The market for default swaps, as measured by the notional amounts of contracts traded per year, has grown from about $50 billion in 1998 to over $400 billion in 2000 (see “Credit Risk” [2000]). This exponential growth has generated significant interest in the fair valuation of default swaps in both the academic and practitioner communities.

The literature investigating the valuation of default swaps gives the impression that simple models for pricing default swaps are available only when credit and market risk are statistically independent (see Hull and White [2000, 2001], Martin, Thompson, and Browne [2000], Wei [2001], and the survey article by Cheng [2001]). Indeed, it is commonly believed that models incorporating correlated market and credit risk are quite complex, requiring burdensome recursive numerical procedures.

For example, from Hull and White:

Like most other approaches, ours assumes that default probabilities, interest rates, and recovery rates are mutually independent. Unfortunately, it does not seem to be possible to relax these assumptions without a considerably more complex model [2000, p. 30].

We provide on the contrary a simple analytic formula for the valuation of default swaps when market and credit risk are correlated. This formula is easy to understand and to compute. It is derived in the context of a reduced-form credit risk model where correlated defaults arise because a firm’s default intensities depend on common macroeconomic factors (see Jarrow [2001]). The common macro factor we use is the spot rate of interest, assumed to follow an extended Vasicek model in the Heath, Jarrow, and Morton [1992] framework.

We illustrate the numerical implementation of this model by deducing the default probability parameters implicit in the term structure of default swap quotes for 22 different companies over the time period August 21–October 31, 2000. The data used for this investigation come from Enron’s web site. The 22 different firms are chosen to stratify various industry groupings: financial, food and beverages, petroleum, airlines, utilities, department stores, and technology.

For comparison purposes, the standard model with statistically independent market and credit risk (a special case of our model) is also calibrated to these market data. One can also easily calibrate our simple model to exactly match the observed default swap quote term structure in the case of correlated market and credit risk.

I. MODEL STRUCTURE

The reduced-form credit risk model in Jarrow [2001] is the basis for valuing default swaps. Trading can take place any time during
the interval \([0, \bar{T}]\). Traded are default-free zero-coupon bonds and risky (defaultable) zero-coupon bonds of all maturities. Markets are assumed to be complete and frictionless, with no arbitrage opportunities.

Let \(p(t, T)\) represent the time \(t\) price of a default-free dollar paid at time \(T\) where \(0 \leq t \leq T \leq \bar{T}\). The instantaneous forward rate at time \(t\) for date \(T\) is defined by \(f(t, T) = -\partial \log p(t, T)/\partial T\). The spot rate of interest is given by \(r(t) = f(t, t)\).

Consider a firm issuing risky debt. Let \(v(t, T)\) represent the time \(t\) price of a promised dollar to be paid by this firm at time \(T\) where \(0 \leq t \leq T \leq \bar{T}\). The debt is risky because if the firm defaults before time \(T\), the promised dollar may not be paid. Let \(\tau\) denote the first time that this firm defaults (\(\tau > \bar{T}\) is possible if the firm does not default). The default time is a random variable.

We let

\[
N(t) = 1_{\{\tau \leq t\}} = \begin{cases} 
1 & \text{if } \tau \leq t \\
0 & \text{otherwise} 
\end{cases}  
\tag{1}
\]

denote the point process indicating whether or not default has occurred prior to time \(t\). It is assumed that the point process has a random intensity \(\lambda(t)\), where \(\lambda(t)\Delta\) gives the approximate probability of default for this firm over the time interval \([t, t+\Delta]\). The intensity process is defined under the risk-neutral probability.

If default occurs, we let the zero-coupon bond receive a fractional recovery of \(\delta(\tau)v(\tau - T)\) dollars, where \(0 \leq \delta(\tau)\), and \(\tau\) represents an instant before default. In this recovery rate structure, the debt is worth only a fraction of its pre-default value.

Under the assumption of no arbitrage and complete markets, standard arbitrage pricing theory implies that there is a unique equivalent probability \(Q\) such that the present values of the zero-coupon bonds are computed by discounting at the spot rate of interest and then taking an expectation with respect to \(Q\).

That is:

\[
p(t, T) = E_Q e^{-\int_{\tau}^{T} f(u)du} 
\tag{2}
\]

and

\[
v(t, T; \delta) = E_Q \left( \delta(\tau)v(\tau - T)e^{-\int_{\tau}^{T} f(u)du} I(\tau \leq T) + \int_{\tau}^{T} e^{\int_{t}^{u} f(\tau) - \lambda(\tau)du} \right) 
\tag{3}
\]

where \(E_Q(\cdot)\) is the conditional expectation with respect to \(Q\) at time \(t\). The risky debt value is composed of two parts. The first part is the present value of the promised payment in default. The second part is the present value of the promised payment if default does not occur.

Credit derivatives are priced using the approach of Lando [1998], assuming a perfectly liquid market. We assume that the point process is modeled as a Cox process with an intensity function \(\lambda(t, X_t)\) where \(\{X_t: t \in [0, T]\}\) is a vector stochastic process representing the state variables underlying the evolution of the economy. These macroeconomic state variables induce the correlation between market and default risk. For example, if \(X_t\) quantifies market risk, and \(\lambda(t, X_t)\) increases as \(X_t\) increases, then as market risk increases, the likelihood of the firm defaulting increases as well.

The cash flows from most credit derivatives can be of three different types:

- The first is a random payment \(Y_T\) at time \(T\), but only if there is no default prior to time \(T\).
- The second is a random payment rate of \(y\,dt\) at time \(t\) for the period \([0, T]\), but only if there is no default prior to the time the payment is received.
- The third is a random payment that occurs only at default of \(\Psi_{\tau}\), and is zero otherwise. This payment is made only if default occurs during the time period \([0, T]\).

Under appropriate integration conditions, the present values of these credit risky cash flows are (proofs are in the appendix):

\[
V_t 1_{\{\tau > t\}} = E_t \left( Y_T 1_{\{\tau > T\}} e^{-\int_{t}^{\tau} f(u)du} \right) 
\tag{4}
\]

\[
= E_t \left( Y_{T_1} e^{-\int_{t}^{T_1} f(u)du} \right) 
\tag{4}
\]

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II. BINARY DEFAULT SWAP

We specialize the general structure to value a binary default swap (sometimes called a credit swap); see Rooney [1998]. In this credit swap there are two counterparties, the protection seller and the protection buyer. We assume that both counterparties are default-free. This assumption can easily be relaxed as discussed in Jarrow and Turnbull [1995].

The swap lasts for a fixed period of time [0, T], the swap’s maturity. The protection seller receives a fixed payment flow of \( c_T \) dollars per unit time from the protection buyer unless there is default on a reference credit [a particular firm with intensity \( \lambda(t) \)]. Default is called a credit event. If default occurs, the protection seller pays the protection buyer $1 on the default date \( \tau \). If there is no default, the protection seller makes no payment to the protection buyer. The swap terminates either after the default event occurs or at the maturity date of the swap.

In traded default swaps, the payments occur at fixed intervals. If default occurs between payment dates, the accrued portion of the payment due at the next payment date is due at the time of default. The continuous payment structure provides a reasonable approximation of this condition.

The value of the swap to the protection seller is:

\[
V_T 1_{\{\tau > T\}} = E_t \left( c_T 1_{\{\tau > T\}} e^{-\int_t^T |r(u)+\lambda(u)| du} \right) -
\]

\[
E_t \left( 1_{\{\tau > T\}} \Psi_T e^{-\int_t^T |r(u)+\lambda(u)| du} \right)
\]

Using Equations (5) and (6), we obtain:

\[
V_T 1_{\{\tau < T\}} = E_t \left( c_T 1_{\{\tau < T\}} \Psi_T e^{-\int_t^T |r(u)+\lambda(u)| du} \right) -
\]

\[
E_t \left( 1_{\{\tau < T\}} \Psi_T e^{-\int_t^T |r(u)+\lambda(u)| du} \right)
\]

We can recognize this as the value of a risky coupon bond of maturity \( T \) paying a continuous cash flow of \( c_T \) dollars per unit time with a zero recovery rate less the cost of a dollar default insurance on the firm; that is:

\[
V_T 1_{\{\tau < T\}} = c_T \int_t^T \nu(t, s; 0) ds -
\]

\[
E_t \left( \int_t^T \lambda(s) e^{-\int_t^s |r(u)+\lambda(u)| du} ds \right)
\]

This simple formula applies when both market risk and credit risk are correlated. Whether or not we obtain a simple analytic formula for the price of a default swap depends on the evaluation of the second term on the right-hand-side of Equation (9).

In standard default swaps, the cash payment \( c_T \), called the default swap rate, is determined at time 0 such that \( V_0 = 0 \); that is:

\[
c_T = \frac{E_0 \left( \int_0^T \lambda(s) e^{-\int_s^T |r(u)+\lambda(u)| du} ds \right)}{\int_0^T \nu(0, s; 0) ds}
\]
III. EMPirical SPECIFICATION

To obtain a simple but realistic empirical formulation of the model, we use a special case of Jarrow [2001], where the economy is Markovian in a single state variable—the spot rate of interest. Although Jarrow includes the cumulative excess return on an equity market index, Janosi, Jarrow, and Yildirim [2000] find that this adds no additional explanatory power in the pricing of corporate debt.

For the spot rate of interest, we use a single-factor model with deterministic volatilities, sometimes called the extended Vasicek model (spot rate evolution):

$$dr(t) = [\bar{r}(t) - r(t)]dt + \sigma_r dW(t)$$  \hspace{1cm} (11)$$

where $\alpha \neq 0, \sigma_r > 0$ are constants, $\bar{r}(t)$ is a deterministic function of $t$ chosen to match the initial zero-coupon bond price curve, and $W(t)$ is a standard Brownian motion under $Q$ initialized at $W(0) = 0.$ The evolution of the spot rate is given under the risk-neutral probability $Q.$

We assume that the intensity function is almost linear with the spot rate of interest.

$$\lambda(t) = \max[\lambda_0(t) + \lambda_1 r(t), 0]$$  \hspace{1cm} (12)$$

where $\lambda_0(t) \geq 0$ is a deterministic function of time, and $\lambda_1$ is a constant. In this formulation, the (pseudo) probability of default per unit time is assumed to be the maximum of a linear function of the spot rate $r(t)$ and zero. The maximum operator is necessary to keep the intensity function non-negative. For analytic convenience, we drop the maximum operator in the empirical implementation. This linear approximation implies that negative default rates ($\lambda(t) < 0$) are possible.

Finally, to price risky debt, we assume that the recovery rate is a constant. In fact, in the valuation of default swaps, this constant recovery rate assumption is unnecessary. It is included only to provide Equation (14) for risky debt prices when the recovery rate is non-zero:

$$\delta(t) = \delta$$

where $\delta$ is a constant.

Given this structure, Jarrow [2001] obtains closed-form solutions for default-free and risky zero-coupon bond prices:

$$p(t, T) = e^{-\mu_1(t, T) + \sigma_1^2(t, T)/2}$$  \hspace{1cm} (13)$$

and

$$r(t, T; \delta) = p(t, T)e^{-\int_0^\tau \lambda_0(s)[1-\delta] - \lambda_1(s)[1-\delta]ds + \left(2\lambda_0(s)[1-\delta] + \lambda_1(s)[1-\delta]^2\right)\sigma_1^2(t, T)/2}$$  \hspace{1cm} (14)$$

where no default has occurred at or prior to time $t$:

$$\mu_1(t, T) = \int_t^T f(u, s)du + \int_t^T b(u, T)^2 du / 2,$$

$$\sigma_1^2(t, T) = \int_t^T b(u, T)^2 du$$

and

$$b(u, t) = \sigma_1 \left(1 - e^{-u(t-u)}\right) / t$$

Substitution of the linear intensity into the default swap’s valuation yields Equation (15). The proof is in the appendix.

$$V_s(t; T; \delta) = e_T \left[ r(t, s; 0) ds - \int_t^T \left[ (\lambda_0(s) + \lambda_1Got\delta_0(t, s) - (1 + \lambda_1)\sigma_0(t, s)]f(t, s; 0) ds \right. \right]$$  \hspace{1cm} (15)$$

where

$$\mu_0(t, s) = f(t, s) + b(t, s)^2 / 2$$

and

$$\sigma_0(t, s) = b(t, s)^2 / 2$$
This is the desired analytic expression for the fair value of a default swap, easily computed using only knowledge of the default-free term structure evolution and the current valuation of the firm’s risky debt prices. If risky debt prices are not available, the risky debt prices can be estimated using the analytic formula given by Equation (14).

Finally, the default swap rate is determined by solving Equation (15):

\[
    c_T = \frac{\int_0^T [\lambda_0(s) + \lambda_1(s) - (1 + \lambda_1(s)) \sigma_{01}(s,s)]e(0,s;0)ds}{\int_0^T v(0,s;0)ds} 
\]  

(16)

IV. DESCRIPTION OF THE DATA

Data from Enron’s web site cover August 21–October 31, 2000. We select 22 different firms to represent various industry groupings: financial, food and beverages, petroleum, airlines, utilities, department stores, and technology. The 22 firms are listed in Exhibit 1. For parameter estimation of the spot rate process, daily U.S. Treasury bond, note, and bill prices were downloaded from Bloomberg for the same time period.

V. ESTIMATION OF SPOT RATE PROCESS PARAMETERS

To implement estimation of the default and liquidity discount parameters, we first need to estimate the parameters for the state variable processes \((r(t))\). The inputs to the spot rate process evolution are the forward rate curves \((f(t, T))\) and the spot rate parameters \((a, \sigma_r)\).

We use a two-step procedure for estimation of the forward rate curves. First, for a given time \(t\), the discount bond prices \([p(t, T)]\) are estimated by solving a minimization problem:

**EXHIBIT 1**
Default Swap Quotes—8/21/00-10/31/00 (basis points)

<table>
<thead>
<tr>
<th>Firm</th>
<th>Avg (c_1)</th>
<th>Avg (c_2)</th>
<th>Avg (c_3)</th>
<th>Avg (c_4)</th>
<th>Avg (c_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>American Airlines Inc. AMR</td>
<td>149.6944</td>
<td>167.0556</td>
<td>175.0278</td>
<td>183.9444</td>
<td>196.2222</td>
</tr>
<tr>
<td>Archer-Daniels-Midland Co. ADM</td>
<td>32.4722</td>
<td>38.2222</td>
<td>43.5000</td>
<td>47.5556</td>
<td>53.5278</td>
</tr>
<tr>
<td>CP &amp; L Energy Inc. CPL</td>
<td>51.3056</td>
<td>57.4167</td>
<td>60.5278</td>
<td>64.0833</td>
<td>68.3889</td>
</tr>
<tr>
<td>Chase Manhattan Corp. CMB</td>
<td>30.3889</td>
<td>35.3889</td>
<td>37.2778</td>
<td>39.0000</td>
<td>41.8333</td>
</tr>
<tr>
<td>Coca-Cola Enterprises CCE</td>
<td>35.3056</td>
<td>40.4167</td>
<td>43.5833</td>
<td>45.0556</td>
<td>49.2778</td>
</tr>
<tr>
<td>Dow Chemical Company DOW</td>
<td>23.5000</td>
<td>29.3333</td>
<td>35.1111</td>
<td>40.6667</td>
<td>46.6389</td>
</tr>
<tr>
<td>Delta Air Lines Inc. DAL</td>
<td>130.1944</td>
<td>138.7222</td>
<td>146.4722</td>
<td>151.2222</td>
<td>161.3333</td>
</tr>
<tr>
<td>Eastman Kodak Co. EK</td>
<td>26.7222</td>
<td>29.1667</td>
<td>31.5833</td>
<td>33.6944</td>
<td>36.6667</td>
</tr>
<tr>
<td>First Union Corp. FTU</td>
<td>32.8333</td>
<td>36.6667</td>
<td>38.2222</td>
<td>41.0278</td>
<td>43.8056</td>
</tr>
<tr>
<td>K Mart Corp. KM</td>
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<td>369.3611</td>
<td>458.2778</td>
<td>471.0833</td>
<td>485.9167</td>
</tr>
<tr>
<td>Lyondell Chemical Company LYO</td>
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<td>349.6389</td>
<td>367.5000</td>
<td>374.6667</td>
<td>387.7222</td>
</tr>
<tr>
<td>Merrill Lynch &amp; Co. MER</td>
<td>27.0278</td>
<td>30.0556</td>
<td>40.0833</td>
<td>43.6389</td>
<td>47.9722</td>
</tr>
<tr>
<td>Phillips Petroleum Co. P</td>
<td>58.5278</td>
<td>67.0556</td>
<td>72.6667</td>
<td>75.1944</td>
<td>79.5000</td>
</tr>
<tr>
<td>Ralston-Ralston Purina Group RAL</td>
<td>61.4444</td>
<td>69.6667</td>
<td>73.9167</td>
<td>76.4167</td>
<td>81.1111</td>
</tr>
<tr>
<td>Sears, Roebuck and Co. S</td>
<td>60.5833</td>
<td>64.7222</td>
<td>71.4722</td>
<td>73.6111</td>
<td>77.0556</td>
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<tr>
<td>Southwest Airlines LUV</td>
<td>34.8611</td>
<td>40.5278</td>
<td>42.7778</td>
<td>44.5278</td>
<td>46.3333</td>
</tr>
<tr>
<td>TXU Corporation TXU</td>
<td>83.3611</td>
<td>89.0278</td>
<td>95.1389</td>
<td>103.0278</td>
<td>104.8611</td>
</tr>
<tr>
<td>Union Oil Co. of California UCL1</td>
<td>59.9444</td>
<td>67.9167</td>
<td>72.3889</td>
<td>74.8889</td>
<td>79.5000</td>
</tr>
<tr>
<td>Wal-Mart Stores Inc. WMT</td>
<td>13.5000</td>
<td>16.2778</td>
<td>17.2778</td>
<td>18.1389</td>
<td>20.4444</td>
</tr>
<tr>
<td>Xerox Corp. XRX</td>
<td>161.2778</td>
<td>169.5278</td>
<td>214.5833</td>
<td>189.8333</td>
<td>227.8056</td>
</tr>
<tr>
<td>Texas Instruments Inc. TXN</td>
<td>40.8889</td>
<td>44.3333</td>
<td>50.2778</td>
<td>52.6944</td>
<td>56.1389</td>
</tr>
</tbody>
</table>
Choose \( p(t, T) \) for all relevant \( T \leq \max\{T_i ; i \in I \} \)
to minimize \[
\sum_{i \in I} \left[ B_i(t, T_i) - B_i(t, T_i)^{\text{bid}} \right]^2
\] (17)

where \( I \) is an index set including the various U.S. Treasury bonds, notes, and bills available at time \( t \); \( B_i(t, T_i) \) is the model price [Equation (4)] for the \( i \)-th bond with maturity \( T_i \) as a function of \( p(t, T) \); and \( B_i(t, T_i)^{\text{bid}} \) is the market bid price for the \( i \)-th bond with maturity \( T_i \).

The discount bond price maturity dates coincide with the maturities of the Treasury bills, and the coupon payments and principal repayment dates are the same as for the Treasury notes and bonds.

Second, we fit a continuous forward rate curve to the estimated zero-coupon bond prices \( p(t, T) \) for all \( T \leq \max\{T_i ; i \in I \} \). We use the maximum-smoothness forward rate curve as developed by Adams and van Deventer [1994] and refined by Janosi and Jarrow [2000]. Briefly, we choose the unique piecewise fourth-degree polynomial with the left and right end points left “dangling” that minimizes

\[
\max\{T_i ; i \in I\}
\int_t^T \left[ \frac{a}{ds} \right]^2 ds
\]

For the spot rate parameter \((a, \sigma)\) estimation, the procedure follows that used in Janosi, Jarrow, and Yildirim [2000]. The procedure is based on an explicit formula for the variance of the default-free zero-coupon bond prices derived using Equation (7); see Heath, Jarrow, and Morton [1992].

For \( \Delta = 1/365 \) (a day), the expression is:

\[
\text{var}[\log(p(t + \Delta, T) / p(t, T)) - r(t)\Delta]
= \left( \sigma^2 \left( e^{-\sigma(T-t)} - 1 \right) / a^2 \right) \Delta
\] (18)

First we fix a time to maturity \( T - t \in \{3 \text{ months}, 6 \text{ months}, 1 \text{ year}, 5 \text{ years}, 10 \text{ years}, \text{the longest time to maturity of an outstanding Treasury bond closest to 30 years} \} \). Then, we compute the sample variance, denoted \( \nu_\Delta \), using the smoothed forward rate curves generated over the sample period.

To estimate the parameters \((\sigma, a)\) we run the non-linear regression:

\[
v_T = \left( \sigma^2 \left( e^{-\sigma(T-t)} - 1 \right) / a^2 \right) \Delta + e_T
\] (19)

across the bond’s time to maturities \( T - t \in \{\frac{1}{4}, \frac{1}{2}, 1, 5, 10, \text{longest time to maturity closest to 30}\} \), where \( e_T \) is the error term.

The parameter estimates (and their standard errors) for this non-linear regression are: \( \sigma = 0.00593 \) (0.0021) and \( a = 0.03450 \) (0.0003).

VI. DEFAULT PARAMETER ESTIMATION

To illustrate the use of Equation (16), we deduce the default parameters consistent with the given term structure of default swap quotes obtained from Enron’s web page. The default swap rates obtained are for contracts of maturities one, two, three, four and five years. This gives five observations per day to estimate the default parameters in Equation (16). The default parameters to estimate are the deterministic function \( \lambda_0(t) \) and the constant \( \lambda_1 \).

Exhibit 1 provides the averages of the default swap quotes for the various companies over the observation period. Notice that for all companies the average swap quotation increases with the maturity of the swap. For example, the American Airlines quotes are 149.6944 basis points for the one-year swap and 196.2222 bp for the five-year swap.

Two formulations of our analytic expression are estimated. Model 1, the standard model, has default risk independent of interest rate risk. This is the model in Martin, Thompson, and Browne [2001]. The second model, Model 2, is a simple two-parameter version of Equation (16) that incorporates correlated interest rate and default risk. We choose the simplest form of the general model to provide a base case for subsequent empirical investigation into the model’s validity.

In Model 1, the standard model, \( \lambda_0(t) \) is piecewise-constant and \( \lambda_1 = 0 \) where:
In this model, the function $\lambda_0(t)$ is calibrated to exactly match the term structure of default swap quotes. Because $\lambda_1 = 0$, default risk and interest rate risk are uncorrelated. Under this structure, Equation (16) becomes:

$$c_k = \frac{\sum_{j=1}^{k} \int e^{-\lambda_0(t)p(0,s)ds}}{\sum_{j=1}^{k} \int e^{-\lambda_0(t)p(0,s)ds}} \text{ for } k = 1, 2, ..., 5$$

We can invert this system for $\lambda_0(t)$. This model gives an exact fit to the term structure of default swap quotes.

In Model 2, for correlated defaults, $\lambda_0(t) = \lambda_1$ and $\lambda_1$ are constants:

$$c_T = \frac{\int_{0}^{T} [\lambda_0 + \lambda_1(s) - (1 + \lambda_1(s))\sigma_0(0,s)]p(0,s)ds}{\int_{0}^{T} p(0,s)ds}$$

This is also a special case of our model because the intercept of the intensity process, a deterministic function, is restricted to be a constant in order to provide a base case for subsequent investigation.

We can invert this valuation formula for both $\lambda_0$ and $\lambda_1$. Since this is only a two-parameter model, there will be errors in matching the term structure of default swap quotes. We choose the parameters to minimize the sum of squared errors between the theoretical and the market quotes.

Exhibit 2 shows the average default intensity parameters for Model 1 over the sample period, measured in basis points. These parameter values exactly match the term structure of default swap quotes.

### Exhibit 2

**Model 1 Average Default Intensity Parameters (bp)**

<table>
<thead>
<tr>
<th>Company</th>
<th>Avg $\lambda_0^1$</th>
<th>Avg $\lambda_0^2$</th>
<th>Avg $\lambda_0^3$</th>
<th>Avg $\lambda_0^4$</th>
<th>Avg $\lambda_0^5$</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>American Airlines Inc. AMR1</td>
<td>149.6944</td>
<td>25.8551</td>
<td>25.9323</td>
<td>25.5596</td>
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<td>36</td>
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<td>32.4722</td>
<td>24.2311</td>
<td>23.8048</td>
<td>24.3386</td>
<td>23.7516</td>
<td>36</td>
</tr>
<tr>
<td>Chase Manhattan Corp. CMB</td>
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<td>24.3501</td>
<td>23.1122</td>
<td>23.5833</td>
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<td>36</td>
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<td>Coca-Cola Enterprises CCE</td>
<td>35.3056</td>
<td>24.0027</td>
<td>24.9629</td>
<td>24.8912</td>
<td>24.9134</td>
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<td>26.7222</td>
<td>19.3607</td>
<td>16.7585</td>
<td>17.6705</td>
<td>17.7911</td>
<td>36</td>
</tr>
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<td>First Union Corp. FTU</td>
<td>32.8333</td>
<td>23.7851</td>
<td>24.5558</td>
<td>24.7381</td>
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**EXHIBIT 4**
American Airlines Parameter Estimates Across Time

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Exhibit 3 provides the average default intensity parameters for Model 2 over the sample period, measured in basis points. As noted, for all firms $\lambda_i$ is positive, indicating that as interest rates rise, the likelihood of default increases as well. The sign of the interest rate coefficient in these intensity functions is consistent with simple economic intuition.

Exhibit 4 graphs the parameter estimates for American Airlines for both Models 1 and 2 over the time period. Exhibit 5 shows the actual swap rate quotes for American Airlines over the period and the computed swap rate curves for Model 2. The simple two-parameter model is able to match the shape of the default swap term structure reasonably well. This is interesting, given that it is only a two-parameter model.

Exhibit 6 presents the average pricing errors and the standard errors for Model 2. Note that the errors tend to be negative at the short end (years 1 and 2) of the default swap term structure and positive at the long end (years 4 and 5). The best matching occurs for the three-year quote. The average pricing errors are $-13.93, -7.75, 3.24, 5.83,$ and $12.61$.

Although there is pricing error present in the estimated parameters in Exhibit 3, it is important to note that these errors can be eliminated completely. Indeed, by calibrating the intercept function of the intensity $\lambda_0(t)$ as in the standard model, one can exactly match the default swap term structure using a model with correlated market and credit risk.

VII. SUMMARY

We have provided a simple analytic formula for valuing default swaps with correlated market and credit risk. We illustrate its implementation by deducing the default probability parameters implicit in default swap quotes for 22 companies over a ten-week period. This simple analytic formula can be calibrated to exactly match the observed default swap term structure.
## Exhibit 6
### Model 2 Errors

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Average standard errors are under each point estimate.
APPENDIX
Proofs and Derivations

Facts about the Spot Rate r(t)

From Equation (7), we define

\[ X_0 = \int_t^T r(s) ds \]
\[ = f(t) + b(t) \int_t^T s^2 ds/2 + \int_t^T p(v,s) dW(v) \]

where \( p(v,s) = \sigma^2 e^{\rho(s-t)} \)

and

\[ b(t) = \int_t^T p(t,v) dv = \frac{\sigma^2 (1-e^{-\rho(s-t)})}{\rho(s-t)} \]
\[ \text{for } a \neq 0 \]
\[ = \int_t^T f(t,s) ds + \int_t^T b(v,T)^2 ds/2 \]
\[ = \int_t^T f(t,s) ds + \int_t^T b(v,T)^2 ds/2 \]

A direct computation gives

\[ \mu_1(t,T) = E_t \left( \int_t^T r(s) ds \right) \]
\[ = \int_t^T f(t,s) ds + \int_t^T b(v,T)^2 ds/2 \]  

and

\[ \mu_0(t,s) = E_t (r(s)) = f(t,s) + b(t,s)^2/2 \]  

Also:

\[ \sigma^2_0(t,s) = \text{Var}_t (r(s)) = \int_t^T p(v,s)^2 dv \]

Define

\[ X_1 = \int_t^T r(s) ds \]
\[ = \int_t^T f(t,s) ds + \int_t^T b(v,T)^2 ds/2 + \int_t^T p(v,s) dW(v) ds \]

Changing the order of integration and a direct computation yields

\[ \int_t^T b(v,s)^2 ds/2 = \int_t^T b(v,T)^2 ds/2 \]

and

\[ \int_t^T b(v,s)^2 ds/2 = \int_t^T b(v,T)^2 ds/2 \]  

Finally, from Heath, Jarrow, and Morton [1992, p. 88]:

\[ p(s,T) = p(t,T) e^{\frac{1}{2} \int_s^T \{ \sigma^2 (s) - \int_s^T (r(v)) dv \} dv} \]

It can be shown that

\[ \text{cov}_t (\int_t^T r(u) du, \int_t^T r(u) du) = \sigma^2_1(t,s) \text{ for } T > s \]  

and

\[ \text{cov}_t (\int_t^T b(u,T) dW(u), \int_t^T r(u) du) = \int_t^T b(u,T) b(u,s) du \]
Facts about Normal Distributions

Given that \((x, y)\) is bivariate-normal, we have (see Hogg and Craig [1970, p. 114]):

\[
E[e^{A+BY}] = e^{\mu_x A + \mu_y B + \frac{1}{2}(\sigma_x^2 A^2 + 2\sigma_{xy} AB + \sigma_y^2 B^2)}
\]

(A-9)

where

\[
\mu_x \equiv E(x), \mu_y \equiv E(y), \sigma_x^2 \equiv Var(x), \sigma_y^2 \equiv Var(y)
\]

and \(\sigma_{xy} \equiv Cov(x, y)\).

Proof of General Valuation Formulas

Let \(E(\cdot)\) denote the conditional expectation with respect to the information set generated by \(\{X_s, N_s\}\) for \(s \leq t\), \(\{X_s\}\) is a vector stochastic process representing the state variables. Let \(E(\cdot | X_T)\) denote the conditional expectation with respect to the information set generated by \(\{X_s, N_s\}\) for \(s \leq t\) and conditioned on the information set \(\{X_s\}\) for \(s \leq T\) as well.

Derivation of Equation (4)

Using the facts about Cox processes, we have that:

\[
E_t(Y_T^{\frac{T-t}{T-t}}) = E_t(E_t(Y_T^{\frac{T-t}{T-t}} | X_T))
\]

(A-10)

Derivation of Equation (5)

Using the facts about Cox processes, we have that:

\[
E_t(Y_T^{\frac{T-t}{T-t}} | X_T) = E_t(E_t(Y_T^{\frac{T-t}{T-t}} | X_T) | X_T)
\]

(A-11)

Derivation of Equation (6)

Using the facts about Cox processes, we have that:

\[
E_t(Y_T^{\frac{T-t}{T-t}} | X_T) = E_t(E_t(Y_T^{\frac{T-t}{T-t}} | X_T) | X_T)
\]

(A-12)

Hence, the probability of default prior to time \(T\) is:

\[
Q_t(\tau \leq T) = 1 - Q_t(\tau > T) = 1 - E_t(e^{-\int \lambda_t \, ds})
\]

(A-13)

The conditional density function is:

\[
Q_t(\tau > T) = E_t(e^{-\int \lambda_t \, ds})
\]

(A-14)

Substitution of the linear intensity into:

\[
I_t(\tau < t) = E_t\left(\int_{\{s < t\}} e^{-\int \lambda(s) \, ds} \right)
\]

(A-15)

gives:

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\[ I(T)\{t \leq T\} = E_t \left[ \int_0^T \left( \lambda_0(s) + \lambda_T(s) \right) e^{-s} ds \right] \]

Algebra and interchanging the order of integration and expectation gives:

\[ I(T)\{t \leq T\} = \left( \int_0^T \lambda_0(s) E_t(e^{-s}) ds \right) + \left( \int_0^T \lambda_1(s) E_t(e^{-s}) ds \right) \]

Using Equation (14) for \( v(t, s; 0) \), we get

\[ \left( \int_0^T \lambda_0(s) e^{-s} ds \right) + \left( \lambda_1 \int_0^T E_t(e^{-s}) ds \right) \]

Using Equation (A-9) with

\[ x \equiv r(t) = X_0, \ y \equiv \int r(u) du = X_1, \ A \equiv 0, \ B \equiv -1 + \lambda_1 \]

and expressions (A-1)-(A-5) gives:

\[ \int_0^T \lambda_0(s) e^{-s} ds + \lambda_1 \int_0^T E_t(e^{-s}) ds \]

\[ [\mu_0(t, s) - (1 + \lambda_1)\sigma_0(t, s)]ds \]

Using Equation (14) gives the result.

ENDNOTES

1See Jarrow and Turnbull [1995]. No-arbitrage guarantees the existence but not the uniqueness of the probability \( Q \). Without any additional hypotheses on the economy, the uniqueness of \( Q \) is equivalent to markets being complete. See Battig and Jarrow [1999].

2The random variables \( Y_s \) and \( r \) and the stochastic process \( \Psi_t \) are measurable with respect to the information set generated by \( \{ X_s; 0 \leq s < T \} \).

3In particular:

\[ f(t) = f(0, t) + \left( \frac{\partial f(0, t)}{\partial t} + \sigma_t^2 \left( 1 - e^{-2at} \right) \right) / 2a | \text{for a } a \neq 0. \]

4Martin, Thompson, and Browne [2001] investigate a procedure for fitting a smoothed intensity function to market data. Although this smoothing procedure could also be used here, we do not focus on this aspect of the estimation.

REFERENCES


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