A generalized coherent risk measure: The firm’s perspective

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Abstract

This note extends the concept of a coherent risk measure to make it more consistent with a firm’s capital budgeting perspective. A coherent risk measure defines the risk of a portfolio to be that amount of cash that must be added to the portfolio such that it becomes acceptable to a regulator. As such, a coherent risk measure implicitly assumes that the firm has already made its capital budgeting decision. Except for a cash infusion, the portfolio composition remains unchanged. We propose a generalized version of a coherent risk measure that also allows the portfolio composition to change as well. Once the investment decisions are fixed, our measure collapses to a coherent risk measure.

Keywords: Coherent risk measures; Capital budgeting; Capital determination

1. Introduction

Risk management is different from the firm’s and the regulator’s perspective. The firm’s goal is to choose the composition of its (assets and liabilities) portfolio so as to maximize its risk/return trade-off, subject to any regulatory capital requirements. In contrast, a regulator takes the firm’s portfolio composition as fixed, and the regulator’s goal is to determine the capital the firm must have in order to limit the consequences of ruin within a given time-
frame. These two different perspectives can be broadly classified as ‘capital budgeting’ and ‘capital determination,’ respectively.

The recently developed coherent risk measure of Artzner, Delbaen, Eber and Heath (1999) (ADEH) is well-suited for the ‘capital determination’ problem, but not the ‘capital budgeting’ problem. Indeed, a coherent risk measure can be defined as the minimum extra cash (capital) that the firm needs to add to its portfolio such that insolvency risk is acceptable to a regulator. In this note we provide a generalization of the coherent risk measure that is better suited to the firm’s capital budgeting decision. Broadly speaking, under a generalized coherent risk measure, the risk of a portfolio is the minimum quantity invested in any marketable security (and not only cash) such that the original portfolio, along with the modified security, becomes acceptable. Thus, in our framework a portfolio’s risk may also be reduced by buying insurance contracts or other assets negatively correlated with the portfolio’s payoffs. Once the portfolio composition is fixed, the regulator’s measure can be applied.

An outline for this note is as follows. Section 2 reviews coherent risk measures. Section 3 presents our generalized coherent risk measure and discusses the notion of an acceptance set and the induced risk measure. Section 3 concludes the note.

2. Coherent risk measure

In their seminal paper ADEH start with the set of four axioms satisfied by a coherent risk measure. We briefly review these axioms for comparison with our generalized coherent risk measure. Fix a finite probability space \( \Omega \) and consider a one period economy in which the future net-worth of any portfolio is denoted as a random variable \( X \). Let \( L^0 \) represent the set of possible random variables. The constant random variable \( r \in L^0 \) is interpreted as cash (or capital). A risk measure \( \rho \) is defined to be any mapping from the set of all random variables \( (L^0) \) into the real line \( (R) \).

**Definition 2.1 (Coherent Risk Measure).** A coherent risk measure satisfies the following four axioms:

1. **Translation Invariance:** \( \rho(X + \alpha r) = \rho(X) - \alpha \) for all \( X \in L^0 \) and \( \alpha \in R \).
2. **Subadditivity:** \( \rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2) \) for all \( X_1 \) and \( X_2 \in L^0 \).
3. **Monotonicity:** \( \rho(X_1) \leq \rho(X_2) \) if \( X_1 \geq X_2 \).
4. **Positive Homogeneity:** \( \rho(\lambda X) = \lambda \rho(X) \) for \( \lambda \geq 0 \).

In addition ADEH also discuss the following axiom in their paper:

5. **Relevance:** \( \rho(X) > 0 \) if \( X \leq 0 \) and \( X \neq 0 \).

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1 We can think of regulator both as an external and an internal entity. An external authority can be government agencies such as the Federal Reserve Bank or FDIC. In the context of a multi-division setting, the head-office may set risk-limits for the divisions and thus the head-office may be viewed as an internal regulator.

2 \( X \geq 0 \) means \( X(\omega) \geq 0 \) for all \( \omega \in \Omega \).
In this framework, the translation invariance axiom gives a coherent risk measure the interpretation of being from the regulator’s perspective. To see this, we need the following definitions from ADEH (1999).

**Definition 2.2.** The acceptance set associated with a risk measure $\rho$ is the set $A_\rho$ defined by

$$A_\rho = \{ X \in L^0 : \rho(X) \leq 0 \}.$$

**Definition 2.3 (Acceptance Set).** Let the cone of non-negative elements in $L^0$ be denoted $L^0_+$ and its negative by $L^0_-$. ADEH argue that all reasonable risk measures have an acceptance set that satisfies the following properties:

1. The acceptance set contains $L^0_+$.
2. The acceptance set does not intersect $L^0_-$. 
3. The acceptance set is convex.
4. The acceptance set is a positively homogeneous cone.

**Definition 2.4.** Given an acceptance set $A$, the risk measure associated with an acceptance set

$$\rho_A(X) = \inf\{ m : m \cdot r + X \in A \}$$

is the minimum amount of capital that must be added to a portfolio $X$ to make it acceptable.

The regulator is interpreted as choosing the set $A$, and the risk measure determines the minimal amount of capital necessary to satisfy the regulator.

### 2.1. Correspondence between the risk measure and the acceptance set

With the above, ADEH prove two important propositions in their paper (Propositions 2.1 and 2.2). The propositions prove that a coherent risk measure inherits the regulator’s perspective.

**Proposition 2.1.** If the set $B$ satisfies the properties of an acceptance set as in Definition 2.3 above, then the risk measure $\rho_B$ is coherent. Moreover, the acceptance set induced by a risk measure $\rho_B$ is the closure of the set $B$.

**Proposition 2.2.** If a risk measure $\rho$ is coherent, then the acceptance set $A_\rho$ is closed and satisfies the axioms of Definition 2.3 above. Moreover, $\rho = \rho_{A_\rho}$.

By virtue of these two propositions, any coherent risk measure has the interpretation of being from the regulator’s perspective.
3. A generalized coherent risk measure

We consider the set up used by ADEH and presented in the previous section. As a slight modification of a risk measure, we define a risk measure \( \rho \) to be any function that maps the random variables, \( L^0 \), into the non-negative real line.\(^3\)

We introduce a norm \( \| \cdot \| \) on this space of random variables \( L^0 \). The norm provides a characterization of when two portfolios are equivalent, i.e., two random variables are viewed as equal if their norm difference is zero. Such a norm is induced in any economy by a specification of investors’ preferences, see Jarrow (1988) and Jarrow et al. (1999) for discussions related to this correspondence.

A risk measure satisfying the following axioms is said to be a generalized coherent risk measure for the norm \( \| \cdot \| \).

**Definition 3.1.** A generalized coherent risk measure for the norm \( \| \cdot \| \) satisfies

1. **Subadditivity:** \( \rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2) \) for all \( X_1, X_2 \in L^0 \).
2. **Monotonicity:** \( \rho(X_1) \leq \rho(X_2) \) if \( X_1 \geq X_2 \).
3. **Positive Homogeneity:** \( \rho(\lambda X) = \lambda \rho(X) \) for \( \lambda \geq 0 \).
4. **Relevance:** \( \rho(X) > 0 \) iff \( X \notin L^0_+ \).

Before stating the final axiom, we introduce a set \( \mathcal{H}_\rho \) defined as

\[
\mathcal{H}_\rho = \{ X \in L^0 : \rho(X) = 0 \}.
\]

Remark: The Subadditivity and Positive Homogeneity properties of \( \rho \) imply that \( \rho \) is a convex function on \( L^0 \), which in turn implies that it is continuous.

Remark: \( \mathcal{H}_\rho \) is closed as it is the zero set of the continuous mapping \( \rho \) on \( \mathbb{R}^n \).

5. **Shortest Path:** For each \( X \in L^0 \), we define the portfolio \( X^* \in \mathcal{H}_\rho \) to be the point of shortest distance\(^4\) from \( X \) to the set \( \mathcal{H}_\rho \). For any scalar \( 0 \leq \lambda \leq \rho(X) \), we have

\[ \rho(X + \lambda \cdot u) = \rho(X) - \lambda, \]

where \( u \) is the unit vector in the direction \( (X^* - X) \).\(^5\)

The key distinction of a generalized coherent risk measure is the shortest path (SP) axiom. The SP axiom exhibits linear risk reduction along the shortest path to the set of acceptable random variables. This induces the firm’s perspective, because the asset added to the portfolio \((u)\) need not be capital, but any risky investment. Coherent risk measures do not have this property.

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\(^3\) This differs from ADEH’s risk measure whose range space is \( \mathbb{R} \) and not \( \mathbb{R}^+ \).

\(^4\) The shortest distance is given by the norm \( \|X - X^*\| \). We can interpret this as the minimum re-balancing of the original portfolio \((X)\) needed such that the resulting portfolio \((X^*)\) belongs to the acceptance set. The existence of \( X^* \) is guaranteed since \( \mathcal{H}_\rho \) is closed.

\(^5\) I.e., \( u = \frac{X^* - X}{|X^* - X|} \)
For any risk $X$, an immediate consequence of the SP axiom is that by adding an amount equal to $\rho(X)$ along the shortest path, the risk of the portfolio reduces to zero, i.e., $\rho(X + \rho(X)u) = \rho(X) - \rho(X) = 0$.

3.1. Induced acceptance sets and risk functions

Following ADEH, we introduce the notion of an acceptance set.

**Definition 3.2 (Acceptable Set).** The set of acceptable portfolio holdings is defined as $L^0_+$. Note that $L^0_+$ is a closed, convex, and positively homogeneous cone.

We next introduce the notion of acceptance set induced by a risk measure (and vice-versa).

**Definition 3.3 (Induced Acceptance Set).** The acceptance set induced by a given $\rho$ equals $A_\rho = \{X \in L^0_+: \rho(X) = 0\}$.

Given an acceptance set $A$, define the distance function for the norm $\| \cdot \|$ by

$$d_A(X) = \inf_{X' \in A} \{\|X - X'\|\}. \quad (2)$$

We now state and prove two important propositions.

**Proposition 3.1 (Generalized Coherency of the Distance Function).** $d_{L^0_+}(\cdot)$ is a generalized coherent risk measure. Moreover $A_{d_{L^0_+}} = L^0_+$.

**Proof.**

(1) **Sub-additivity:** Consider two random variables $X$ and $Y$ and let $X^*$ and $Y^*$ respectively be the closest portfolio on the acceptance set $L^0_+$ under the given norm. Since $L^0_+$ is closed, $X^*$ and $Y^*$ exist. In other words, $X^* = \arg\inf\{\|X - X'\|: X' \in L^0_+\}$. Similarly for $Y^*$. Therefore,

$$d_{L^0_+}(X) = \|X - X^*\| \quad \text{and} \quad d_{L^0_+}(Y) = \|Y - Y^*\|. \quad (3)$$

Consider the portfolio $(X + Y)$ and let $(X + Y)^*$ be the closest acceptable portfolio for $(X + Y)$. Also note that $X^* + Y^*$ belongs to the acceptance set $L^0_+$. Thus, $\|X + Y - (X + Y)^*\| \leq \|X + Y - (X^* + Y^*)\|$. Thus, $d_{L^0_+}(X + Y) \leq \|X - X^* + Y - Y^*\| \leq \|X - X^*\| + \|Y - Y^*\| = d_{L^0_+}(X) + d_{L^0_+}(Y)$.

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6 To streamline the presentation, we restrict our attention to $L^0_+$ rather than an arbitrary “acceptance” set as in ADEH. The generalization to an arbitrary closed and convex set is left to subsequent research.
(2) **Monotonicity:** Consider two random variables $X$ and $Y \in L^0$ and let $X \leq Y$. Consider a random variable $Y - X$. Since $Y - X \geq 0$, it follows that $Y - X \in L^0$. Since $Y = X + Y - X$, by sub-additivity of the induced risk measure (proved above), it follows that $d_{L^0}(Y) \leq d_{L^0}(X) + d_{L^0}(Y - X)$. Using the fact that $d_{L^0}(Y - X) = 0$ (because this random variable is already in the set $L^0$), we get $d_{L^0}(Y) \leq d_{L^0}(X)$.

(3) **Positive Homogeneity:** Consider a risk $X$ and a scalar $\lambda \geq 0$. Define $X^\ast$ as in the proof of subadditivity.

$$d_{L^0}(\lambda X) = \|\lambda X - (\lambda X)^\ast\| \leq \|\lambda X - \lambda X^\ast\| = \lambda d_{L^0}(X).$$

For the reverse direction, using the same reasoning:

$$d_{L^0}(X) \leq \frac{1}{\lambda} d_{L^0}(\lambda X).$$

(4) **Relevance:** Consider a risk $X \notin L^0$. It must be proved that $d_{L^0}(X) > 0$. Using $d_{L^0}(X) = \inf_{X' \in L^0_+} \|X - X'\|$ it follows immediately that $d_{L^0}(X) > 0$. The other direction is trivial.

(5) **Equality of $L^0_+$ and $A_{d_{L^0_+}}$:** A direct interpretation of Definitions 3.3 and a distance function.

(6) **Shortest Path:** The set $H_{d_{L^0_+}}$ is defined as the set of all random variables $X$ such that $d_{L^0_+}(X) = 0$. Thus from the above proof, it is easy to see that the set $H_{d_{L^0_+}}$ is same as $L^0_+$. Thus to prove the shortest path axiom for $d_{L^0_+}$ we need to show that for every $X$ there exists a point $X^\ast \in L^0_+$ such that there is a linear reduction in risk along the path $X^\ast - X$. Since $L^0_+$ is a closed and convex set and $X$ a point outside this set, there exists a point $X^\ast$ on the boundary of $L^0_+$ such that $\|X - X^\ast\|$ is the unique minimum distance. □

**Proposition 3.2** (Induced Acceptance Set). If a risk measure $\rho$ is a generalized coherent risk measure according to Definition 3.1, then the induced acceptance set in Definition 3.3 $A_\rho = L^0_+$. Moreover, $\rho = d_{L^0_+}$.

**Proof.** Let $\rho(X)$ satisfy the axioms of Definition 3.1.

(1) By the relevance axiom $X \in A_\rho$, i.e., $\rho(X) = 0$ iff $X \in L^0_+$.

(2) Now we need to show that $\rho(X) = d_{L^0_+}(X)$ for any $X \in L^0_+$.

Consider a generalized coherent risk measure $\rho$. For any $X \in L^0_+$, the proof is trivial since $X \in L^0_+$ implies that $\rho(X) = 0$ as well as $d_{L^0_+}(X) = 0$.

Suppose that $X \notin L^0_+$. In this case let $X^\ast$ be a random variable as defined in the shortest path axiom of Definition 3.1. Thus $d_{L^0_+}(X) = \|X - X^\ast\|$. Define $u$ as the unit vector in the direction of $X^\ast - X$. Note that $X + \|X - X^\ast\|u \in L^0_+$ which means that $\rho(X + \|X - X^\ast\|u) = 0$. This, with shortest path axiom, implies that $\rho(X) - \|X - X^\ast\| = 0$, i.e., $\rho(X) = \|X - X^\ast\| = d_{L^0_+}(X)$. □
This proposition provides a complete characterization of a generalized coherent risk measure. Any generalized coherent risk measure is equivalent to the distance function determined by the norm $\| \cdot \|$. In our setting, a coherent risk measure is defined by

$$d_{L_0^+}^{\text{coh}}(X) = \inf_{X' \in L_0^+, \|X' - X\| = k,r} \{\|X - X'\|\}$$

for some constant $k$. Under the ADEH framework, the risk of a portfolio is measured as the distance along the path of the riskless rate. Therefore, it immediately follows that $d_{L_0^+}^{\text{coh}}(X) \geq d_{L_0}(X)$.

4. Conclusion

In this note, we study a generalization of the coherent risk measure well-suited for the capital budgeting decisions. Once the investment decisions are made, the subsequent capital determination decision can be made using the original coherent risk measure.

References

