\[
\pi[c] = \frac{\varphi}{1+r} \int_\Omega c(\omega) q(\omega) \mathbb{P}(d\omega) + \frac{1-\varphi}{1+r} \int_\Omega c(\omega) \lambda(d\omega) \quad \text{for all } c \in C,
\]

where \( 0 \leq \varphi \leq 1 \), \( q(\omega) \) is a risk-neutral density defining an equivalent risk-neutral measure \( Q \) by \( dQ/d\mathbb{P} = q \), and \( \lambda \) is a finitely additive measure on \( (\Omega, \mathcal{F}) \).

Assuming that there are no Jarrow–Madan arbitrage opportunities, then \( \pi[1_A] > 0 \) for \( A \in \mathcal{F} \) if and only if \( \mathbb{P}(A) > 0 \), where \( 1_A \) is the indicator function of the set \( A \).

The proof is in the Appendix.

This theorem shows that the stronger definition of no finite arbitrage opportunities allows for the existence of a price operator that is decomposable into an intrinsic-value component (the first term on the right-hand side of (5)) and a monetary-value component (the second term on the right-hand side of (5)). This monetary-value component is long lasting and is due to investors’ preferences (or lack thereof) concerning the similarity of two random variables. If investors do not view NFLVR arbitrage opportunities as “real” arbitrage opportunities, then stocks can have a monetary-value component. If they do, then stocks cannot have a monetary value. This is an empirical question, not discernible by assumption.

The removal of Jarrow–Madan arbitrage opportunities guarantees only that the pricing operator is equivalent to the probability \( \mathbb{P} \) in the sense that Arrow–Debreu securities, paying a dollar when event \( A \) occurs, have positive prices if

\(^8\) Delbaen and Schachermayer (1994) provide an example showing that NFLVR cannot be eliminated.

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and only if the event $A$ occurs with strictly positive probability. This is a mild additional restriction. It does not alter the existence or otherwise of a monetary-value component to the stock’s price.

Because this theorem is so abstract, it is useful to consider an example of such a pricing operator.

**Example: Market Value for Unlimited Loss Exposure**

This example provides a pricing operator satisfying expression (5), where the monetary value can be interpreted as payment for the avoidance of unlimited losses.

Let the state space $\Omega = (0, 1)$ be the unit interval. We let the true probability measure be any probability distribution over $(0, 1)$ that assigns positive probability to any open subset of $(0, 1)$. We let $q(\omega) = 1$, the uniform risk-neutral density function. We also let $\lambda(A) = \lim_{\omega \to 0} 1_A(\omega)$, a finitely additive measure that is not countably additive.

Given the above, the price operator is given by

$$
\pi(c) = \frac{1}{2(1 + r)} \int_0^1 c(\omega) \, d\omega + \frac{T[c]}{2(1 + r)},
$$

where $T[c] = \lim_{\omega \to 0} c(\omega)$ when this limit exists (otherwise it is the Banach limit).

For this example, the price has two components and the second term is the monetary value. To understand the economic interpretation of this monetary value, it helps to evaluate a particular cashflow. Consider the digital put with strike $K$ and a time-1 payoff given by

$$
d_K(\omega) = \begin{cases} 
1 & \text{if } \omega \leq K, \\
0 & \text{otherwise}.
\end{cases}
$$

The digital put provides protection for losses when the state $\omega$ is close to 0.

The price of the digital put is

$$
\pi(d_K) = \frac{1}{2(1 + r)} \int_0^K d\omega + \frac{1}{2(1 + r)} \lim_{\omega \to 0} 1_{\omega \leq K} = \frac{K + 1}{2(1 + r)} > 0.
$$

As the strike price approaches zero, the digital put’s payoff approaches 0 pointwise, i.e., $\lim_{K \to 0} d_K(\omega) = 0$ for all $\omega \in (0, 1)$. But in the “uniform closeness” measure, the distance between the digital put and the zero payout is always 1, because $|d_K(\omega) - 0| = 1$ for all $\omega \in (0, K]$. So, the digital put with a small strike price is not viewed as a good approximation to a zero payout when individuals possess the stronger preferences.

Prices in this example are consistent with preferences exhibiting this view of uniform closeness. Indeed, the limit of the digital put’s value is strictly positive:

$$
\lim_{K \to 0} \pi(d_K) = \frac{1}{2(1 + r)} > 0.
$$

If preferences were otherwise, the limiting value of this digital put’s price would
be zero. Thus, in this example, the monetary value has an interpretation of being
the market value for avoiding unlimited losses. This completes the example.

4. MONETARY VALUES AND OPTION PRICING

This section discusses the relevance of monetary values for option-pricing
theory. Current option-pricing theory is derived under the standard hypotheses
of the traditional first fundamental theory of asset pricing, and, as such, current
option-pricing theory excludes monetary values by fiat. Nonetheless, the
standard hypothesis may not be satisfied in the actual markets, and investors
may not view the weaker notion of infinite-asset arbitrage opportunities as
“real” arbitrage opportunities. If this is the case, then the existing option-pricing
models are misspecified.

For example, an application of the generalized first fundamental theorem of
asset prices, where the risk-neutral density is a lognormal distribution (the
Black–Scholes formula), gives the call’s value on a stock $S_t(\omega)$ with strike price
$K$ and maturity $T$ as

$$\pi(\text{call}) = \varphi(\text{Black–Scholes}) + \frac{1 - \varphi}{1 + r} \int_{\Omega} \max(S_T(\omega) - K, 0) \lambda(d\omega)$$

for some finitely additive measure $\lambda$, where Black–Scholes denotes the Black–
Scholes formula.

This monetary component value biases the call option price away from the
Black–Scholes value. It is possible that the smile effects or moneyness biases
observed in Black–Scholes option pricing may be partially explained by such
monetary values. The resolution of this conjecture, however, is left for future
research.

Recent papers on option pricing attempt to infer the risk-neutral density
from the market prices of traded options (see Rubinstein 1994; Derman and
Kani 1994; Dupire 1994; Aït-Sahalia and Lo 1995; Abken, Madan, and
Ramamurtie 1996). All of these papers assume the existence of the risk-neutral
density, and hence exclude a monetary-value component. Yet, as shown
previously, investor preferences may not satisfy the necessary hypothesis.
Assets may have value as private money. These monetary valuations could
generate fat tails in the estimated risk-neutral densities to accommodate the
misspecified model. The source of the fat tails may be a positive value placed
on puts and calls due to payoffs associated with limiting or extreme downward
or upward price movements.

5. CONCLUSIONS

This paper reevaluates, in the context of current market events, the meaning of
the first fundamental theorem of asset pricing. It is well known that the existence
of an equivalent martingale measure implies the absence of arbitrage opportunities, but the implication does not go in the reverse direction. For this direction, one needs to exclude weak notions of infinite-asset arbitrage opportunities. The market, however, may not view these weak notions of infinite-asset arbitrage opportunities as "real". In this case, stock prices (and, more generally, market prices) may exhibit a deviation from intrinsic value that is long lasting—a monetary value. This monetary value cannot be arbitrated away and it can explain the high price-to-earnings ratios existing in current markets. This monetary value may also contribute to the well-known biases present in the Black–Scholes formula.

One remaining question to be considered is why monetary values appear to expand and contract through time. Under our structure, such monetary values are based on assessments of limiting cashflows that, in mathematical terms, typically amount to extracting a limit from a selected subsequence. The limit of the subsequence selected can fluctuate between the liminf and the limsup of the original sequence of cashflows. Translated back into economic terms, the monetary values generated by this process can fluctuate considerably within these bounds as a result of changes in consumer confidence in the economy and stocks' future prospects. These fluctuations in consumer confidence may explain the onset and collapsing of monetary values through time.

APPENDIX

**Determination of equilibrium price in expression (2)**

The economy is in equilibrium if the investor's initial endowment represents his optimal consumption bundle. It follows that, for all \( c \in C \), the utility \( f(x) \) of buying \( x \) units of \( c \) at its market price \( p[c] \) is

\[
f(x) = U(\omega + xc - xp[c]e_0),
\]

where \( e_0 = (1, 0, 0, \ldots) \) is a sequence whose first entry is unity and whose remaining entries are zeros. For \( \omega \) to be optimal, the function \( f(x) \) has a maximum at \( x = 0 \) for all \( c \). It follows that

\[
\nabla U(\omega)(c - p[c]e_0) = 0,
\]

where \( \nabla U(\omega) \), a linear functional on \( C \), is the gradient of \( U \) evaluated at \( \omega \).

Solving this equation for the price functional gives

\[
p[c] = \frac{\nabla U(\omega)[c]}{\nabla U(\omega)[e_0]}.
\]

The gradient of \( U \) at \( \omega \) is defined on \( h \in C \) by the condition

\[
\lim_{x \to 0} \left| \frac{U(c + xh) - U(c)}{x} - \nabla U(c)[h] \right| = 0.
\]
Evaluating this limit for expression (1) gives

$$\nabla U(c)[h] = \sum_{i=0}^{\infty} \beta^i \frac{1}{c_i} h_i + \frac{1}{T[c]} T[h].$$

So, $\nabla U(\omega)[e_0] = 1$ and $\nabla U(\omega)[c] = \sum_{i=0}^{\infty} \beta^i c_i + T[c]$. Substitution gives expression (2).

**Proof of the generalized fundamental theorem of asset pricing**

The space $C$ is a Banach space. Under NA, the linear subspace $C_0$ does not intersect the interior of the positive orthant of $C$, a nonempty open convex subset of $C$. It follows from the Hahn–Banach theorem (see, e.g., Horvath 1966, Theorem 2, p. 177) that there exists a hyperplane $H$ containing $C_0$ that has the interior of $C^+$ lying above it. Further, it is shown below that, under NJMA, there is an open convex set $W$ containing $C^+$ that does not meet $C_0$ and, in this case, $W$ lies above $H$. Since $0 \in C_0$, the hyperplane is, in fact, a maximal proper linear subspace.

The hyperplane $H$ is therefore defined as the linear subspace on which a continuous linear functional on $C$ is null. Hence, for some continuous linear functional $\Gamma$ on $C$,

$$H = \{c \in C : \Gamma(c) = 0\}. \quad \text{(A.1)}$$

The map $\phi : C \to L_{\infty}([\Omega, \mathcal{F}, \mathbb{P}])$, defined by $c \mapsto c/V$, is a topological isomorphism. It follows that $\Gamma$ induces a continuous linear map $\Psi$ on $L_{\infty}([\Omega, \mathcal{F}, \mathbb{P}])$ defined by

$$\Psi(y) = \Gamma(\phi^{-1}(y)) = \Gamma(yV). \quad \text{(A.2)}$$

The dual space of $y \in L_{\infty}([\Omega, \mathcal{F}, \mathbb{P}])$ is $ba([\Omega, \mathcal{F}, \mathbb{P}])$ (see Dunford and Schwartz 1988, Chap. IV, §8.14, Theorem 16, p. 296), the space of bounded finitely additive measures on $\mathcal{F}$ that vanish on sets of $\mathbb{P}$ measure zero. It follows that there exists a finitely additive measure $\rho$ on $\mathcal{F}$ such that

$$\Psi(y) = \int_{\Omega} y(\omega) \rho(d\omega). \quad \text{(A.3)}$$

Hence, the hyperplane $H$ may now be defined as the set of $c \in C$ such that

$$\Gamma(c) = \int_{\Omega} \frac{c(\omega)}{V(\omega)} \rho(d\omega) = 0. \quad \text{(A.4)}$$

Note further that, under NJMA, $1_A V \in C^+$ for all $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$, and hence it is in the open convex set $W$. It follows that

$$\Gamma(1_A V) = \int_{\Omega} 1_A \rho(d\omega) = \rho(A) > 0. \quad \text{(A.5)}$$

Therefore, under NJMA, the measures $\mathbb{P}$ and $\rho$ are equivalent.
By the Yosida–Hewitt theorem on the decomposition of bounded additive measures (Bhaskara Rao and Bhaskara Rao 1983, Theorem 10.2.1, p. 241), the finitely additive measure \( \rho \) has a unique decomposition as

\[
\rho = \rho_c + \rho_p,
\]

where \( \rho_c \) is a countably additive measure and \( \rho_p \) is a pure charge, in that there is no countably additive measure \( \nu \) on \( \mathcal{F} \) such that \( \nu \leq \rho_p \) on all of \( \mathcal{F} \). Heuristically, \( \rho_c \) is the largest countably additive measure that is dominated by \( \rho \). We observe from the proof provided by Bhaskara Rao and Bhaskara Rao (1983), that, when \( \rho \) is nonnegative, then so are both the components \( \rho_c \) and \( \rho_p \). By the Radon–Nikodym theorem (Dunford and Schwartz 1988, Chap. III, §10.2, Theorem 2, p. 176), there exists \( f \in L_1(\Omega, \mathcal{F}, \mathbb{P})^+ \) such that, for all \( A \in \mathcal{F} \),

\[
\rho_c(A) = \int_A f(\omega) \, d\mathbb{P}(d\omega).
\]

The risk-neutral density will be defined in terms of \( f \), but asset prices will be seen to include valuation components under \( \rho_p \) as well.

Consider the strategy that shorts the risk-free asset with payoff \( I_\Omega \) in \( C \) and time-0 price \( (1 + r)^{-1} \) in sufficient quantity to finance the purchase of the asset with price \( \pi_i \) and time-1 payoff \( D_i(\omega) \). We short \( (1 + r)^{-1} \pi_i \) units of the risk-free asset for the purpose of obtaining one unit of the \( i \)th asset. The payoff to this zero-cost portfolio is

\[
c(\omega) = D_i(\omega) - (1 + r)\pi_i \cdot I_\Omega.
\]

By construction, this payoff belongs to \( C_0 \), and hence to \( H \), and so \( \Gamma(c) = 0 \). Note also that, as \( I_\Omega \) is in the interior of \( C^+ \), we have \( \Gamma(I_\Omega) > 0 \). It follows that

\[
\pi_i = \frac{1}{1 + r} \frac{\Gamma(D_i)}{\Gamma(I_\Omega)}.
\]

By equation (A.4), we then see that we must have

\[
\pi_i = \frac{1}{1 + r} \frac{1}{\alpha V(\omega)} \int_\Omega D_i(\omega) \rho(\omega) d\omega,
\]

where \( \alpha = \Gamma(I_\Omega) = \int_\Omega [1/V(\omega)] \rho(\omega) d\omega \).

Now we employ the decomposition of \( \rho \) given by (A.6) and, assuming that both components are positive, i.e.,

\[
\theta = \int_\Omega [1/V(\omega)] \rho_c(\omega) d\omega > 0 \quad \text{and} \quad \eta = \int_\Omega [1/V(\omega)] \rho_p(\omega) d\omega > 0,
\]

we may write

\[
\pi_i = \frac{1}{1 + r} \left[ \frac{\theta}{\alpha} \int_\Omega \frac{D_i(\omega)}{\theta V(\omega)} \rho_c(\omega) d\omega + \frac{\eta}{\alpha} \int_\Omega \frac{D_i(\omega)}{\eta V(\omega)} \rho_p(\omega) d\omega \right].
\]
Define the density of \( \mathcal{Q} \) with respect to \( \mathcal{P} \) using the density \( f \) of \( \rho_c \) with respect to \( \mathcal{P} \) by
\[
q(\omega) = \frac{1}{\theta} \frac{f(\omega)}{V(\omega)}.
\] (A.10)

We may ensure the positivity of the measure induced by \( q \), if the probability measure \( \mathcal{P} \) is defined on a Borel \( \sigma \)-field, as the charge component \( \rho_p \) is identically zero for \( \rho \) absolutely continuous with respect to the countably additive measure \( \mathcal{P} \) on bounded Borel sets (see Bhaskara Rao and Bhaskara Rao 1983, p. 245).

The asset price \( \pi_t \) can now be written
\[
\pi_t = \frac{\theta}{\alpha} \int_{\Omega} D_s(\omega) q(\omega) \mathcal{P}(d\omega) + \frac{\eta}{\alpha} \frac{B}{1 + r},
\] (A.11)

where the monetary-value component \( B \) is
\[
B = \int_{\Omega} D_s(\omega) \lambda(d\omega)
\] (A.12)

and the finitely additive measure \( \lambda \) is defined by
\[
\lambda(d\omega) = \frac{1}{\eta V(\omega)} \rho_p(d\omega).
\]

The result (A.11) extends to other traded cashflows; the linearity of \( \Gamma \) and (5) follows for all traded \( c(\omega) \) with \( \psi = \theta/\alpha \). □

**Proof of the existence of \( \mathcal{W} \)**

By the NJMA hypothesis, as \( 0 \notin \overline{C_0 - G} \),
\[
\varepsilon = \inf \{ \| u - w \|_\infty : u = c/V, w = g/V \text{ for } c \in C_0, g \in G \} > 0.
\] (A.13)

Now let \( U \) be the open convex set defined by
\[
U = \{ u \in L_\infty([\Omega, \mathcal{F}, \mathcal{P}]) : d(u, G) < \varepsilon \},
\] (A.14)

where \( d(u, G) = \inf_{g \in G} \{ \| u - g/V \|_\infty \} \). The convexity of \( U \) follows on noting that, if \( u_1 \) and \( u_2 \) are in \( U \), then there exist \( g_1 \) and \( g_2 \) in \( G \) such that \( \| u_1 - g_1/V \|_\infty \) and \( \| u_2 - g_2/V \|_\infty \) are both less than \( \varepsilon \). For \( 0 < \lambda < 1 \), consider \( g = \lambda g_1 + (1 - \lambda)g_2 \) and note, from the convexity of \( G \), that \( g \in G \). It follows that
\[
d(\lambda u_1 + (1 - \lambda)u_2, G) \leq \| \lambda u_1 + (1 - \lambda)u_2 - g/V \|_\infty \\
\leq \lambda \| u_1 - g_1/V \|_\infty + (1 - \lambda)\| u_2 - g_2/V \|_\infty \\
\leq \varepsilon.
\]

Hence \( U \) is convex. Let \( \mathcal{W}' \) be the open convex cone generated by all positive scalar multiples of elements of \( U \). Now, \( \phi(G) \) is contained in \( U \) and hence in \( \mathcal{W}' \). Let \( \mathcal{W} \) be \( \phi^{-1}(\mathcal{W}') \). For any set \( A \) of positive probability, the function
\[ \frac{1}{P(A)} I_A V \text{ is an element of } G, \text{ and hence } \frac{1}{P(A)} I_A \in U \subseteq W'. \text{ It follows that } C^+ \subseteq W = \phi^{-1}(W'). \]

To see that \( W \) fails to meet \( \overline{C_0} \), suppose that it did and \( c \in W \cap \overline{C_0} \). Let \( u = c/V \) and we have \( u = \phi(c) \in W' \). Hence, for some positive scalar \( \theta \), \( u = \theta \tilde{u} \) for \( \tilde{u} \in U \). Since \( \overline{C_0} \) is a subspace, \( \tilde{u} V \in \overline{C_0} \) and \( \tilde{u} \) is within \( \varepsilon \) of \( \phi(G) \) by virtue of being in \( U \). This contradicts the definition of \( \varepsilon \) in (A.13). \( \square \)

REFERENCES


